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ON COMPACTLY AND WEAKLY COMPACTLY STRONGLY EXPOSED PROPERTIES AND CRITERIA FOR THEM IN ORLICZ SEQUENCE SPACES EQUIPPED WITH THE ORLICZ NORM

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In this paper, we introduce two new geometric properties in Banach spaces, namely compactly strongly exposed property and weakly compactly strongly exposed property and we prove that the first one is the dual property of property S and the second one is the dual property of property WS. Criteria for these two properties in Orlicz sequence spaces l^{0}_{M} eqiupped with the Orlicz norm are given. Criteria for property S and property WS of the subspace h_M of order continuous elements of the Orlicz space l_M equipped with the Luxemburg norm are obtained, too.

Key Words : Compactly Strongly Exposed Property; Weakly Compactly Strongly Exposed Property; Property *S*; Orlicz Sequence Space

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a real Banach space and X^* be its dual space. By $S(X)$ we denote the unit sphere of X. For any $x \in S(X)$, we denote by $Grad(x)$ the set of all support functional at x, that is, $Grad(x) = \{f \in S(X^*) : f(x) = \|x\|\}$.

Before starting with our results, we need to recall some geometric notions.

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A Banach space X is said to be rotund (R for short) if for any x and y in $S(X)$ with $\|x+y\| = 2$, we have $x=y$ (see [5]).

A Banach space X is called locally uniformly rotund (LUR for short) (resp. weakly locally uniformly rotund ($WLUR$ for short)) if for each $x \in S(X)$ and each sequence $\{x_n\}$ in $S(X)$ such that $\|x_n+x\| \to 2$ as $n \to \infty$, we have $x_n \to x$ (resp. $\left(x_n \xrightarrow{w} x\right)$ (see [5] and [1]).

We say a Banach space X have property WM if for each $x \in S(X)$ and each sequence $\{x_n\}$ in $S(X)$ such that $\|x_n+x\| \to 2$ as $n \to \infty$, there exist $f \in Grad(x)$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $f\left(x_{n_i}\right) \to 1$ (see [10]).

A Banach space X is said to be strongly exposed (SE for short) (resp. weakly strongly exposed (WSE for short)) if for any $x \in S(X)$ and any sequence $\{x_n\}$ in $S(X)$, if $f\left(x_n\right) \to 1$ for some $f \in Grad(x)$, then $x_n \to x$ (resp. $\left(x_n \xrightarrow{w} x\right)$ (see [2] and [1]).

It is obvious that a Banach space X is LUR ($WLUR$) if and only if X has property WM and X is SE (resp. WSE).

Afterwards, the notion of LUR and $WLUR$ were generalized in the following way. A Banach space X is said to be compactly locally uniformly rotund ($CLUR$ for short) (resp. weakly compactly locally uniformly rotund ($WCLUR$ for short)) if for each $x \in S(X)$ and each sequence $\{x_n\}$ in $S(X)$ such that $\|x_n+x\| \to 2$ as $n \to \infty$, the set $\{x_n : n \in \mathcal{N}\}$ is relatively compact in the norm topology (resp. in the weak topology) (see [8] and [1]).

It is easy to prove that a Banach space X is LUR ($WLUR$) if and only if X is R and $CLUR$ (resp. $WCLUR$ and WM).

Analogously, we introduce the notions of compactly strongly exposed property and weakly compactly strongly exposed property of Banach spaces. We say a Banach space is compactly strongly exposed (CSE for short) (resp. weakly compactly strongly exposed ($WCSE$ for short)) if for any $x \in S(X)$ and any sequence $\{x_n\}$ in $S(X)$, the condition $f\left(x_n\right) \to 1$ for some $f \in Grad(x)$, implies that the set $\{x_n : n \in \mathcal{N}\}$ is relatively compact in the norm topology (resp. in the weak topology). As we will see below, these notions are dual to the properties S and WS, respectively, and a Banach space X is SE (WSE) if and only X is CSE (resp. $WCSE$} and R.

