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ON UNSTABLE NEUTRAL DIFFERENCE EQUATIONS OF HIGHER ORDER

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We study the neutral difference equation of the form

$$\Delta^m (x_n - px_{n-\tau}) = q_n x_{n-\sigma} .$$

Existence of a positive unbounded solution is established. Sufficient conditions for all bounded solutions to be oscillatory are obtained.

Key Words: Neutral Difference Equation; Oscillation; Nonoscillation; Bounded Solution

1. INTRODUCTION

In this paper we consider the neutral difference equation of the form

$$\Delta^m (x_n - px_{n-\tau}) = q_n x_{n-\sigma}, \qquad n = 1, 2, \ldots \qquad \ldots \text{(E)}$$

where $m \geq 2, p$ is a nonnegative number, τ, σ are nonnegative integers, $\tau \geq 1$ and $\{q_n\}$ is a sequence of real numbers such that $q_n \geq 0$ for all $n \geq 1$ and not identically zero. By a solution of equation (E) we mean a sequence $\{x_n\}$ which is defined for $n \geq -\max\{\tau, \sigma\}$ and which satisfies equation (E) for all $n = 1, 2, \ldots$. A nontrivial solution $\{x_n\}$ is called oscillatory, if for every $n_0 \geq 1$ there exists $n \geq n_0$ such that $x_n x_{n+1} \leq 0$. Otherwise, it is called nonoscillatory.

In the last few years, there has been an increasing interest in the study of oscillatory and nonoscillatory behaviour of solutions of neutral difference equations, see for example [4-11] and the references cited therein. In particular, Lalli and Zhang in [4] studied the equation

$$\Delta^2 (x_n - px_{n-\tau}) = q_n x_{n-\sigma}, \qquad n = 1, 2, \ldots \qquad \ldots \text{(E')}$$

and proved that equation (E') always has a positive unbounded solution. In this paper we obtained such results for equation (E). So, it is not possible to find criteria for all solutions of equation (E) to be oscillatory. From the result of Zafer in [11, Theorem 5] it follows that if m is even, $p > 1$ and

$$\sum_{n=1}^{\infty} n^{\underline{m-1}} q_n = \infty, \qquad \text{... (1)}$$

then every bounded solution of equation (E) is oscillatory. Note, that for $0 < p \leq 1$ it may not be true any more, as the following example shows.

Example 1 — Consider the difference equation

$$\Delta^2 \left(x_n - \frac{1}{3} x_{n-1} \right) = \frac{1}{24} x_{n-1}. \qquad \text{... (2)}$$

Here, condition (1) is satisfied, but the above equation has nonoscillatory, bounded solution $x_n = \frac{1}{2^n}$.

In this paper we present conditions that imply oscillation of all bounded solutions of equation (E), when m is even and $0 < p \leq 1$.

For the sake of convenience, throughout this paper, we use the convention $\sum_{i=k}^{j} q_i \equiv 0$ whenever $j < k$, and

$$n^{\underline{k}} = n(n-1)\ldots(n-k+1) \text{ with } n^{\underline{0}} = 1.$$

To obtain our results we need following lemmas.

Lemma 1 (see [1, Th. 1.8.11]) — Let $\{x_n\}$ be a sequence of positive real numbers and let $\Delta^m x_n$ be of constant sign and not identically zero. Then, there exists an integer $l, 0 \leq l \leq m$ with $m + l$ odd for $\Delta^m x_n \leq 0$ or $m - l$ even for $\Delta^m x_n \geq 0$ and such that

$$\Delta^i x_n > 0 \text{ for all large } n, 1 \leq i \leq l - 1,$$

$$(-1)^{l+i} \Delta^i x_n > 0 \text{ for all } n, l \leq i \leq m - 1. \qquad \text{... (3)}$$

