Indian J. pure appl. Math., **37**(1) : 3–7, February 2006

DISCRIMINANTS OF INVOLUTIONS

V. SURESH

Department of Mathematics and Statistics
University of Hyderabad
P.O. Central University
Gachibowli
Hyderabad 500 046
E-mail address: vssm@uohyd.ernet.in

(*Received* 31 January 2006; *accepted* 21 May 2006)

Work of Knus, Parimala and Sridharan on the discriminants of involutions is described.

Key words: Discriminants; Involutions

1. Introduction

In this article we describe the work of Knus, Parimala and Sridharan on the discriminants of involutions. In the eighties, Knus, Parimala and Sridharan were interested in classification of low rank quadratic forms and also the splitting of involutions on central simple algebras of rank 16. During this period, they attached an invariant with values in $k^* = k \setminus \{0\}$ modulo k^{*2} called *discriminant* (rather *pfaffian discriminant*) to an involution on a central simple algebra over a field k. In fact they also showed that this invariant precisely gives the obstruction to the splitting of an involution on a rank 16 central simple algebra into a tensor product of involutions on quaternion subalgebras.

In the second section, we recall a definition of the discriminant of an involution. In the third section, we discuss the splitting of involutions on rank 16 central simple algebras. In the last section we discuss some natural questions on the discriminants of involutions on central simple algebras. Most of the definitions, theorems and their proofs are taken from the papers [1-5].

2. Discriminant of involution

Let k be a field of characteristic not equal to 2. Let A be a central simple algebra over k with an involution σ (i.e. σ is an anti-automorphism of order 2 of A and $\sigma(\lambda) = \lambda$ for all $\lambda \in k$). For example let $M_n(k)$ be the algebra of square matrices over k of size n. For any $X \in M_n(k)$, let $\sigma_0(X) = X^t$, where X^t denotes the transpose of the matrix X. Then σ_0 is an involution on $M_n(k)$ Let $n = 2m$ be even and $\tau_0(X) = JX^tJ^{-1}$ for $X \in M_n(k)$, where J is the standard alternating matrix consisting of diagonal blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then τ_0 is also an involution on $M_n(k)$. Let A be a

(This paper is dedicated to Prof. C. Musili).

central simple algebra with an involution σ. It is well known that there exists a field extension L of k such that $A \otimes_k L \simeq M_n(L)$ and $\sigma \otimes id$ is either σ_0 or τ_0. In the first case σ is called an *orthogonal* involution and in the second case it is called a *symplectic* involution. Let

$$S_\sigma^+ = \{x \in A \mid \sigma(x) = x\}$$

and

$$S_\sigma^- = \{x \in A \mid \sigma(x) = -x\}.$$

The elements of S_σ^+ are called the *symmetric* elements and the elements of S_σ^- are called the *antisymmetric* or *skew-symmetric* elements for σ. Since the dimension of the space of symmetric matrices of size n is $n(n+1)/2$ and the dimension of skew-symmetric matrices of size n is $n(n-1)/2$. We have ([6], p.303)

$$\dim(S_\sigma^\pm) = \begin{cases} \frac{n(n\pm 1)}{2} & \text{if } \sigma \text{ is orthogonal} \\ \frac{n(n\mp 1)}{2} & \text{if } \sigma \text{ is symplectic,} \end{cases}$$

where the rank of A is n^2.

In [1], the discriminant of involution is defined using the reduced pfaffians. In fact in that paper the discriminant is defined for involutions of Azumaya algebras over rings. In [4], they showed that their definition of discriminant is equivalent to the following:

Definition: *Let A be a central simple algebra over k, with an involution σ. The discriminant of σ, denoted by $disc(\sigma)$, is defined as the square class of the reduced norm of any antisymmetric element if σ is orthogonal and the square class of reduced norm of any symmetric element if σ is symplectic.*

In particular, if σ is a symplectic involution, then its discriminant is trivial.