A Banach space X is said to have property S (resp. property WS) if for any $x \in S(X)$ and any sequence $\{f_n\}$ in $S(X^*)$ with $f_n(x) \to 1$ as $n \to \infty$, we have that the set $\{x_n : n \in \mathcal{N}\}$ is relatively compact in the norm topology (resp. in the weak topology) (see [9] and [10]).

A Banach space X is said to have property H if every sequence from the unit sphere weakly convergent to a point of the sphere is convergent in norm (see [1]).

It is easy to see that if a Banach space X has property WS and X^* has property H, then X has property S.

A map $M: \mathcal{R} \to [0, \infty)$ is said to be a N-function if it is vanishing only at zero, even, convex, $\lim\limits_{u \to 0} \frac{M(u)}{u} = 0$ and $\lim\limits_{u \to \infty} \frac{M(u)}{u} = \infty$. For any N-function M we define its complementary function $N: \mathcal{R} \to [0, \infty)$ by the formula

$$N(v) = \sup_{u > 0} \{u|v| - M(u)\}$$

for every $v \in \mathcal{R}$. The complementary function N is also a N-function (see [1] and [6]).

Let us denote by l^0 the space of all real sequences. For a given N-function M, we define the Orlicz sequence space l_M by

$$l_M = \{x \in l^0 : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

where

$$\rho_M(x) = \sum_{i=1}^{\infty} M(x(i)) \qquad (\forall\, x = (x(i)) \in l^0).$$

This space equipped with the Luxemburg norm

$$\|x\| = \inf\left\{\lambda > 0 : \rho_M\left(\frac{x}{\lambda}\right) \le 1\right\}$$

or with the equivalent one

$$\|x\|^0 = \inf_{k > 0} \frac{1}{k}(1 + \rho_M(kx)),$$

called the Orlicz (or the Amemiya) norm, is a Banach space (see [1], [6], [8] and [11]).

By h_M we denote the subspace of l_M denned by

$$h_M = \{x \in l^0 : \rho_M(\lambda x) < \infty \text{ for any } \lambda > 0\}.$$

To simplify notations, we put $l_M = \left(l_M, \|\cdot\| \right)$ and $l_M^0 = \left(l_M, \|\cdot\|^0 \right)$.

We say that M satisfies the Δ_2-condition ($M \in \Delta_2$ for short) if there are $u_0 > 0$ and $K > 0$ such that $M(2u) \le K M(u)$ whenever $|u| \le u_0$ (see [1]).

Let $\theta_M(x) = \inf \left\{ \lambda > 0 : \rho_M\left(\frac{x}{\lambda} \right) < \infty \right\}$ for any $x \in l_M$. It is clear that $h_M = \{x \in l_M : \theta_M(x) = 0\}$. Moreover, θ_M is the norm in the quotient space l_M / h_M.

2. Some General Results

In this part, we will discuss some relationships between the properties *S, SE, CSE, R, WSE, WCSE WM, WCLUR* and *WS*. Some general results will be given.

***Theorem* 1** — *A Banach space X is SE if and only if X is CSE and R.*

Proof : *Necessity* : By the definitions of *SE* and *CSE*, the implication *SE* $\Rightarrow$ *CSE* is obvious. It remains to prove that any *SE* Banach space X is R. Let $x, y \in S(X)$ and $\|x+y\| = 2$. By the Hahn-Banach theorem there exists $f \in S(X^*)$ such that $f(x+y) = \|x+y\| = 2$. Consequently, $f(x) = f(y) = 1$. Since x is *SE*, we have $x = y$. This proves that X has property R.

