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GENERALIZED JORDAN TRIPLE HIGHER DERIVATIONS ON PRIME RINGS

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We prove that every generalized Jordan triple higher derivation on a 2-torsion free prime ring is a generalized higher derivation.

Key Words: Higher Derivations; Jordan Triple Higher Derivations; Generalized Jordan Triple Higher Derivations

1. INTRODUCTION

Throughout this paper, R will represent an associative ring. Let $a, b, c \in R$, the element $abc - cba$ will be denoted by $[a, b, c]$.

A derivation (resp. Jordan derivation) is an additive map $\delta : R \to R$ satisfying $\delta(ab) = \delta(a)\,b + a\,\delta(b)$ for all $a, b \in R$ (resp. $\delta(a^2) = \delta(a)\,a + a\,\delta(a)$ for all $a \in R$). An additive map $\delta : R \to R$ is called a Jordan triple derivation if $\delta(aba) = \delta(a)\,ba + a\delta(b)\,a + ab\delta(a)$ for all $a, b \in R$.

It is obvious that every derivation is a Jordan derivation. But the converse is in general not true. Herstein [8] proved that the converse is true on 2-torsion free prime rings and latter on. Brešar [5] extended this result to 2-torsion free semiprime rings. In [4], he also generalized his result by proving that every Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Of course, it turns out that every Jordan derivation on a 2-torsion free ring is a Jordan triple derivation [4].

In [3], Brešar defined the following concept. An additive map $f : R \to R$ is called a generalized derivation if there exists a derivation $\delta : R \to R$ such that $f(ab) = f(a)\,b + a\,\delta(b)$ for all $a, b \in R$. This notion is found in [11] and other properties of generalized derivations were given by Hvala [9] and Quadri *et al.* [12]. We call an additive map $f : R \to R$ a generalized Jordan derivation if there exists a Jordan derivation $\delta: R \to R$ such that $f(a^2) = f(a)\,a + a\,\delta(a)$ for all $a \in R$ (see [2] and [10]).

For example, let $a, b \in R$ be such that one of them is not zero. Define $f(x) = ax + xb$ for all $x \in R$. Then for all $x, y \in R$, we have $f(x+y) = f(x) + f(y)$ and

$$\begin{aligned} f(x^2) &= ax^2 + x^2 b \\ &= (ax + xb)\,x + x\,(-bx - x\,(-b)) \\ &= f(x)\,x + x\,\delta(x), \end{aligned}$$

where δ defined by $\delta(x) = -bx - x(-b)$ for all $x \in R$ is a Jordan derivation on R. That is, f is a generalized Jordan derivation on R.

Recently, Ashraf and Rehman [1] showed that in a 2-torsion free ring R which has a commutator nonzero divisor, every generalized Jordan derivation on R is a generalized derivation, and Jing and Lu [10] proved this result on a 2-torsion-free prime ring. Furthermore, they [10] proved that every generalized Jordan triple derivation on a 2-torsion free prime ring is a generalized derivation, where an additive map $f: R \to R$ is said to be a generalized Jordan triple derivation if there exists a Jordan triple derivation $\delta: R \to R$ satisfying $f(aba) = f(a)\,ba + a\,\delta(b)\,a + ab\,\delta(a)$ for all $a, b \in R$.

On the other hand, higher derivations have been studied in rings (mainly in commutative rings), but also in noncommutative rings. In particular, Ferrero and Haetinger [7] extended the above Brešars results in [4] and [5] to higher derivations.

Cortes and Haetinger [6] improved the Ashraf and Rehmen's result [1] to a generalized Jordan higher derivation, that is, if R is a 2-torsion free ring which has a commutator right nonzero divisor and U is a square closed Lie ideal of R, then every generalized Jordan higher derivation of U into R is a generalized higher derivation of U into R.

As pointed in [6. Examples, p. 8-9], we remark that the assumption that a ring has a commutator nonzero divisor and the assumption that a ring is semiprime are independent each other.

