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OSCILLATION OF FIRST ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

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In this paper, the authors establish the oscillation of the first order nonlinear neutral delay difference equation

$$\Delta (x_n + px_{n-r}) + q_n f(x_{n-k}) = 0$$

by comparing it with the second order non-neutral non-delay linear difference equation

$$\Delta^2 y_{n-1} + \frac{2(m+1)^m}{m^{m+1}} \left[Q_n - \frac{m^m}{(m+1)^{m+1}} \right] y_n = 0.$$

Here, $p > 1$, $\{q_n\}$ is a sequence of nonnegative real numbers, k and r are positive integers such that $m = k - r$. is positive, and $\{Q_n\}$ is a sequence of nonnegative real numbers related to the sequence $\{q_n\}$.

1. INTRODUCTION

Consider the neutral difference equation

$$\Delta (x_n + px_{n-r} + q_n f(x_{n-k}) = 0, \qquad n = 0, 1, 2, \ldots \qquad \ldots (1)$$

where $p \in (1, \infty)$, $\{q_n\}$ is a nonnegative sequence of real numbers, $f: \mathbb{R} \to \mathbb{R}$ is continuous, $uf(u) > 0$ for $u \neq 0$, $f(u)/u \geq M > 0$, k and r are positive integers, and $m = k - r > 0$. The forward difference operator Δ is defined as $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 x_n = \Delta(\Delta x_n)$. By a solution of equation (1), we mean a real sequence $\{x_n\}$ denned for $n \geq -\theta$, where $\theta = \max\{k, r\}$, and which satisfies equation (1) for $n \geq 0$. A solution $\{x_n\}$ of equation (1) is said to be nonoscillatory if the terms x_n are either eventually all positive or eventually all negative, and oscillatory otherwise.

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For some time there has been an increasing interest in the study of oscillation and nonoscillation of solutions of neutral delay difference equations, and as recent examples we refer the reader to the survey monographs of Agarwal *et al.* [1]-[3], the papers [4]-[12], and the references contained therein. In a recent paper, Zhang and Zhou [11] proved some very interesting oscillation results for the first order linear delay difference equation

$$\Delta x_n + q_n x_{n-k} = 0, \qquad n = 0, 1, 2, \dots, \qquad \dots (2)$$

based on known oscillation results for the second order linear non-delay equation

$$\Delta^2 y_{n-1} + \frac{2(k+1)^k}{k^{k+1}} \left[q_n - \frac{k^k}{(k+1)^{k+1}} \right] y_n = 0, \qquad n = 0, 1, 2, \dots . \qquad \dots (3)$$

In this paper, we show how this approach can also be used to obtain the oscillation of all solutions of the nonlinear neutral delay difference equation (1) by comparing it with a second order non-neutral non-delay linear equation.

2. Some Preliminary Lemmas

In this section, we present some lemmas that will be used in the proof of our main results. The first lemma is a well-known result that compares a second order linear difference inequality to a corresponding equality.

Lemma 1 ([10]) — Assume that $\{a_n\}$ is a real sequence with $a_n \geq 0$. Then the difference inequality

$$\Delta^2 x_n + a_n x_{n-k} \leq 0, \qquad n = 0, 1, 2, \dots$$

has an eventually positive solution if and only if the difference equation

$$\Delta^2 x_n + a_n x_{n-k} = 0, \qquad n = 0, 1, 2, \dots$$

has an eventually positive solution.

The next lemma is due to Zhang and Zhou.

Lemma 2 ([11, Lemma 5]) — Assume that $\{b_n\}$ is a real sequence with $b_n \geq 0$ and $b_n \not\equiv 0$ for all large n. Let $\{v_n\}$ be an eventually positive solution of the equation

$$\Delta \left(v_n - \frac{1}{k} \sum_{i=n-k}^{n-1} v_i \right) + b_n v_{n-k} = 0$$

and set

$$u_n = v_n - \frac{1}{k} \sum_{i=n-k}^{n-1} v_i.$$

Then,

$$\Delta u_n \le 0 \quad \text{and} \quad u_n > 0$$

for all large n.

Our next lemma provides some information about the behaviour of solutions of the nonlinear neutral eq. (1).

Lemma 3 — Let $\{x_n\}$ be a positive solution of eq. (1). Then, $z_n = x_n + px_{n-r}$ is positive, nonincreasing, and satisfies

$$x_{n-k} \ge \left(\frac{p-1}{p^2} \right) z_{n+r-k}$$

for all sufficiently large n.