Lemma 2 (see [3]) — Let $\{a_n\}$ be any real sequence. Then

$$\sum_{n=N_0}^{N} \sum_{i_{m-1}=n}^{N} \cdots \sum_{i_1=i_2}^{N} a_{i_1} = \sum_{j=N_0}^{N} \frac{(j-N_0+m-1)^{\underline{m-1}}}{(m-1)!} a_j.$$

Lemma 3 — Let $\{a_n\}$ be any real sequence. Then

$$\sum_{i_m=N_0}^{n-1} \sum_{i_{m-1}=N_0}^{i_m-1} \cdots \sum_{i_1=N_0}^{i_2-1} a_{i_1} = \sum_{j=N_0}^{n-1} \frac{(n-j-1)^{\underline{m-1}}}{(m-1)!} a_j. \qquad \text{... (4)}$$

PROOF : It is known (see [2], Lemma 7.16) that

$$\sum_{i=N_0}^{n-1} \sum_{j=N_0}^{i-1} a_j = \sum_{j=N_0}^{n-1} (n-j-1)\, a_j.$$

Assume that for $k \geq 2$

$$\sum_{i_k=N_0}^{n-1} \sum_{i_{k-1}=N_0}^{i_k-1} \cdots \sum_{i_1=N_0}^{i_2-1} a_{i_1} = \sum_{j=N_0}^{n-1} \frac{(n-j-1)^{\underline{k-1}}}{(k-1)!} a_j.$$

Then, using the above equality, we get

$$\sum_{i_{k+1}=N_0}^{n-1} \sum_{i_k=N_0}^{i_{k+1}-1} \cdots \sum_{i_1=N_0}^{i_2-1} a_{i_1} = \sum_{i_{k+1}=N_0}^{n-1} \sum_{j=N_0}^{i_{k+1}-1} \frac{(i_{k+1}-j-1)^{\underline{k-1}}}{(k-1)!} a_j$$

$$= \frac{1}{(k-1)!} \Bigg[1^{\underline{k-1}} a_{N_0} + \left(2^{\underline{k-1}} a_{N_0} + 1^{\underline{k-1}} a_{N_0+1} \right) + \ldots +$$

$$+ \left((n-N_0-2)^{\underline{k-1}} a_{N_0} + (n-N_0-3)^{\underline{k-1}} a_{N_0+1} + \ldots + 1^{\underline{k-1}} a_{n-3} \right) \Bigg]$$

$$= \frac{1}{(k-1)!} \Bigg[\left(1^{\underline{k-1}} + 2^{\underline{k-1}} + \ldots + (n-N_0-2)^{\underline{k-1}} \right) a_{N_0}$$

$$+ \left[\left(1^{\underline{k-1}} + 2^{\underline{k-1}} a_{N_0} + \ldots + (n-N_0-3)^{\underline{k-1}} \right) a_{N_0+1} + \ldots + 1^{\underline{k-1}} a_{n-3} \right]$$

$$= \frac{1}{(k-1)!}\left[\sum_{i=1}^{n-N_0-2} i^{\underline{k-1}} a_{N_0} + \sum_{i=1}^{n-N_0-3} i^{\underline{k-1}} a_{N_0+1} + \ldots + 1^{\underline{k-1}} a_{n-3}\right]$$

$$= \frac{(n-N_0-1)^{\underline{k}}}{k!} a_{N_0} + \frac{(n-N_0-2)^{\underline{k}}}{k!} a_{N_0+1} + \ldots + \frac{2^{\underline{k}}}{k!} a_{n-3}.$$

Therefore, we obtain

$$\sum_{i_{k+1}=N_0}^{n-1} \sum_{i_k=N_0}^{i_{k+1}-1} \ldots \sum_{i_1=N_0}^{i_2-1} a_{i_1} = \sum_{j=N_0}^{n-1} \frac{(n-j-1)^{\underline{k}}}{k!} a_j,$$

and by mathematical induction (4) holds for all $m \geq 2$.