Let R be a 2-torsion-free prime ring. Then we see that every generalized higher derivation on R is a generalized Jordan higher derivation in view of the fact that every generalized derivation is a Jordan generalized derivation, and every generalized Jordan higher derivation on R is a generalized Jordan triple higher derivation as in the proof of [7, Theorem 1.3].

In this paper we investigate that, by showing that every generalized Jordan triple higher derivation on a 2-torsion free prime ring is a generalized higher derivation (which generalizes the Jing and Lu's result [10, Theorem 3.5]). the notions of generalized higher derivation, generalized

Jordan higher derivation and generalized Jordan triple higher derivation on a 2-torsion free prime ring are equivalent to each other.

2. Definitions and Technical Lemmas

Let $\mathbb{N}_0$ be the set of all nonnegative integers.

Definition 2.1 — Let $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ be a family of additive maps on a ring R such that $\delta_0 = id_R$. Δ is said to be:

(*i*) a higher derivation if for each $n \in \mathbb{N}_0$,

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a)\,\delta_j(b) \quad \text{for all } a, b \in R;$$

(*ii*) a Jordan higher derivation if for each $n \in \mathbb{N}_0$,

$$\delta_n(a^2) = \sum_{i+j=n} \delta_i(a)\,\delta_j(a) \quad \text{for all } a \in R;$$

(*iii*) a Jordan triple higher derivation if for each $n \in \mathbb{N}_0$,

$$\delta_n(aba) = \sum_{i+jk=n} \delta_i(a)\,\delta_j(b)\,\delta_k(a) \quad \text{for all } a, b \in R;$$

Definition 2.2 — Let $F = (f_i)_{i \in \mathbb{N}_0}$ be a family of additive maps on a ring R such that $f_0 = id_R$. F is said to be:

(*i*) a generalized higher derivation if there exists a higher derivation $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$,

$$f_n(ab) = \sum_{i+j=n} f_i(a)\,\delta_j(b) \quad \text{for all } a, b \in R;$$

(*ii*) a generalized Jordan higher derivation if there exists a Jordan higher derivation $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$,

$$f_n(a^2) = \sum_{i+j=n} f_i(a)\,\delta_j(a) \quad \text{for all } a \in R;$$

(*iii*) a generalized Jordan triple higher derivation if there exists a Jordan triple higher derivation $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$,

$$f_n(aba) = \sum_{i+j+k=n} f_i(a)\, \delta_j(b)\, \delta_k(a) \quad \text{for all } a, b \in R.$$

We precede the proof of our main theorem by a series of lemmas.

Lemma 2.3 ([4, Lemma 1.2]) — Let $G_1, G_2, \ldots, G_n$ be additive groups and R be a semiprime ring. Suppose that mappings $S : G_1 \times G_2 \times \ldots \times G_n \to R$ and $T : G_1 \times G_2 \times \ldots \times G_n \to R$ are additive in each argument. If we have

$$S(a_1, a_2, \ldots, a_n)\, xT(a_1, a_2, \ldots, a_n) = 0$$

for all $x \in R, a_i \in G_i, i = 1, 2, \ldots, n$, then we get

$$S(a_1, a_2, \ldots, a_n)\, xT(b_1, b_2, \ldots, b_n) = 0$$

for all $x \in R, a_i b_i \in G_i, i = 1, 2, \ldots, n$.

Given a generalized Jordan triple higher derivation $F = (f_i)_{i \in \mathbb{N}_0}$ on a ring R, and a Jordan triple higher derivation $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ satisfying the definition 2.2(*iii*), we put, for all $a, b, c \in R$ and $n \in \mathbb{N}_0$.

$$P_n(a, b, c) = f_n(abc) - \sum_{i+j+k=n} f_i(a)\, \delta_j(b)\, \delta_k(c).$$

The following result remarks two elementary properties of the just defined expression. Whereas linearity in each argument (*ii*) is an immediate consequence of the definition, the equality (*i*) can be easily obtained from the definition of generalized Jordan triple higher derivation by linearity.