3. Involutions on rank 16 central simple algebras

Let k be a field of characteristic not equal to 2. Let $a, b \in k^*$. Let H be the algebra generated by i and j with relations $i^2 = a$, $j^2 = b$ and $ij = -ji$. Then it is easy to see that H is a rank 4 central simple algebra over k, known as a *quaternion* algebra. Every element of H can be written as $a + bi + cj + dij$, with $a, b, c, d \in k$. Define $\tau(a + bi + cj + dij) = a - bi - cj - dij$. Then τ is a symplectic involution on H. Any orthogonal involution on H is given by $int(x) \circ \tau$, with $\tau(x) = -x$, where for any $x \in H^*$, $int(x)$ denotes the inner automorphism given by x, i.e. the automorphism $y \mapsto xyx^{-1}$ of H.

Let A be a finite dimensional central simple algebra over k. The *rank* of A is defined to be the dimension of A over k. Since $A \otimes_k L \simeq M_n(L)$ for some field extension L of k, rank of A is n^2. Therefore the first non-trivial example of a central simple algebra has rank 4. Let A be a finite dimensional central simple algebra over k with an involution σ. Then it follows that the rank of A must be even. Therefore after the quaternions, the next central simple algebra with an involution is of rank 16.

Let H_1 and H_2 be two quaternion algebras over k and σ_1, σ_2 involutions on H_1, H_2 respectively. Let $A = H_1 \otimes H_2$ and $\sigma = \sigma_1 \otimes \sigma_2$. Then A is a rank 16 central simple algebra over k and σ is an involution on A.

The following is a classical result of Albert [7].

Theorem. *Let A be a rank 16 central simple algebra over k with an involution. Then there exist quaternion subalgebras H_1 and H_2 such that $A = H_1 \otimes_k H_2$.*

It is natural to ask the following

Question *Let A be a rank 16 central simple algebra over k with an involution σ. Does there exist quaternion subalgebras H_1, H_2 and involutions σ_1, σ_2 on H_1 and H_2 respectively, such that $A = H_1 \otimes H_2$ and $\sigma = \sigma_1 \otimes \sigma_2$.*

It is not that difficult to see that the above question has an affirmative answer if and only if A admits a quaternion subalgebra over k which is invariant under σ (see Lemma 1 below).

Amitsur, Rowen and Tignol [8] have constructed examples of rank 16 central simple algebras with involutions which do not admit invariant quaternion subalgebras, giving a negative answer to the above question. Rowen [9] also proved that if A is a rank 16 central simple algebra over k and if σ is a symplectic involution, then A admits a σ invariant quaternion subalgebras. Thus if σ is symplectic, then the above question has an affirmative answer.

We have the following theorem of Knus, Parimala and Sridharan [2].

Theorem. *Let A be a rank 16 central simple algebra over k with an involution σ. Then $A = H_1 \otimes H_2$ and $\sigma = \sigma_1 \otimes \sigma_2$ for some quaternion subalgebras H_1, H_2 of A with involutions σ_1, σ_2 if and only if $disc(\sigma)$ is trivial.*

Lemma 1. *Let A and σ be as in the above theorem. Then $A = H_1 \otimes H_2$ and $\sigma = \sigma_1 \otimes \sigma_2$ if and only if there is a quaternion subalgebra of A over k, which is invariant under σ.*

Proof. Assume that A admits a quaternion subalgebra H_1 over k, which is invariant under σ. Let H_2 be the commutant of H_1 in A. Then $A = H_1 \otimes H_2$ ([6] p. 292). By comparing the ranks we see that H_2 is also a quaternion algebra over k. Since H_1 is invariant under σ, H_2 is also invariant under σ. Let σ_1 be the restriction of σ to H_1 and σ_2 the restriction of σ to H_2. Then $\sigma = \sigma_1 \otimes \sigma_2$. The converse is trivial. □

Lemma 2. *Let A be a central division algebra over k of rank 16, with an involution. Let σ be an involution on A and $x \in A^*$ such that $\sigma(x) = -x$. Suppose that the reduced norm $Nrd(x)$ of x is a square in k. Then there exists an element $y \in A$ such that $\sigma(y) = -y$, $x + y \neq 0$ and $(x+y)^2 \in k$.*