PROOF : From our hypothesis, we see that $\{z_n\}$ is eventually positive, say for $n \ge N$, and from equation (1), we then have $\{z_n\}$ is nonincreasing for $n \ge N$. Now,

$$px_{n-r} = z_n - x_n, \qquad \text{... (4)}$$

so

$$z_{n+r} = x_{n+r} + px_n.$$

Since $\{z_n\}$ is nonincreasing, we have

$$z_n \ge z_{n+r} \ge px_n$$

for $n \ge N + r$, and so (4) yields

$$p^2 x_{n-r} \ge pz_n - z_n.$$

Thus,

$$x_{n-r} \ge \frac{p-1}{p^2} z_n,$$

or

$$x_{n-k} \ge \frac{p-1}{p^2} z_{n+r-k}$$

for large n. □

3. Oscillation Results

In this section, we first establish a comparison theorem that will be the basis for obtaining some new oscillation conditions for eq. (1).

***Theorem* 1** — *Assume that*

$$Q_n = \frac{M(p-1)}{p^2} q_n \geq \frac{m^m}{(m+1)^{m+1}} \text{ for all large } n. \quad \dots (5)$$

If every solution of the linear difference equation

$$\Delta^2 y_{n-1} + \frac{2(m+1)^m}{m^{m+1}} \left[Q_n - \frac{m^m}{(m+1)^{m+1}} \right] y_n = 0, \; n = 0, 1, 2, \dots, \quad \dots (6)$$

is oscillatory, then every solution of the nonlinear neutral delay difference equation (1) is oscillatory.

Proof : Assume that $\{x_n\}$ is an eventually positive solution of eq. (1) and let $z_n = x_n + px_{n-r}$. By Lemma 3, $\{z_n\}$ is eventually positive and nonincreasing, say for $n \geq N_1$ for some $N_1 \geq 0$. Also from Lemma 3, we have

$$x_{n-k} \geq \left(\frac{p-1}{p^2} \right) z_{n+r-k} \quad \dots (7)$$

for $n \geq N_2$, for some $N_2 \geq N_1$, so substituting into eq. (1) and using the fact that $f(u)/u \geq M > 0$ for $u \neq 0$, we obtain

$$\Delta z_n + M \left(\frac{p-1}{p^2} \right) q_n z_{n+r-k} \leq 0, \quad \dots (8)$$

or

$$\Delta z_n + Q_n z_{n-m} \leq 0 \quad \dots (9)$$

for $N \geq N_2$. Notice that in view of (5), there exists an integer $N_3 \geq N_2$ such that

$$a_n = \left(\frac{m+1}{m} \right)^{m+1} \left(Q_n - \frac{m^m}{(m+1)^{m+1}} \right) \geq 0, \quad \dots (10)$$

for $n \geq N_3$. If we set

$$v_n = \left(1 + \frac{1}{m} \right)^n z_n,$$

then $v_n > 0$ for $n \geq N_3$, and so from (9), we have

$$\Delta\left(v_n - \frac{1}{m}\sum_{i=n-m}^{n-1} v_i\right) + a_n v_{n-m} \le 0 \qquad \ldots (11)$$

for $n \ge N_4 = N_3 + m$.

Now define

$$u_n = v_n - \frac{1}{m}\sum_{i=n-m}^{n-1} v_i. \qquad \ldots (12)$$

for $n \ge N_4$. Then, by Lemma 2, (11), and (12), there is an integer $N_5 \ge N_4 + m$ such that

$$u_n > 0 \quad \text{and} \quad \Delta u_n \le 0 \qquad \ldots (13)$$

for $n \ge N_5$.

Now let $c = \frac{1}{2}\min\{v_n : N_5 - m \le n \le N_5\}$; we claim that

$$v_n > c \qquad \ldots (14)$$

for all $n \ge N_5 - m$. Obviously, (14) holds for $N_5 - m \le n \le N_5$, so if (14) does not hold for all $n \ge N_5 - m$, let $N = \inf\{n > N_5 : v_n \le c\}$. Then $v_n > c$ for $N_5 - m \le n \le N$ and $v_N \le c$. From (12) and (13), we have

$$c \ge v_N = u_N + \frac{1}{m}\sum_{i=N-m}^{N-1} v_i > c,$$

which is a contradiction, and so (14) holds.

