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NEW PARTITION THEORETIC FACTS OF TWO IDENTITIES OF EULER

S. BHARGAVA*, CHANDRASHEKAR ADIGA* AND N. ANITHA

Department of Studies in Mathematics, University of Mysore, Manasa Gangotri, Mysore 570 006, *India*

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We obtain two partition identities and show their equivalence to two identities of Euler. Our work is a sequel to the recent work of J. P. O. Santos [3].

Key Words : Partitions; Combinatorial Identities.

1. INTRODUCTION

Partition theoretic identities are important in the theory of q-series as, in many instances, they lead to important q-series identities. In Theorem 2.1 and Theorem 3.1 we prove two partition identities which are respectively equivalent to the two of Euler's identities

$$\frac{1}{(q\,;q^2)_\infty} = \sum_0^\infty \frac{q^n}{(q^2\,;q^2)_n}, \quad |q|<1, \qquad \text{... (1.1)}$$

and

$$\frac{1}{(q^2\,;q^2)_\infty} = \sum_0^\infty \frac{q^{2n}}{(q^2\,;q^2)_n}, \quad |q|<1, \qquad \text{... (1.2)}$$

Here, as usual,

$$(a,\ q)_0 := 1$$

$$(a\,;q)_n := \prod_{k=0}^{n-1} (1-aq^k), \qquad n = 1, 2, ...,$$

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$$(a\,;q)_\infty := \prod_{k=0}^{\infty} (1-aq^k), \quad |q|<1.$$

The identities (1.1) and (1.2) are special cases of Euler's celebrated theorem [1, p.19],

$$\frac{1}{(t\,;q)_\infty} = \sum_{n=0}^{\infty} \frac{t^n}{(q:q)_n}, \quad |q|<1. \qquad \text{... (1.3)}$$

An analytic proof of the general identity (1.3) can be found in [1, p.19]. Another, employing the partition function $p(m, n)$, the number of partitions of n with exactly m parts, and partition analysis can be found in [2, p. 564]. However, our proof of the special cases (1.1) and (1.2) is new and we believe that it throws further light on (1.3).

2. Combinatorial Proof of (1.1)

We first obtain a partition theoretic result. In what follows we always assume that the members of a partition are in non-increasing order.

***Theorem* 2.1** — *Let A_n be the collection of partitions of n into odd parts and B_n the collection of partitions $(\lambda_1, \lambda_2, \ldots \lambda_s)$ of n such that $n-\lambda_1 \le \lambda_1 - \lambda_2$.* Then $\# A_n = \# B_n$.

Proof : If $(\mu_1, \mu_2, \ldots \mu_r) \equiv (2k_1-1, 2k_2-1, \ldots 2k_r-1) \in A_n$, define

$$f(\mu_1, \mu_2, \ldots, \mu_r) = \left(\lambda_1, \lambda_2, \ldots, \lambda_{k_1}\right) \text{ by}$$

$$\lambda_1 := \frac{1}{2}(n+r) = k_1 + k_2 + \ldots + k_r$$

and, for $2 \le k \le k_1$,

$\lambda_k :=$ number of parts of $(\mu_1, \mu_2, \ldots \mu_r)$ which are greater than or equal to $2k-1$.

Clearly, from these definitions,

$$\lambda_2 \le r = 2\lambda_1 - n$$

or,

$$n - \lambda_1 \le \lambda_1 - \lambda_2$$

and hence, $\left(\lambda_1, \lambda_2, \ldots, \lambda_{k_1}\right) = f(\mu_1, \mu_2, \ldots, \mu_r) \in B_n$.

In the other direction, suppose $(\lambda_1, \dots \lambda_s) \in B_n$ so that $n = \lambda_1 + \dots + \lambda_s$ and $n - \lambda_1 \le \lambda_1 - \lambda_2$. Then define $\frac{\mu_i + 1}{2} := k_i :=$ number of parts of $(\lambda_1, \lambda_2, \dots, \lambda_s)$ which are greater than or equal to i for $i = 1, 2, \dots, \lambda_2$ $\mu_i := 1$ for $i = \lambda_2 + 1, \dots, \lambda_2 + \lambda_1 - (k_1 + k_2 + \dots + k_{\lambda_2})$. Then clearly, $(\mu_1, \mu_2, \dots \mu_r) = f^{-1}(\lambda_1, \lambda_2, \dots, \lambda_s)$ with $r = \lambda_2 + \lambda_1 - (k_1 + \dots + k_{\lambda_2})$.

Thus there is a 1-1 correspondence (namely f) between the members of A_n and those of B_n and hence the theorem has been proved.

Remark 2.1 : In view of the Euler's identity "the number of partitions of n into distinct parts equals the number of partitions of n into odd parts"

(i) the Theorem 2.1 can be extended as follows:

$$\# A_n = \# B_n = \# C_n$$

where C_n is the collection of partitions of n into distinct parts.