Proof. If $x^2 \in k$, then, let $y = 0$ and we are done. Assume that $x^2 \notin k$. Let $L = k(x) \subset A$ be the field generated by x over k. Then L/k is of degree 4. Since $\sigma(x) = -x$ and σ is identity on k, the minimal polynomial of x over k is of the form $X^4 + aX^2 + b$, for some $a, b \in k$. Let E be the splitting field of the polynomial $X^4 + aX^2 + b$ over k. Let $y \in E$ be a conjugate of x over k, which is not equal to x and $-x$. Then $x, -x, y, -y$ are all the conjugates of x over k and $x^2y^2 = b$. Since $Nrd(x) = b$ is a square in k, we have $b = c^2$ for some $c \in k$. In particular $xy = \pm c \in k$. This implies that $y \in L$ and $\sigma(y) = -y$. Since $x^2 + y^2 = a \in k$, we have $(x+y)^2 = x^2 + y^2 + 2xy \in k$. Since $x \neq -y$, we also have $x + y \neq o$. □

Proof of the Theorem. By the Lemma 1, it is enough to show that the discriminant of σ is trivial if and only if A admits a quaternion subalgebra over k which is invariant under σ. We give the proof only in the case where A is a central division algebra over k. Assume that A is a central simple algebra over k of rank 16.

Suppose that A admits a quaternion algebra H over k, which is invariant under σ. If σ is a symplectic involution, then as we noted earlier the discriminant is always trivial. Assume that σ is orthogonal. Let $x \in H$ be such that $\sigma(x) = -x$. Then the reduced norm of $x \in A$ is a square of the reduced norm of $x \in H$ ([6], p.298). Since $disc(\sigma)$ is the reduced norm of $x \in A$ modulo squares, it follows that $disc(\sigma)$ is trivial.

Assume that $disc(\sigma)$ is trivial. First assume that σ is an orthogonal involution. Let $x \in A^*$ be such that $\sigma(x) = -x$. Then $Nrd(x) = disc(\sigma)$ modulo squares in k. Since the $disc(\sigma)$ is trivial, $Nrd(x)$ is a square in k. By Lemma 2, there exists an element $y \in A$ such that $\sigma(y) = -y$ and $(x+y)^2 \in k$. Assume that σ is a symplectic involution. By the theorem of Albert, we have $A = H_1 \otimes H_2$ for some quaternion subalgebras H_1 and H_2. Let τ_1 and τ_2 be the symplectic involutions on H_1 and H_2 respectively. Then $\tau = \tau_1 \otimes \tau_2$ is an orthogonal involution on A of trivial discriminant. Therefore $\tau = int(x)\sigma$ for some $x \in A^*$ with $\sigma(x) = -x$ ([6], p. 304). Since τ is orthogonal involution of trivial discriminant and $\tau(x) = -x$, we have $Nrd(x) \in k^{*2}$. Now by Lemma 2, there exists $y \in A$ such that $\sigma(y) = -y$ and $(x+y)^2 \in k^*$. Therefore in both the cases we have obtained a non-zero element $z \in A$ such that $\sigma(z) = -z$ and $z^2 \in k$. Let $L = k(z)$. Since $\sigma(z) = -z$, the restriction of σ to L is the non-trivial automorphism of L over k. By ([3], 3.4), there is a quaternion subalgebra H of A over k such that the restriction of σ to H is the symplectic involution. This completes the proof of the theorem. □

4. Discriminants of involutions

Let D be a rank 16 central division algebra over a field k, with an involution. By the theorem mentioned in the previous section, there is an involution σ on D which does not admit an invariant quaternion subalgebra if and only if the $disc(\sigma)$ is non-trivial. Thus it is natural to ask the description of the set of discriminants of involutions on D. Note that the set of discriminants is a subset of the group of reduced norms of non-zero elements of D modulo squares in k^*. More generally let D be a central simple algebra over a field k, with an involution. The following question concerning discriminants of involutions on D was raised by many mathematicians including Knus, Lam, Rowen, Saltman, Tignol and Yanchevskii.

Question *Is the set of discriminants of involutions on D equal to the full group of reduced norms of non-zero elements of D modulo squares in k?*

Let k be a field of characteristic not equal to 2 and D a quaternion algebra over k. Let D be generated by i and j. Then the set of discriminants of involutions on D is the set of reduced norm of elements of D of the form $ai + bj + cij$ ("trace zero elements"), with $a, b, c \in k$. It is not that difficult give an example of a quaternion algebra for which the above question has a negative answer.