Lemma 2.4 — Let R be a ring equipped with a generalized Jordan triple higher derivation F. Then, with the above notation, for all $a, b, c \in R$ and $n \in \mathbb{N}_0$, we have

(*i*) $P_n(a, b, c) + P_n(c, b, a) = 0$;

(*ii*) $P_n(a, b, c)$ is additive in each argument.

Lemma 2.5 — Let R be a semiprime ring equipped with a generalized Jordan triple higher derivation F. Then, with the above notation, if $n \in \mathbb{N}_0$ is such that $P_m(a, b, c) = 0$ for all $a, b, c \in R$ and $m < n$, we have

$$P_n(a, b, c)\, x\, [a, b, c] = 0 \text{ for all } a, b, c \in R.$$

PROOF : Since R is semiprime, it follows from [7, Theorem 1.2] that $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ is a higher derivation on R. Hence it is easy to see that

$$\delta_n(abc) = \sum_{i+j+k=n} \delta_i(a)\, \delta_j(b)\, \delta_k(c) \quad \text{for all } a, b, c \in R.$$

Let $W = f_n(abcxcba + cbaxabc)$. We compute W in two ways. On the one hand we have by Lemma 2.4(i)

$$
\begin{aligned}
W &= \sum_{i+j+k=n} f_i(a)\, \delta_j(bcxcb)\, \delta_k(a) \quad \sum_{i+j+k=n} f_i(c)\, \delta_j(baxab)\, \delta_k(c) \\
&= \sum_{i+j+k=n} f_i(a) \left(\sum_{p+q+r=j} \delta_p(b)\, \delta_q(cxc)\, \delta_r(b) \right) \delta_k(a) \\
&\quad + \sum_{i+j+k=n} f_i(c) \left(\sum_{p+q+r=j} \delta_p(b)\, \delta_q(axa)\, \delta_r(b) \right) \delta_k(c) \\
&= \sum_{i+p+q+r+k=n} f_i(a)\, \delta_p(b)\, \delta_q(cxc)\, \delta_r(b)\, \delta_k(a) \\
&= \sum_{i+p+q+r+k=n} f_i(c)\, \delta_p(b)\, \delta_q(axa)\, \delta_r(b)\, \delta_k(c) \\
&= \sum_{i+p+q+r+k=n} f_i(a)\, \delta_p(b) \left(\sum_{l+s+t=q} \delta_l(c)\, \delta_s(x)\, \delta_t(c) \right) \delta_r(b)\, \delta_k(a) \\
&\quad + \sum_{i+p+q+r+k=n} f_i(a)\, \delta_p(b) \left(\sum_{l+s+t=q} \delta_l(a)\, \delta_s(x)\, \delta_t(a) \right) \delta_r(b)\, \delta_k(c) \\
&= \sum_{i+p+l+s+t+r+k=n} f_i(a)\, \delta_p(b)\, \delta_l(c)\, \delta_s(x)\, \delta_t(c)\, \delta_r(b)\, \delta_k(a) \\
&\quad + \sum_{i+p+l+s+t+r+k=n} f_i(c)\, \delta_p(b)\, \delta_l(a)\, \delta_s(x)\, \delta_t(a)\, \delta_r(b)\, \delta_k(c)
\end{aligned}
$$

$$= \sum_{\substack{i+p+l+s+t+r+k=n \\ s+t+r+k\neq 0}} f_i(a)\, \delta_p(b)\, \delta_l(c)\, \delta_s(x)\, \delta_t(c)\, \delta_r(b)\, \delta_k(a)$$