2. Main Results

***Theorem* 1** — Equation (E) always has a positive solution which tends to infinity as $n \to \infty$.

PROOF : For a given $q_n \geq 0$ one can find a sequence $\{h_n\}$ such that $\sum_{i=1}^{\infty} q_i h_i = \infty$ and

$$\frac{q_n}{\sum_{i=1}^{n} q_i h_i} \to 0 \quad \text{as} \quad n \to \infty.$$

We define a sequence $\{u_n\}$ as

$$u_n = \prod_{l=1}^{n} \sum_{k=1}^{l} \prod_{j=1}^{k} \sum_{i=1}^{j} q_i h_i.$$

It is easy to check that

$$\frac{u_{n-\tau}}{u_n} \to 0 \quad \text{as} \quad n \to \infty. \qquad \ldots (5)$$

By Stolz's Theorem, we have

$$\frac{1}{u_n} \sum_{i=1}^{n} q_i u_{i-\sigma} \to 0 \quad \text{as} \quad n \to \infty. \qquad \ldots (6)$$

Consider the Banach space l_∞^N of all real bounded sequences $y = \{y_n\}$ where $n \ge N$ with the sup norm $\|y\| = \sup_{n \ge N} \{|y_n|\}$. We define a subset S of l_∞^N as

$$S = \left\{ y \in l_\infty^N : 0 \le y_n \le 1, n \ge N \right\}.$$

We define an operator T on S

$$(Ty)_n = \begin{cases} \dfrac{1}{u_n} \displaystyle\sum_{i=N_1}^{n-1} \dfrac{(n-i-1)^{\underline{m-1}}}{(m-1)!} q_i y_{i-\sigma} u_{i-\sigma} + \dfrac{1}{2u_n} + p \dfrac{u_{n-\tau}}{u_n} y_{n-\tau} \text{ if } n \ge N_1, \\ 1 \text{ if } N \le n < N_1 \end{cases} \tag{7}$$

where $N = N_1 - \max\{\delta, \tau\}$ and N_1 is chosen so large that

$$u_n \ge 1, \; p\frac{u_{n-\tau}}{u_n} + \frac{1}{u_n} \sum_{i=N_1}^{n-1} \frac{(n-i-1)^{\underline{m-1}}}{(m-1)!} q_i u_{i-\sigma} \le \frac{1}{2} \qquad \dots (8)$$

for $n \ge N_1$.

We note, that in view of (5) and (6) such an integer N_1 does exist. By (7) and (8) we have

$$0 \le (Ty)_n \le 1 \quad \text{for} \quad n \ge N,$$

which implies that $TS \subseteq S$.

Let $y, y^* \in S$, then

$$\left| (Ty)_n - (Ty_n^*) \right| \le p\frac{u_{n-\tau}}{u_n} \left| y_n - y_n^* \right| + \frac{1}{u_n} \sum_{i=N_1}^{n-1} \frac{(n-i-1)^{\underline{m-1}}}{(m-1)!} q_i u_{i-\sigma}$$

$$\times \left| y_{i-\sigma} - y_{i-\sigma}^* \right| \le \frac{1}{2} \|y - y^*\|, \; n \ge N_1.$$

Hence

$$\|Ty - Ty^*\| = \sup_{n \ge N} |(Ty)_n - (Ty^*)_n| = \sup_{n \ge N_1} |(Ty)_n - (Ty^*)_n|$$

$$\le \frac{1}{2} \|y - y^*\|,$$

which shows that T is a contraction on S.