For division algebras of rank 16, a positive answer to the above question was given by Knus, Lam, Shapiro and Tignol [10]. Parimala, Sridharan and Suresh, proved the following [5].

Theorem. *Let k be a field of characteristic not equal to 2. Let D be a central division algebra of rank ≥ 16, with an involution. The group of reduced norms of non-zero elements of D modulo squares in k^* coincides with the set of discriminants of orthogonal involutions on D.*

Proof. We give a proof of the theorem in the rank 16 case. The general case is a little technical and we refer the reader for a proof to the original paper [5].

Let D be a central simple division algebra over k of rank 16 with an involution. Let $u \in D^*$. We have to show that the square class of the reduced norm $Nrd(u)$ of u is the discriminant of an involution on D. Let τ be a symplectic involution on D. The the dimension of S_τ^- is 10. Since k is not contained in S_τ^-, we have $dim(S_\tau^- + k) = 11$. Since τ is symplectic, $disc(\tau) = 1$ and if $u \in S_\tau^+$, then $Nrd(u) \in k^{*2}$ and by ([5], 1.3), there is an orthogonal involution of discriminant $Nrd(u)$. Assume that $u \notin S_\tau^+$. Then $(uS_\tau^+) \cap (S_\tau^- + k) \neq \{0\}$ and hence there exists $v \in S_\tau^+$, $v \neq 0$, such that $uv \in S_\tau^- + k$. Let $\lambda \in k$ and $w \in S_\tau^-$ be such that $uv = w + \lambda$. Since $u \notin S_\tau^+$, $w \neq 0$. Since $Nrd(v)$ is a square, we have $Nrd(u) = Nrd(w + \lambda)$ modulo squares in k^*. The subfields $k(w+\lambda)$ and $k(w)$

of D generated by $w + \lambda$ and w coincide. Since $\tau(w) = -w$ and $w \neq 0$, the degree of $k(w)$ over $k(w^2)$ is 2. By ([5], 1.3), there exists an orthogonal involution σ on D of trivial discriminant, which is identity on $k(w)$. Let $\sigma' = int(w + \lambda)\sigma$. Then we have $disc(\sigma') = Nrd(w + \lambda)disc(\sigma) = Nrd(u)$ modulo squares in k^*.

REFERENCES

(1) Knus, M.-A., Parimala, R., Sridharan R., A classification of rank 6 quadratic spaces via pfaffians. *J. Reine Angew. Math.*, **398**,(1989), 187-218.

(2) Knus, M.-A., Parimala, R., Sridharan R. Pfaffians, central simple algebras and similitudes. *Math. Z.*, **206**, (1991), 589-604.

(3) Knus, M.-A., Parimala, R., Sridharan R., Involutions on rank 16 central simple algebras. *J. Indian Math. Soc.*, **57**, (1991), 143-151.

(4) Knus, M.-A., Parimala, R., Sridharan R., On the discriminant of an involution. *Bull. Soc. Math. Belg.*, **43**, (1991), 89-98.

(5) Parimala, R., Sridharan R., Suresh V., A question on the discriminants of involutions of central division algebras. *Math. Ann.*, **297**, (1993), 575-580.

(6) Scharlau, W., *Quadratic and hermitian forms. Grundl.* Math. Wiss. Vol. 270, Berlin, Heidelberg, New York: Springer 1985.

(7) Albert, A.A., *Structure of algebras*, A.M.S. Coll. Pub. 24, New York, 1939.

(8) Amitsur, S.A., Rowen, L.H., Tignol, J.-P., Division algebras of degree 4 and 8 with involutions. *Isr. J. Math.* **33**, (1979), 133-148.

(9) Rowen L.H., Central simple algebras. *Isr. J. Math.* **29**, (1978), 285-301.

(10) Knus, M.-A., Lam T. Y., Sjaporo, D. B., Tignol, J.-P., *Discriminants of involutions on biquaternion algebras. K-theory and algebraic geometry: connections with quadratic forms and division algebras* (Santa Barbara, CA, 1992).