Since $\{u_n\}$ is positive and nonincreasing, $u_n \to \sigma \ge 0$ as $n \to \infty$. If $\sigma = 0$, then there is an integer $N_6 \ge N_5$ such that $u_n < c/4$ for $n \ge N_6$. Hence, for any $N_7 \ge N_6$, we have

$$v_n > \frac{c}{2} \ge \frac{2}{m+1}\sum_{i=N_7}^{n+m} u_i \quad \text{for} \quad n \in \{N_7, N_7+1, \ldots, N_7+m+1\}.$$

If $\sigma > 0$, then $u_n \ge \sigma$ for $n \ge N_5$, so (12) and (14) imply

$$v_n \ge \sigma + \frac{1}{m}\sum_{i=n-m}^{n-1} v_i \ge \sigma + c$$

for $n \geq N_5$. By induction, we see that

$$v_n \geq j\sigma + c \text{ for } n \geq N_5 + (j-1)m, \qquad j = 1, 2, \dots .$$

Thus, $v_n \to \infty$ as $n \to \infty$, so there exists $N_8 \geq N_5$ such that

$$v_n > \frac{2}{m+1} \sum_{i=N_8}^{n+m} u_i \qquad \text{for} \qquad n \in \{N_8, N_8+1, \dots, N_8+m+1\}.$$

Hence, in either case, for any $N_9 \geq N_5$,

$$v_n > \frac{2}{m+1} \sum_{i=N_9}^{n+m} u_i \qquad \text{for} \qquad n \in \{N_9, N_9+1, \dots, N_9+m+1\}.$$

Next, we show that

$$v_n > \frac{2}{m+1} \sum_{i=N_9}^{n+m} u_i$$

for $n \geq N_9$. If this is not the case, let

$$\hat{N} = \inf \left\{ n \geq N_9 + m + 1 : v_n \leq \frac{2}{m+1} \sum_{i=N_9}^{n+m} u_i \right\}.$$

Then,

$$v_n > \frac{2}{m+1} \sum_{i=N_9}^{n+k} u_i \qquad \text{for} \qquad n \in \{N_9, N_9+1, \dots, \hat{N}-1\}$$

and

$$v_{\hat{N}} \leq \frac{2}{m+1} \sum_{i=N_9}^{\hat{N}+m} u_i.$$

Then, (12) and (13) imply

$$\frac{2}{m+1} \sum_{i=N_9}^{\hat{N}+m} u_i \geq u_{\hat{N}} = u_{\hat{N}} + \frac{1}{m} \sum_{i=\hat{N}-m}^{\hat{N}-1} v_i$$

$$> u_{\hat{N}} + \frac{2}{m(m+1)} \sum_{i=\hat{N}-m}^{\hat{N}-1} \sum_{j=N_9}^{i+m} u_j$$

$$\geq u_{\hat{N}} + \frac{2}{(m+1)} \sum_{i=N_9}^{\hat{N}+m} u_i$$

$$- \frac{2}{m(m+1)} u_{\hat{N}} \sum_{j=\hat{N}}^{\hat{N}+m} (i-\hat{N}) = \frac{2}{(m+1)} \sum_{i=N_9}^{\hat{N}+m} u_i,$$

which is a contradiction. Thus, (15) holds and so

$$v_{n-m} > \frac{2}{m+1} \sum_{i=N_9}^{n} u_i \qquad \text{... (16)}$$

for $n \geq N_9 + m$.

Finally, we set $Y_n = \sum_{i=N_9}^{n} u_i$ Then, $\Delta Y_{n-1} = u_n$ and $\Delta^2 Y_{n-1} = \Delta u_n$, so (11), (12), and (16), yield

$$\Delta^2 Y_{n-1} + 2 \frac{(m+1)^m}{m^{m+1}} \left[Q_n - \frac{m^m}{(m+1)^{m+1}} \right] Y_n \leq 0 \qquad \text{... (17)}$$

for $n \geq N_9 + m$. Lemma 1 implies that (6) has an eventually positive solution, and this contradiction completes the proof of the theorem. □

We will now show how the approach used here will allow us to obtain oscillation criteria for the first order nonlinear neutral delay eq. (1) by applying known oscillation criteria for second order non-neutral linear equations. First, we quote some well known oscillation results for second order linear difference equations.

Lemma 4 ([4]) — If

$$\liminf_{n \to \infty} n \sum_{i=n+1}^{\infty} a_i > \frac{1}{4},$$

then every solution of $\Delta^2 y_{n-1} + a_n y_n = 0$ oscillates.

Lemma 5 ([12]) — If

$$n \sum_{i=n+1}^{\infty} a_i \leq \frac{1}{4}$$

for all large n, then the equation $\Delta^2 y_{n-1} + a_n y_n = 0$ has a nonoscillatory solution.

One possible oscillation result is the following.