$$+ \sum_{i+p+l=n} f_i(a)\, \delta_p(b)\, \delta_l(c)\, xcba$$

$$+ \sum_{\substack{i+p+l+s+t+r+k=n \\ s+t+r+k\neq 0}} f_i(c)\, \delta_p(b)\, \delta_l(a)\, \delta_s(x)\, \delta_t(a)\, \delta_r(b)\, \delta_k(c)$$

$$+ \sum_{i+p+l=n} f_i(c)\, \delta_p(b)\, \delta_l(a)\, xabc.$$

On the other hand, using the assumption $P_m(a, b, c) = 0$ for all a, b, c and $m < n$, we get

$$W = f_n((abc)\, x\, (cba)) + f_n((cba)\, x\, (abc))$$

$$= \sum_{u+v+w=n} f_u(abc)\, \delta_v(x)\, \delta_w(cba) + \sum_{u+v+w=n} f_u(cba)\, \delta_v(x)\, \delta_w(abc)$$

$$= \sum_{u+v+w=n} f_u(abc)\, \delta_v(x) \left(\sum_{e+g+h=w} \delta_e(c)\, \delta_g(b)\, \delta_h(a) \right)$$

$$+ \sum_{u+v+w=n} f_u(cba)\, \delta_v(x) \left(\sum_{e+g+h=w} \delta_e(a)\, \delta_g(b)\, \delta_h(c) \right)$$

$$= \sum_{u+v+e+g+h=n} f_u(abc)\, \delta_v(x)\, \delta_e(c)\, \delta_g(b)\, \delta_h(a)$$

$$+ \sum_{u+v+e+g+h=n} f_u(cba)\, \delta_v(x)\, \delta_e(a)\, \delta_g(b)\, \delta_h(c)$$

$$= \sum_{\substack{u+v+e+g+h=n \\ v+e+g+h\neq 0}} f_u(abc)\, \delta_v(x)\, \delta_e(c)\, \delta_g(b)\, \delta_h(a) + f_n(abc)\, xcba$$

$$+ \sum_{\substack{u+v+e+g+h=n \\ v+e+g+h\neq 0}} f_u(cba)\, \delta_v(x)\, \delta_e(a)\, \delta_g(b)\, \delta_h(c) + f_n(cba)\, xabc$$

$$= \sum_{\substack{u+v+e+g+h=n \\ v+e+g+h\neq 0}} \left(\sum_{\alpha+\beta+\gamma=u} f_\alpha(a)\, \delta_\beta(b)\, \delta_\gamma(c) \right) \delta_v(x)\, \delta_e(c)\, \delta_g(b)\, \delta_h(a)$$

$$+ f_n(cba)\, xabc$$

$$= \sum_{\substack{u+v+e+g++h=n \\ v+e+g+h\neq 0}} \left(\sum_{\alpha+\beta+\gamma=u} f_\alpha(c)\,\delta_\beta(b)\,\delta_\gamma(a) \right) \delta_v(x)\,\delta_e(a)\,\delta_g(b)\,\delta_h(c)$$

$$+ f_n(cba)\,xabc$$

$$= \sum_{\substack{\alpha+\beta+\gamma+v+e+g+h=n \\ v+e+g+h\neq 0}} f_\alpha(a)\,\delta_\beta(b)\,\delta_\gamma(c)\,\delta_v(x)\,\delta_e(c)\,\delta_g(b)\,\delta_h(a)$$

$$+ f_n(abc)\,xcba$$

$$= \sum_{\substack{\alpha+\beta+\gamma+u+e+g+h=n \\ v+e+g+h\neq 0}} f_\alpha(c)\,\delta_\beta(b)\,\delta_\gamma(a)\,\delta_v(x)\,\delta_e(a)\,\delta_g(b)\,\delta_h(c)$$

$$+ f_n(cba)\,xabc.$$

Comparing the last terms of two expressions so obtained for W, we obtain

$$P_n(a,b,c)\,xcba + P_n(c,b,a)\,xabc = 0 \text{ for all } a,b,c,x \in R.$$

Since $P_n(a,b,c) = -P_n(c,b,a)$ by Lemma 2.4(i), we see that

$$P_n(a,b,c)\,x\,[a,b,c] = 0 \text{ for all } a,b,c,x \in R.$$

This completes the proof of the lemma. □

3. The Proof of Main Theorem

Now we are ready to prove our main theorem.