Sufficiency : Let us assume that X is *CSE* and R and let $x \in S(X)$, $\{x_n\} \subset S(X)$ and $f(x_n) \to 1$ for some $f \in Grad(x)$. By *CSE* of X, we conclude that there exist a subsequence $\left\{x_{n_i}\right\}$ of $\{x_n\}$ and $x' \in S(X)$ such that $x_{n_i} \to x'$. Clearly $f(x') = 1$ and consequently $\|x + x'\| = 2$. It follows, by rotundity of X, that $x' = x$. Therefore $x_{n_i} \to x$ and, by the double extract subsequence theorem, $x_n \to x$. So the theorem is proved.

An analogous characterization of Banach spaces that are weakly compactly strongly exposed gives the following theorem.

***Theorem* 2** — *A Banach space X is WSE if and only if X is WCSE and R.*

Proof : The proof is similar to the proof of Theorem 1, so we omit it here.

***Theorem* 3** — *A Banach space X is CSE if and only if X is WCSE and X has property H.*

PROOF : Let X be *CSE*. Clearly X is then *WCSE*. Now we will show that X has property H. Let $x \in S(X)$, $\{x_n\} \subset S(X)$ and $x_n \xrightarrow{w} x$. Let $f \in S(X^*)$ be such that $f(x) = \| x \| = 1$. Then $f(x_n) \to f(x) = 1$ as $n \to \infty$. By property *CSE* of X, $\{x_n\}$ has a convergent subsequence. Also any subsequence of $\{x_n\}$ has a convergent subsequence. In view of $x_n \xrightarrow{w} x$, this shows that $x_n \to x$ as $n \to \infty$.

Conversely, let us suppose that X has properties *WCSE* and H. Let $x \in S(X)$, $\{x_n\} \subset S(X)$ and $f(x_n) \to 1$ for some $f \in Grad(x)$. By *WCSE* of X, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $x' \in (X)$ such that $x_{n_i} \xrightarrow{w} x'$. Obviously, $f\left(x_{n_i}\right) \to f(x')$ as $i \to \infty$. By the uniqueness of the limit, we conclude that $f(x') =$ I. Hence, in view of the lower semicontinuity of the norm with respect to the weak topology, we get

$$1 = f(x') \leq \| x' \| \leq \liminf_{i \to \infty} \left\| x_{n_i} \right\| \leq 1,$$

i.e. $x' \in S(X)$. Taking into account that X has property H, we have that $x_{n_i} \to x'$, and the results follows.

Combining Theorems 1, 2 and 3, it is easy to conclude the following corollary.

Corollary 1 — A Banach space X is *SE* if and only if X is *WSE* and X has property H.

***Theorem* 4** — *If X has property WM, then X is WCLUR if and only if X is WCSE.*

PROOF : The necessity of *WCSE* for *WCLUR* is obvious. To prove the sufficiency of *WCSE* for *WCLUR* when X is *WM* assume that X is *WCSE*, $x \in S(X)$ and $\{x_n\}$ is a sequence in $S(X)$ such that $\| x_n + x \| \to 2$ as $n \to \infty$. Since X has property *WM*, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $f \in Grad(x)$ such that $f\left(x_{n_i}\right) \to 1$. Since X is *WCSE*, this implies that the sequence $\{x_n\}$ is relatively compact in the weak topology, which finishes the proof.

***Theorem* 5** — *If X has the WM-property, then X is CLUR if and only if X is CSE.*

PROOF : We can prove this theorem using similar argumentation as in the proof Theorem 4.

***Theorem* 6** — *If X^* is WCSE, then X has property WS.*

PROOF : Let X^* be *WCSE*, $x \in S(X)$ and $\{f_n\}$ be a sequence in $S(X^*)$ such that $f_n(x) \to 1$ as $n \to \infty$. Take any $f \in Grad(x)$. Denote by $\mathcal{H}$ the canonical embedding of X into X^{**}. Then $\mathcal{H}x(f_n) = f_n(x) \to 1$ and $\mathcal{H}x(f) = f(x) = 1$. Now, by the assumption that X^* is *WCSE*, we get that the sequence $\{f_n\}$ is relatively compact in the weak topology, which completes the proof.