Therefore there exists an element $y \in S$ such that $Ty = y$. Hence, we have

$$y_n = \begin{cases} \dfrac{1}{u_n} \displaystyle\sum_{i=N_1}^{n-1} \dfrac{(n-i-1)^{\underline{m-1}}}{(m-1)!} q_i y_{i-\sigma} u_{i-\sigma} + \dfrac{1}{2u_n} + p\dfrac{u_{n-\tau}}{u_n} y_{n-\tau} & n \geq N_1 \\ 1, \ N \leq n < N_1. \end{cases}$$

Now we set $x_n = y_n u_n$. Then

$$x_n = \begin{cases} \displaystyle\sum_{i=1}^{n-1} \dfrac{(n-i-1)^{\underline{m-1}}}{(m-1)!} q_i x_{i-\sigma} + p x_{n-\tau} + \dfrac{1}{2}, & n \geq N, \\ u_n, \ N \leq n < N_1. \end{cases}$$

Therefore, using Lemma 3, we have

$$\Delta^m (x_n - p x_{n-\tau}) = \Delta^m \left(\sum_{i=1}^{n-1} \frac{(n-i-1)^{\underline{m-1}}}{(m-1)!} q_i x_{i-\sigma} + \frac{1}{2} \right)$$

$$= \Delta^m \left(\sum_{i_m=1}^{n-1} \sum_{i_{m-1}=1}^{i_m - 1} \cdots \sum_{i_1=1}^{i_2 - 1} q_{i_1} x_{i_1 - \sigma} + \frac{1}{2} \right)$$

$$= q_n x_{n-\sigma}, \ n \geq N_1,$$

that is $\{x_n\}$ is a positive solution of equation (E). Now we shall show that $\lim_{n\to\infty} x_n = \infty$. In fact, set $z_n = x_n - p x_{n-\tau}$. Then

$$z_n = \sum_{i=1}^{n-1} \frac{(n-i-1)^{\underline{m-1}}}{(m-1)!} q_i x_{i-\sigma} + \frac{1}{2} > 0,$$

$$\Delta z_n = \sum_{i=1}^{n-1} \frac{(n-i-1)^{\underline{m-2}}}{(m-2)!} q_i x_{i-\sigma} > 0,$$

$$\Delta^2 z_n = \sum_{i=1}^{n-1} \frac{(n-i-1)^{\underline{m-3}}}{(m-3)!} q_i x_{i-\sigma} > 0.$$

Hence, $\lim_{n\to\infty} z_n = \infty$. But $x_n > z_n$, so $\lim_{n\to\infty} x_n = \infty$. This completes the proof.

In view of Theorem 1, equation (E) has always non oscillatory solution. Therefore we only need to find conditions for all bounded solutions of (E) to be oscillatory.

Theorem 2 — *Assume that m is even and* $0<p<1$. *Let*

$$\limsup_{n\to\infty} \sum_{j=n-\sigma}^{n-1} \frac{(j-n+\sigma+m-1)^{\underline{m-1}}}{(m-1)!} q_j > 1. \qquad \text{... (9)}$$

Then every bounded solution of equation (E) is oscillatory.

PROOF : Assume to contrary that $\{x_n\}$ is an eventually positive bounded solution of equation (E), say $x_n > 0$ for $n \geq n_1$. Define

$$z_n = x_n - px_{n-\tau}. \qquad \text{... (10)}$$

Then we have $x_n \geq z_n$.

Equation (E) becames

$$\Delta^m z_n = q_n x_{n-\sigma} > 0 \quad \text{for} \quad n \geq n_1 + \sigma. \qquad \text{.. (11)}$$

There are two cases to consider:

(A) $z_n > 0$,

(B) $z_n < 0$,

eventually.