The proof of our main theorem is based on [7] to use the corresponding property for Jordan triple higher derivations and on the result in [10], which is essentially, a first step in our induction argument.

Theorem — *Let R be a 2-torsion free prime ring. Then every generalized Jordan triple higher derivation on R is a generalized higher derivation.*

PROOF : We first intend to prove that $P_n(a,b,c) = 0$ for all $a,b,c \in R$ and $n \in \mathbb{N}_0$ by induction.

In case $n = 0$, we get $P_0(a,b,c) = 0$ for all $a,b,c \in R$. If $n = 1$, then it follows from [10, Theorem 3.5] that $P_1(a,b,c) = 0$ for all $a,b,c \in R$. Thus. we assume that $P_m(a,b,c) = 0$ for all $a,b,c \in R$ and $m < n$. Then, from Lemma 2.3 and Lemma 2.5, we see that

$$P_n(a,b,c)\,x\,[w,y,z] = 0 \text{ for all } a,b,c,w,x,y,z \in R. \qquad \ldots (1)$$

We have two cases:

Case I: There exist $w_0, y_0, z_0 \in R$ satisfying $[w_0, y_0, z_0] \neq 0$ (1).

The primeness of R with Lemma 2.5 yields $P_n(a,b,c) = 0$ for all $a,b,c \in R$ and $n \in \mathbb{N}$.

Case II: $[w, y, z] = 0$ is fulfilled for all $w, y, z \in R$ in (1).

It is easily proved that R is then commutative.

Set $W = f_n(c^3 ba + abc^3)$. We recall that $\Delta = (\delta_n)_{n \in \mathbb{N}_0}$ a generalized higher derivation by the semiprimeness of R. Let us compute W in two ways. On the one hand, applying Lemma 2.4(i). we get

$$\begin{aligned}
W &= f_n(c^3 ba + abc^3) \\
&= \sum_{i+j+k=n} f_i(c^3)\,\delta_j(b)\,\delta_k(a) + \sum_{i+j+k=n} f_i(a)\,\delta_j(b)\,\delta_k(c^3) \\
&= \sum_{i+j+k=n} \left(\sum_{h+u+v=i} f_h(c)\,\delta_u(c)\,\delta_v(c) \right) \delta_j(b)\,\delta_k(a) \\
&\quad + \sum_{i+j+k=n} f_i(a)\,\delta_j(b) \left(\sum_{\alpha+\beta+\gamma=k} \delta_\alpha(c)\,\delta_\beta(c)\,\delta_\gamma(c) \right) \\
&= \sum_{h+u+v+j+k=n} f_h(c)\,\delta_u(c)\,\delta_v(c)\,\delta_j(b)\,\delta_k(a) \\
&\quad + \sum_{i+j+\alpha+\beta+\gamma=n} f_i(a)\,\delta_j(b)\,\delta_\alpha(c)\,\delta_\beta(c)\,\delta_\gamma(c) \\
&= \sum_{h+u+v+j+k=n} f_h(c)\,\delta_u(c)\,\delta_v(c)\,\delta_j(b)\,\delta_k(a) \\
&\quad + \sum_{\substack{i+j+\alpha+\beta+\gamma=n \\ \beta+\gamma\neq 0}} f_i(a)\,\delta_j(b)\,\delta_\alpha(c)\,\delta_\beta(c)\,\delta_\gamma(c) \\
&\quad + \sum_{i+j+\alpha=n} f_i(a)\,\delta_j(b)\,\delta_\alpha(c)\,c^2.
\end{aligned}$$