***Theorem* 7** — *If X^* has property WS , then X is WCSE.*

PROOF : Let $x \in S(X)$ and $\{x_n\} \subset S(X)$ be such that $f(x_n) \to 1$ for some $f \in Grad(x)$. Since X^* has property *WS*, by Remark 1 in [6], we have that X is reflexive. Consequently, there exist a subsequence $\left\{x_{n_i}\right\}$ of $\{x_n\}$ and $x' \in X$ such that $x_{n_i} \xrightarrow{w} x'$. Thus X has property *WS*.

Analogously, there is a complete duality between properties *CSE* and *S*.

Corollary 2 — If X^* is *CSE*, then X has property *S*.

PROOF : Let X^* be *CSE*. Then X^* is *WCSE*. From Theorem 6, it follows that X has property *WS*. By Theorem 3, X has property *H*. Consequently X has properties *WS* and *H*, so X has property *S*.

Corollary 3 — If X^* has property *S*, then X is *CSE*.

PROOF : If X^* has property *S*, then X^* has property *WS* and X has property *H* (see Theorem 3.2 in [2]). By Theorem 7, X is *WCSE*. Hence, by Theorem 3, we conclude that X is *CSE*.

3. PROPERTIES *S* AND *WS* FOR h_M AND PROPERTIES *CSE* AND *WCSE* FOR l_M^0

***Theorem* 8** — For the space h_M, the following statements are equivalent:

(1) h_M has property S,

(2) h_M has property WS,

(3) $N \in \Delta_2$.

PROOF : The implication (1) $\Rightarrow$ (2) is obvious. We will prove now that (2) $\Rightarrow$ (3). If $N \notin \Delta_2$, then $\overset{0}{h}_N \neq \overset{0}{l}_N$. By the Bishop-Phelps theorem, there is $f \in S\left(\overset{0}{l}_N \setminus \overset{0}{h}_N\right)$ and $x \in S(h_M)$ such that $f(x) = \|f\|^0 = 1$. Put

$$f_n = (f(1), f(2), ..., f(n), 0, ...) \quad n = 1, 2, \dots .$$

Then

$$f_n(x) = \sum_{i=1}^{n} f(i)\, x(i) \to f(x) = 1.$$

Since $f \notin \overset{0}{h}_N$, by the Hahn-Banach theorem, there exists a singular functional ϕ such that $\phi(f) \neq 0$. Since $f_n \notin \overset{0}{h}_N$, by separability of h_M, we have $\phi(f_n) = 0$. Since $f_n \to f$ coordinatewise, we may assume without loss of generality (passing to a subsequence if necessary) that $f_n \overset{w}{\to} f$. But $\phi(f - f_n) = \phi(f) \neq 0$ for any $n \in \mathcal{N}$, which contradicts the fact that $f_n \overset{w}{\to} f$. This contradiction shows that $N \notin \Delta_2$.

(3) $\Rightarrow$ (1). For any $x \in S(h_M)$, we have $\theta_M(x) = 0 < 1$. By Theorem 3 in [3], we conclude that x is an S point. So h_M has property S.

***Theorem* 9** — *The following statements are equivalent:*

(i) $\overset{0}{l}_M$ *is CSE*

(ii) $\overset{0}{l}_M$ *is WCSE,*

(iii) $M \in \Delta_2$ *and* $N \in \Delta_2$.