Case (A) — Assume $z_n > 0$ for $n \geq n_2$. From Lemma 1 and (11), there exists a positive, even number l such that (3) holds. If $l \geq 2$, then $\lim_{n\to\infty} z_n = \infty$, which contradicts the boundedness of $\{x_n\}$. Hence, there exists an integer $n_3 \geq n_1$ such that

$$(-1)^i \Delta^i z_n > 0 \quad \text{for} \quad i = 0, 1, \ldots, m \quad \text{and } n \geq n_3. \qquad \text{... (12)}$$

Summing both sides of (11) from N to $n-1$, we have

$$-\Delta^{m-1} z_N \geq \sum_{i_1=N}^{n-1} q_{i_1} x_{i_1-\sigma}, \quad n \geq n_3.$$

Summing again, in N, repeatedly $(m-2)$ times and using (12), we get

$$-\Delta z_N \geq \sum_{i_{m-1}=N}^{n-1} \sum_{i_{m-2}=i_{m-1}}^{n-1} \cdots \sum_{i_1=i_2}^{n-1} q_{i_1} x_{i_1-\sigma}.$$

Next, summing the above inequality, and using Lemma 2, we obtain

$$-z_n + z_{n-\sigma} \geq \sum_{j=n-\sigma}^{n-1} \frac{(j-n+\sigma+m-1)^{\underline{m-1}}}{(m-1)!} q_j x_{j-\sigma}.$$

Therefore

$$z_{n-\sigma-1} \left\{ \sum_{j=n-\sigma}^{n-1} \frac{(j-n+\sigma+m-1)^{\underline{m-1}}}{(m-1)!} q_j - 1 \right\} + z_n - \Delta z_{n-\sigma-1} \leq 0,$$

which contradicts (9).

Case (B) — Assume $z_n < 0$ for $n \geq n_4$. By Lemma 1, $\Delta z_n < 0$. In this case

$$x_n < px_{n-\tau} < p^2 x_{n-2\tau} < \ldots < p^k x_{n-k\tau}$$

for $n \geq n_4 + k\tau$, which implies that $\lim_{n\to\infty} x_n = 0$. Consequently, $\lim_{n\to\infty} z_n = 0$.

This is a contradiction, which completes the proof.

For $m = 2$ Theorem 2 reduces to Theorem 4.1 of [4]. Note, that for equation (2) condition (9) does not hold.

We can improve this result, providing such criterion which includes coefficient p explicity.

***Theorem* 3** — *Assume that m is even and $0 < p < 1$. Let there exists an integer $k \geq 0$ such that*

$$\limsup_{n\to\infty} \sum_{j=n-\sigma}^{n-1} \frac{(j-n+\sigma+m-1)^{\underline{m-1}}}{(m-1)!} q_j > \frac{1-p}{1-p^{k+1}}. \quad \ldots (13)$$

Then every bounded solution of equation (E) is oscillatory.

PROOF : Assume to contrary that $\{x_n\}$ is an eventually positive bounded solution of equation (E), say $x_n > 0$ for $n \geq n_1$. As in the proof of Theorem 2, we consider the two cases (A) and (B). In case (A) we have $z_n > 0, \Delta z_n < 0$ for $n \geq n_3$. Using (10) in (11) we obtain

$$\Delta^m z_n = q_n z_{n-\sigma} + pq_n x_{n-\tau-\sigma}.$$

Repeated this procedure we get

$$\Delta^m z_n = q_n \sum_{i=0}^{k} p^i z_{n-\sigma-i\tau} + q_n p^{k+1} x_{n-\sigma-(k+1)\tau}.$$

Therefore

$$\Delta^m z_n \geq q_n \sum_{i=0}^{k} p^i z_{n-\sigma-i\tau}.$$

For simplicity we denote $\sum_{i=0}^{k} p^i = K$. Then using monotonicity of $\{z_n\}$ one gets

$$\Delta^m z_n \geq K q_n z_{n-\sigma}.$$

Using the procedure of (A) from the proof of Theorem 2 we get

$$z_{n-\sigma-1} \left\{ K \sum_{j=n-\sigma}^{n-1} \frac{(j-n+\sigma+m-1)^{\underline{m-1}}}{(m-1)\,!} q_j - 1 \right\} \leq 0. \qquad \text{... (14)}$$

which contradicts (13). The rest of the proof is similar to that of Theorem 2.