On the other hand, we have, by Lemma 2.4(i), the induction assumption, and commutativity of R,

$$\begin{aligned}
W &= f_n(cc\,(cba) + (cba)\,cc) \\
&= \sum_{i+j+k=n} f_i(c)\,\delta_j(c)\,\delta_k(cba) + \sum_{i+j+k=n} f_i(cba)\,\delta_j(c)\,\delta_k(c) \\
&= \sum_{i+j+k=n} f_i(c)\,\delta_j(c)\,\delta_k(cba) + \sum_{i+j+k=n} f_i(abc)\,\delta_j(c)\,\delta_k(c) \\
&= \sum_{i+j+k=n} f_i(c)\,\delta_j(c)\left(\sum_{p+q+r=k} \delta_p(c)\,\delta_q(b)\,\delta_r(a)\right) \\
&\quad + \sum_{\substack{i+j+k=n \\ j+k\neq 0}} \left(\sum_{l+s+t=i} f_l(a)\,\delta_s(b)\,\delta_t(c)\right)\delta_j(c)\,\delta_k(c) + f_n(abc)\,c^2 \\
&= \sum_{i+j+p+q+r=n} f_i(c)\,\delta_j(c)\,\delta_p(c)\,\delta_q(b)\,\delta_r(a) \\
&\quad + \sum_{\substack{l+s+t+j+k=n \\ j+k\neq 0}} f_l(a)\,\delta_s(b)\,\delta_t(c)\,\delta_j(c)\,\delta_k(c) + f_n(abc)\,c^2.
\end{aligned}$$

The comparison of two expressions for W yields

$$\left(f_n(abc) - \sum_{i+j+\alpha=n} f_i(a)\,\delta_j(b)\,\delta_\alpha(c)\right)c^2 = 0,$$

that is,

$$P_n(a, b, c)\,c^2 = 0 \text{ for all } a, b, c \in R.$$

Then we obtain

$$P_n(a, b, c)\,cx\,P_n(a, b, c)\,c = P_n(a, b, c)\,c^2\,xP_n(a, b, c) = 0 \text{ for all } a, b, c, x \in R$$

which implies that

$$P_n(a, b, c)\,c = 0 \text{ for all } a, b, c \in R \qquad \dots (2)$$

according to the semiprimeness of R.

Linearizing (2) (by using Lemma 2.4(*ii*)) leads to

$$P_n(a, b, c)\,y + P_n(a, b, y)\,c = 0 \text{ for all } a, b, c, y \in R.$$

Then it follows from (3) that

$$P_n(c, b, c)\,yz\,P_n(a, b, c)\,y$$

$$= -P_n(c,b,y)\, cz\, P_n(a,b,c)\, y$$

$$= -P_n(c,b,y)\, ycz\, P_n(a,b,c)$$

$$= 0$$

for all $a, b, c, y, z \in R$ which gives $P_n(a,b,c)\, y = 0$ for all $a, b, c, y \in R$. Hence we have $P_n(a,b,c) = 0$ for all $a, b, c \in R$ and $n \in \mathbb{N}_0$ by the primeness of R.

It remains to prove that $F = (f_n)_{n \in \mathbb{N}_0}$ is a generalized higher derivation.

Let $\Theta_n(a,b) = f_n(ab) - \sum_{i+j=n} f_i(a)\,\delta_j(b)$ for all $a, b \in R$ and $n \in \mathbb{N}_0$. It is trivial that $\Theta_n(a,b)$ is additive in each argument. If $n = 0$, then $\Theta_0(a,b) = 0$. If $n = 1$, then we see from [10: Theorem 3.5] that $\Theta_1(a,b) = 0$. By induction, we assume that $\Theta_m(a,b) = 0$ for all $a, b \in R$ and $m < n$. Let $W = f_n(abxba)$ for all $a, b, x \in R$. On the one hand we have