PROOF : The implication *(i)* $\Rightarrow$ *(ii)* follows immediately from the definition of properties *CSE* and *WCSE*. To prove the implication *(ii)* $\Rightarrow$ *(iii)*, suppose that $\overset{0}{l}_M$ is *WCSE*. By Theorems 6 and 8, we conclude that $M \in \Delta_2$. Now, we are going to prove that $N \in \Delta_2$. Assume that $N \notin \Delta_2$. Then there is a sequence $\left\{ \upsilon_n \right\}_{n=1}^{\infty}$ of positive numbers such that $\upsilon_n \searrow 0$ as $n \to \infty$ and

$$N\left(\left(1+\frac{1}{n}\right)\upsilon_n\right) > 2^{n+1} N(\upsilon_n), \quad \text{and} \quad N(\upsilon_n) < \frac{1}{2^n} \ (n = 1, 2, \ldots).$$

For each $n \in \mathcal{N}$ choose a positive integer m_n such that

$$m_n N(\upsilon_n) \leq \frac{1}{2^n} < (m_n + 1) N(\upsilon_n).$$

Namely, in order to get this, for any $n \in \mathcal{N}$ we define m_n as the largest natural number such that $m_n N(\upsilon_n) \leq \frac{1}{2^n}$. Pick $I_n \subset Z$ with $Card(I_n) = m_n$ such that $I_n \cap I_m = \phi$ and define

$$y = \begin{cases} a & \text{for } i \in I_1, \\ \upsilon_n & \text{for } i \in I_1, n = 2, 3, \ldots \\ 0 & \text{for } i \notin \bigcup_{i=1}^{\infty} I_n, \end{cases}$$

where a is chosen in such a way that $\rho_N(y) = 1$. Then $\| y \|_N = 1$. Since $\rho((1+\tau) y) = \infty$ for any $\tau > 0$, we have $\theta_N(y) = 1$. Consequently $\| y \|_N = \theta_N(y) = 1$. Let $\{ \upsilon_n \}_{n=1}^{\infty}$ be a sequence of positive numbers such that $q^-(\upsilon_n) \leq u_n \leq q(\upsilon_n)$ for $n = 2, 3, \ldots$ and $q^-(a) \leq u_1 \leq q(a)$ and let us define

$$x = \begin{cases} u_n & i \in I_n, n = 1, 2, \ldots \\ 0 & i \notin \bigcup_{i=1}^{\infty} I_n. \end{cases}$$

Since $\| x \|^0 = \sup\{\langle x, z \rangle : \| z \|_N \leq 1\}$ (see [4]), $\langle x, y \rangle = \rho_M(x) + \rho_N(y) = \rho_M(x) + 1$, and $\langle x, z \rangle \leq \rho_M(x) + 1$ for any $z \in l_N$ with $\| z \|_N \leq 1$, we have $\langle x, y \rangle = \| x \|^0$, that is, $y \in Grad(x)$.

Since $\theta_N(y) = 1$, we get $\left\| [y]_n^{\infty} \right\| = 1$ for any $n \in \mathcal{N}$ where $[y]_n^{\infty} = (0, 0, \ldots, 0, y(n+1), y(n+2), \ldots)$.

Denote $i_1 = 0$. Since $\left\| [y]_{i_1}^{\infty} \right\| = 1 > 0$, there exists $i_2 > i_1$ such that $\left\| [y]_{i_1}^{i_2} \right\| > 0$.

Since $\left\| [y]_{i_2}^{\infty} \right\| = 1 > 1 - \frac{1}{2}$, there exists $i_3 > i_2$ such that $\left\| [y]_{i_2}^{i_3} \right\| > 1 - \frac{1}{2}$.

Continuing this procedure in such a way, we can get a sequence $\{i_n\}$ of natural numbers with $i_1 < i_2 < \ldots$ such that $\left\| [y]_{i_n}^{i_{n+1}} \right\| > 1 - \frac{1}{n}$ $(n = 1, 2, \ldots)$.