(ii) it would be interesting to try and provide similar combinatorial proof of the identity

$$(-q\,;q)_\infty = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q\,;q)_\infty}.$$

We are now in a position to prove (1.1) combinatorially, employing Theorem 2.1.

***Theorem* 2.2** — *The Identity (1.1) holds.*

PROOF : Theorem 2.1 is easily seen to assert the following two facts:

(i) If n is even, then the coefficient of q^n in the power series expansion of

$$\frac{1}{(1-q)(1-q^3)(1-q^5)\dots(1-q^{2k-1})\dots}$$

equals the coefficient of $q^{n/2}$ in the power series expansion of

$$\sum_{k=0}^{\infty} \frac{q^k}{(1-q)(1-q^2)\dots(1-q^{2k})}.$$

(ii) If n is odd, then the coefficient of q^n in the power series expansion of

$$\frac{1}{(1-q)(1-q^3)(1-q^5)\dots(1-q^{2k-1})\dots}$$

equals the coefficient of $q^{\frac{n+1}{2}}$ in the power series expansion of

$$\sum_{k=1}^{\infty} \frac{q^k}{(1-q)(1-q^2)\dots(1-q^{2k-1})}.$$

Now, if C_n is the coefficient of q^n in the power series expansion of

$$\frac{1}{(1-q)(1-q^3)\dots(1-q^{2k-1})\dots},$$

we have, from the above two facts,

$$\sum_{n=0}^{\infty} C_{2n} q^n = \sum_{k=0}^{\infty} \frac{q^k}{(q\,;q)_{2k}}$$

and

$$\sum_{n=1}^{\infty} C_{2n-1} q^n = \sum_{k=1}^{\infty} \frac{q^k}{(q\,;q)_{2k-1}},$$

so that

$$\sum_{n=0}^{\infty} C_{2n} q^{2n} + \sum_{n=1}^{\infty} C_{2n-1} q^{2n-1}$$

$$= \sum_{k=0}^{\infty} \frac{q^{2k}}{(q^2\,;q^2)_{2k}} + \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(q^2\,;q^2)_{2k-1}},$$

or, what is the same,

$$\frac{1}{(q\,;q^2)_\infty} \left(=: \sum_{n=0}^{\infty} C_n q^n \right) = \sum_{k=0}^{\infty} \frac{q^k}{(q^2\,;q^2)_k}$$

This completes the proof of Theorem 2.2.

In the article [3] even though Santos states an equivalent of Theorem 2.1 and gives a graphical illustration of his theorem he has not realized the equivalence of the partition theoretic Theorem 2.1 with the analytic identity (1.1).

3. Combinatorial Proof of (1.2)

Following theorem is the counter part of Theorem 2.1.

Theorem 3.1 — *Let A_n be the collection of partitions of n into even parts and B_n the collection of partitions $(\lambda_1, \lambda_2, ..., \lambda_s)$ of n such that $n - \lambda_1 \leq \lambda_1 - 2\lambda_2$. Then $\# A_n = \# B_n$.*

PROOF : The proof is similar to that of Theorem 2.1. In fact, define

$$f : A_n \to B_n$$

by

$$f(\mu_1, \mu_2, ..., \mu_r) = (\lambda_1, \lambda_2, ..., \lambda_{\mu_1/2})$$

where

$$\lambda_1 := n/2 + r = (k_1 + k_2 + \ldots + k_r) + r \text{ (with } \mu_j = 2k_j,\ j = 1, ..., r),$$

and, for $2 \leq k \leq k_1$

$\lambda_k :=$ number of parts of $(\mu_1, ..., \mu_r)$ which are greater than or equal to $2k$.

Clearly, these definitions imply,

$$\lambda_2 \leq r = \lambda_1 - n/2 \quad \text{or,} \quad 2\lambda_2 \leq 2\lambda_1 - n \quad \text{or,} \quad n - \lambda_1 \leq \lambda_1 - 2\lambda_2.$$

Thus $(\lambda_1, ..., \lambda_{\mu_1/2})$ indeed belongs to B_n.

In the other direction, starting from $(\lambda_1, \lambda_2, ..., \lambda_s)$ in B_n, define, for $i = 1, 2, ..., \lambda_2$,

$\frac{\overline{\mu}_i}{2} := k_i :=$ number of parts of $(\lambda_1, ..., \lambda_s)$ which are greater than or equal to i and

$$\mu_i = 2 \text{ for } i = \lambda_2 + 1, ..., \lambda_2 + n/2 - (k_1 + \ldots + k_{\lambda_2}).$$

Then clearly,

$$(\mu_1, \mu_2, \ldots \mu_r) = f^{-1}(\lambda_1, \lambda_2, ..., \lambda_s)$$

with

$$r = \lambda_2 + n/2 - (k_1 + \ldots + k_{\lambda_2}).$$

This completes the proof of the theorem.

We are now in a position to prove (1.2) combinatorially.

Theorem **3.2** — *The Identity (1.2) holds.*

PROOF : The proof follows from Theorem 3.1 in exactly the same way as Theorem 2.2 from Theorem 2.1 and hence we omit the details.

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