The conclusion of Theorem 2 can be strengthened as follows.

***Theorem* 4** — *Assume that m is even,* $0 < p < 1$.

Let

$$\limsup_{n\to\infty} \sum_{j=n-\sigma}^{n-1} \frac{(j-n+\sigma+m-1)^{\underline{m-1}}}{(m-1)\,!} q_j > 1-p. \qquad \text{... (15)}$$

Then every bounded solution of equation (E) is oscillatory.

PROOF : Denote $a = \limsup_{n\to\infty} \sum_{j=n-\sigma}^{n-1} \frac{(j-n+\sigma+m-1)^{\underline{m-1}}}{(m-1)\,!} q_j$.

Let integer k be chosen such that

$$a > \frac{1-p}{1-p^{k+1}}.$$

Then the assertion of this theorem follows immediately from Theorem 3.

Example 2 — Consider the neutral difference equation

$$\Delta^2 (x_n - p x_{n-\tau}) = \left(c + \frac{1}{n} \right) x_{n-1}, \qquad \text{... (16)}$$

where $p \in (0,1)$, $c \in R_+$ and $\tau = 0, 1, 2, \ldots$

Condition (15) for equation (16) reduces to

$$c > 1 - p \tag{17}$$

and so for example for $p = \frac{2}{3}$ and $c = \frac{1}{2}$ condition (17) is fulfilled and therefore all bounded solutions of equation (16) are oscillatory. On the other hand, the criterion (9) fails.

***Theorem* 5** — *Assume that m is even and p = 1. Let there exists an integer k > 0 such that*

$$\limsup_{n \to \infty} \sum_{j=n-\sigma}^{n-1} \frac{(j-n+\sigma+m-1)^{\underline{m-1}}}{(m-1)\,!} q_j > \frac{1}{k}.$$

Then every bounded solution of equation (E) is oscillatory.

PROOF : Assume that $\{x_n\}$ is an eventually positive bounded solution of equation (E). Define $z_n = x_n - x_{n-\tau}$. Then $\{z_n\}$ is bounded, too. Similar as in the proof of Theorem 2, we have two cases to consider:

(A) $z_n > 0, \quad \Delta z_n < 0 \quad$ for $\; n \geq n_1$;

(B) $z_n < 0, \quad \Delta z_n < 0 \quad$ for $\; n \geq n_2$.

In case (A) we led to (15) with $K = k$, which contradicts the assumptions.

In case (B), we have $\lim_{n \to \infty} z_n = -c$, where $c > 0$ is a finite number. So, there exists $n_3 \geq n_2$ such that

$$-c < z_n < -\frac{c}{2} \quad \text{for} \quad n \geq n_3.$$

Thus

$$-c < x_n - x_{n-\tau} < -\frac{c}{2} \quad \text{for} \quad n \geq n_3.$$

Consequently

$$x_n < -\left(\frac{c}{2}\right) + x_{n-\tau} < -2\left(\frac{c}{2}\right) + x_{n-2\tau} < \ldots < -k\left(\frac{c}{2}\right) + x_{n-k\tau}$$

for $\quad n \geq n_3 + k\tau$.

Let us chose a sequence $\{n_k\}$ such that $n_k = n_3 + k\tau$. Then

$$x_{n_k} = x_{n_3} + k\tau < -k\left(\frac{c}{2}\right) + x_{n_3}.$$

Therefore $\lim_{k \to \infty} x_{n_k} = -\infty$ which is a contradiction with the boundedness of $\{x_n\}$. This completes the proof.

Combining our previous results we have

Corollary 1 — Assume that m is even and $0 < p \leq 1$.

$$\limsup_{n \to \infty} \sum_{j=n-\sigma}^{n-1} \frac{(j-n+\sigma+m-1)^{\underline{m-1}}}{(m-1)!} q_j > 1-p,$$

then every bounded solution of equation (E) is oscillatory.

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