$$\begin{aligned} W &= f_n(a\,(bxa)\,b) \\ &= \sum_{i+j+k=n} f_i(a)\,\delta_j(bxa)\,\delta_k(b) \\ &= \sum_{i+j+k=n} f_i(a) \left(\sum_{l+s+t=j} \delta_l(b)\,\delta_s(x)\,\delta_t(a) \right) \delta_k(b) \\ &= \sum_{i+l+s+t+k=n} f_i(a)\,\delta_l(b)\,\delta_s(x)\,\delta_t(a)\,\delta_k(b) \\ &= \sum_{\substack{i+l+s+t+k=n \\ s+t+k\neq 0}} f_i(a)\,\delta_l(b)\,\delta_s(x)\,\delta_t(a)\,\delta_k(b) \\ &\quad + \sum_{i+l=n} f_i(a)\,\delta_l(b)\,xab. \end{aligned}$$

On the other hand we obtain, by using inductive hypothesis,

$$\begin{aligned} W &= f_n((ab)\,x\,(ab)) \\ &= \sum_{i+j+k=n} f_i(ab)\,\delta_j(x)\,\delta_k(ab) \end{aligned}$$

$$= \sum_{i+j+k=n} f_i(ab)\,\delta_j(x)\left(\sum_{u+v=k} \delta_u(a)\,\delta_v(b)\right)$$

$$= \sum_{i+j+u+v=n} f_i(ab)\,\delta_j(x)\,\delta_u(a)\,\delta_v(b)$$

$$= \sum_{\substack{i+j+u+v=n\\ j+u+v\neq 0}} f_i(ab)\,\delta_j(x)\,\delta_u(a)\,\delta_v(b) + f_n(ab)\,xab$$

$$= \sum_{\substack{i+j+u+v=n\\ j+u+v\neq 0}} \left(\sum_{p+q=i} f_p(a)\,\delta_q(b)\right)\delta_j(x)\,\delta_u(a)\,\delta_v(b) + f_n(ab)\,xab$$

$$= \sum_{\substack{p+q+j+u+v=n\\ j+u+v\neq 0}} f_p(a)\,\delta_q(b)\,\delta_j(x)\,\delta_u(a)\,\delta_v(b) + f_n(ab)\,xab.$$

Comparing the last terms of two expressions for W, we have

$$\left(f_n(ab) - \sum_{i+l=n} f_i(a)\,\delta_l(b)\right) xab = 0,$$

i.e.,

$$\Theta_n(a, b)\,xab = 0 \text{ for all } a, b, x \in R. \qquad \ldots (3)$$

Furthermore, the linearization of (3) is

$$\Theta_n(a, b)\,xab = -\Theta_n(y, b)\,xab,$$

and so (3) yields

$$(\Theta_n(a, b)\,xyb)\,z\,(\Theta_n(a, b)\,xyb) = -\Theta_n(a, b)\,x\,(ybz\,\Theta_n(y, b))\,xab = 0$$

for all $a, b, x, y, z \in R$. Hence we obtain

$$\Theta_n(a, b)\,xyb = 0 \text{ for all } a, b, x, y \in R. \qquad \ldots (4)$$

Linearizing (4) and using the same approach as just done, we also have

$$\Theta_n(a, b)\,xyw = 0 \text{ for all } a, b, x, y, w \in R.$$

In particular, we see that

$$\Theta_n(a, b)\,xy\;\Theta_n(a, b)\,x = 0 \text{ for all } a, b, x, y \in R.$$

Therefore, we get

$$\Theta_n(a, b)\, x = 0 \text{ for all } a, b, x \in R.$$

From the primeness of R, we conclude that

$$\Theta_n(a, b) = 0 \text{ for all } a, b \in R,$$

that is, $F = (f_i)_{i \in \mathbb{N}_0}$ is a generalized higher derivation and completes the proof. □

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