For each $n \in \mathcal{N}$, noticing that $[y]_{i_n}^{i_{n+1}} \in h_N$, there exists $x_n \in S\left(l_M^0 \right)$ such that $\langle x_n, [y]_{i_n}^{i_{n+1}} \rangle = \left\| [y]_{i_n}^{i_{n+1}} \right\|$ and $x_n = [x_n]_{i_n}^{i_{n+1}}$. Therefore

$$1 \geq \langle y, x_n \rangle = \sum_{i=i_n}^{i_{n+1}} y(i) x_n(i)$$

$$= \langle x_n, [y]_{i_n}^{i_{n+1}} \rangle$$

$$= \left\| [y]_{i_n}^{i_{n+1}} \right\| > 1 - \frac{1}{n} \quad n = 1, 2, \ldots,$$

whence $\langle y, x_n \rangle \to 1$ as $n \to \infty$. But for any $m, n \in \mathcal{N}, m > n$, we have

$$\| x_m - x_n \|^0 \geq \| x_m \|^0 = 1,$$

which means that the set $\{x_n : n \in N\}$ is not relatively compact in $S\left(l_M^0 \right)$. Hence l_M^0 is not *CSE*. Moreover, thanks to $M \in \Delta_2$, l_M^0 has property *H*. So it yields, by Theorom 3, that l_M^0 is not *WCSE*. This finishes the proof of this implication.

(iii) $\Rightarrow$ *(i)*. Suppose that $x \in S\left(l_M^0 \right)$ and $\{x_n\}$ is a sequence in $S\left(l_M^0 \right)$ such that

$f(x_n) \to 1$ for some $f \in Grad(x)$. Since, by $M \in \Delta_2$ and $N \in \Delta_2$, $\overset{0}{l}_M$ is reflexive, whence we conclude that there are a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $x' \in \overset{0}{l}_M$ such that $x_{n_i} \overset{w}{\to} x'$. In particular, $f(x_{n_i}) \to f(x')$. By $f(x_{n_i}) \to 1$, we have $f(x') = 1$. Therefore, by lower semicontinuity of the norm $\|\cdot\|^0 = \overset{0}{\|\cdot\|}_M$ with respect to the weak topology, we have

$$1 = f(x') \le \|x'\|^0 \le \liminf_{i \to \infty} \left\| x_{n_i} \right\|^0 \le 1,$$

which implies that $\|x'\|^0 = 1$. In virtue of $M \in \Delta_2$, $\overset{0}{l}_M$ has property H (see [1]). Hence $x_{n_i} \to x'$, which completes the proof of the theorem.

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REFERENCES

1. S. Chen, Geometry of Orlicz spaces, *Dissertations Math.*, **356** (1996).
2. Y. Cui, H. Hudzik and H. Zhu, On the weakly strongly exposed property and some smoothness properties of Orlicz spaces, *Bull. Austral. Math. Soc.*, **54** (1996), 431-40.
3. C. Hao, D. Ji and T. Wang, On S points of Orlicz spaces, *Northeast, Math. J.*, **14** (1998), 75-80.
4. H. Hudzik and L. Maligranda, Amemiya norm equals Orlicz norm in general, *Indagationes Math., N.S.* **11**(4) (2000), 573-85.
5. A. Lovaglia, Locally uniformly convex Banach spaces, *Trans. Amer. Math. Soc.*, **78** (1995). 225-38.
6. M. Krasnoselskii and Ya Rutickii, *Convex Function and Orlicz Spaces,* P. Nordhoff Ltd (Groningen, 1961), translation.
7. L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars de Mathematica, Vol. 5, Campinas (Brazil) Univ. Estadual de Campinas, 1989.
8. J. Musielak, *Orlicz Spaces an Modular Spaces*, Lecture Notes in Math., 1034, Springer-Verlag (1983).
9. B. Panda and P. Kapoor, A generalization of local uniform convexity of the norm, *J. Math. Anal. Appl.*, **52** (1975), 300-308.
10. M. Rao and Z. Ren, *Theory of Orlicz Spaces*, Marcel Dekker Inc. (1991).
11. J. Wang and C. Nan, On the dual spaces of the S-spaces and WKR spaces, *Chinese Ann. Math.*, **13**A(1) (1992), 65-69.