***Theorem* 2** — *Suppose that condition (5) holds.*

(i) If

$$\liminf_{n \to \infty} \; n \sum_{i=n+1}^{\infty} \left[Q_n - \frac{m^m}{(m+1)^{m+1}} \right] > \frac{m^{m+1}}{8\,(m+1)^m},$$

every solution of (1) oscillates.

(ii) If

$$n \sum_{i=n+1}^{\infty} \left[Q_n - \frac{m^m}{(m+1)^{m+1}} \right] \leq \frac{m^{m+1}}{8\,(m+1)^m},$$

for large n, then eq. (1) has a nonoscillatory solution.

As an example of Theorem 2, we have the following.

Example 1 — Consider the nonlinear neutral delay difference equation

$$\Delta\,(x_n + 2x_{n-r}) + \left[\frac{4m^m}{(m+1)^{m+1}} + \frac{c}{(n+1)^{\alpha}} \right] x_{n-k} \left(1 + x_{n-k}^2 \right) = 0,$$

$$n = 0, 1, 2, \ldots, \qquad \ldots (18)$$

where $c > 0$ is a constant and α is a real number. Here we have $M = 1$ and $p = 2$. By Theorem 2, if $\alpha < 2$ and $c > 0$, then all solutions of (18) are oscillatory, while if $\alpha = 2$ and $c > \dfrac{m^{m+1}}{2\,(m+1)^m}$, then eq. (18) has a nonoscillatory solution. It does not appear that this conclusion can be obtained from previously know results.

Now it should be clear how to use this approach to obtain further oscillation criteria for eq. (1). In fact, any known oscillation result for (6) will yield a corresponding result for eq. (1). For example, if we let

$$p_n = 2\,\frac{(m+1)^m}{m^{m+1}} \left(Q_n - \frac{m^m}{(m+1)^{m+1}} \right) \geq 0,$$

then eq. (6) becomes

$$\Delta^2 y_{n-1} + p_n y_n = 0, \qquad \text{... (19)}$$

where we have

$$Q_n = \frac{M(p-1)}{p^2} q_n \geq \frac{m^m}{(m+1)^{m+1}}. \qquad \text{... (20)}$$

If we then apply an oscillation criteria for this equation, such as assuming that $\sum_{i=n}^{\infty} p_i = \infty$, then every solution of (1) oscillates.

We can also obtain results in case $\sum_{i=n}^{\infty} p_i < \infty$ by letting

$$R_n^0 = \sum_{i=n}^{\infty} p_i,$$

and, provided they exist, let

$$R_n^j = \sum_{i=n}^{\infty} \frac{\left(R_i^{j-1}\right)^2}{1 + R_i^{j-1}} + R_n^0, \qquad j \geq 1.$$

We then have the following theorem that is based on a result of Erbe and Zhang [4].

***Theorem* 3** — *Every solution of eq. (1) oscillates if either one of the following conditions holds*:

(*i*) *There exists a positive integer j such that* $R_n^0, \ldots, R_n^{j-1}$ *exist and*

$$\sum_{i=n}^{\infty} \frac{\left(R_i^{j-1}\right)^2}{1 + R_i^{j-1}} = \infty;$$

(*ii*) *There exists an integer* $n_0 \geq 0$ *such that* $\limsup_{j \to \infty} R_{n_0}^j = \infty$.

We also have the following results.

***Theorem* 4** — *If* $\lim_{n \to \infty} \sup n \sum_{j=n}^{\infty} p_j > 1$, *then every solution of eq. (1) oscillates.*

PROOF : The proof can be modeled after the proof of Corollary 2 in [8] and hence the details are omitted. □

***Theorem* 5** — *Assume that* $\sum_{n=n_0}^{\infty} p_n < \infty$. *If* $\liminf_{n\to\infty} n \sum_{s=n}^{\infty} p_s > \frac{1}{4}$, *then every solution of eq. (1) oscillates.*

PROOF : The proof of this result can be modeled after results in [1]. □

Various other oscillation criteria for eq. (19) can be found, for example, in Agarwal [1], Agarwal, Grace, and O'Regan [2], and Agarwal and Wong [3]. To formulate our final result we define $H_n^{(0)} = \sum_{i=n}^{\infty} p_i$ and $H_n^{(j)} = \sum_{i=n}^{\infty} ip_i H_n^{(j-1)}$ provided the right hand side of this expression exists.

***Theorem* 6** — *If* $H_n^{(j)} < \infty$ *for* $j = 1, 2, \ldots, N$ *and* $\sum_{i=1}^{\infty} ip_i H_i^{(N)} = \infty$, *then every solution of eq. (1) oscillates.*

PROOF : The proof of this theorem is easily modeled after that in [5] and so we omit the details. □

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