Indian J. pure appl. Math., **37**(2): 63-73, April 2006

COMBINED HARVESTING OF TWO COMPETITIVE SPECIES HAVING A RESOURCE DEPENDENT CARRYING CAPACITY

M. AGARWAL AND P. PANDEY

Department of Mathematics and Astronomy, Lucknow University, Lucknow 226 007

(*Received* 27 *October* 2004; *after final revision* 20 *January* 2006; *accepted* 1 *June* 2006)

In this paper, a non-linear mathematical model is proposed and analyzed to study the combined harvesting of a system involving two competitive species having a resource dependent carrying capacity. Bounds for existing solutions of the system and criteria for the existence and stability of positive equilibria are obtained. The dynamical behavior of the system is studied in both the local and global senses. We also obtain the optimal harvesting policy for sustainable development of the species.

Key Words : Combined Harvesting; Competitive Species; Resource Stability

1. INTRODUCTION

The study of biological system with harvesting is a subject of mathematical bioeconomics. The basic idea and techniques related to this field were first introduced by Clark [1]. Bhattacharya and Begum [2] discussed the feasible bionomic equilibrium points for a two species system. Pradhan and Chaudhuri [3] studied the dynamics of a two species fishery taking tax as a control instrument and obtained its optimal harvesting policy. They [4] also investigated exploitation of a single schooling fish species with a realistic catch rate function. Mesterton-Gibbon [5] described a technique to find the optimal harvesting policy for two competitive species when the harvest rate is proportional to the harvesting effort and either a single stock is selectively harvested or both stocks are harvested together. When a resource is present in the system, it works as an alternative source of intake and this shows a slight difference in stability. Chattopadhyay *et al.* [6] discussed a resourced-based competitive system in a three species a fishery. Dubey *et al.* [7] in their paper discussed the dynamics of a fish population dependent on a logistically growing resource with harvesting effort as a dynamic variable proportional to the net revenue of the harvester.

Keeping all of the above in mind, in the present paper, a nonlinear mathematical model on the combined harvesting policy for two competitive species having resource dependent carrying capacity is proposed and analyzed. Conditions for the existence of nonnegative equilibria and criteria for their local and global stability are obtained. Finally, a combined harvesting policy for two competitive species is discussed by using Pontryagin's maximum principle.

2. The Mathematical Model

The system of differential equations governing the biological community is as follows:

$$\begin{aligned} \frac{dR}{dt} &= R(R_0 - aR) - \frac{\alpha_1 R x}{R+\beta_1} - \frac{\alpha_2 R y}{R+\beta_2}; \\ \frac{dx}{dt} &= r_1 x \left[1 - \frac{x}{K_1(R)}\right] - q_1 E x - mxy, \\ \frac{dy}{dt} &= r_2 y \left[1 - \frac{y}{K_2(R)}\right] - q_2 E y - nxy, \end{aligned} \tag{2.1}$$

with initial conditions $R(0) > 0, x(0) >$ and $y(0) > 0$.

Here $R(t)$ denotes the concentration of the resource biomass. $x(t)$ and $y(t)$ are the concentrations of the competing species. It is assumed that the resource biomass grows logistically with the supply rate of the external resource input to the system a constant R_0 and its density reduces due to certain degradation factors present in the environment at a rate aR. Both competing species also grow logistically with their growth rate coefficients r_1, r_2 and resource dependent carrying capacities $K_1(R), K_2(R)$ respectively such that

$$K_i(0) = K_{i0} > 0, \quad K_i'(R) > 0, \quad i = 1, 2 \tag{2.2}$$

The uptake of resource biomass by the species is taken as a Holling type-II functional form to represent the feeding rate of competitive species where α_i and β_i are the rates of uptake and half saturation constants respectively for $i = 1, 2$. E is the combined effort to harvest both competitive species with catchability coefficients q_1 and q_2 respectively, m and n are the competitive coefficients of the two species.

Since when $r_1 < q_1 E$ and $r_2 < q_2 E$, $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} < 0$, then both competitive species approach to zero. So throughout in this paper, we assume

$$r_1 - q_1 E > 0 \quad \text{and} \quad r_2 - q_2 E > 0 \tag{2.3}$$

Now, we shall prove the boundedness of the system (2.1).

Lemma 1 — The set

$$\Omega = \left\{ (R, x, y) : 0 \le R \le \frac{R_0}{a}, 0 \le x \le K_1\left(\frac{R_0}{a}\right), 0 \le y \le K_2\left(\frac{R_0}{a}\right) \right\}$$

attracts all solutions initiating in the interior of the positive orthant.

PROOF : From (2.1)

$$\frac{dR}{dt} \leq R(R_0 - aR)$$

On integrating,we get

$$R(t) \leq \frac{R_0/a}{\left[1 - \left(\frac{R(0)-R_0/a}{R(0)}\right) e^{-R_0 t}\right]}.$$

When $t \to \infty$, we have

$$R(t) \leq \frac{R_0}{(a)}$$

Similarly, we can prove the other parts of the lemma.

3. Mathematical Analysis

a. Existence of Equilibrium Points

It is easy to check that there are eight non-negative equilibria namely $P_0(0,0,0)$, $P_1(R_0/a, 0, 0)$, $P_2\left(0), \left(1 - \frac{q_1 E}{r_1}\right) K_{10}, 0\right)$, $P_3\left(0, 0, \left(1 - \frac{q_2 E}{r_2}\right) K_{20}\right)$, $P_4(\bar{R}, \bar{x}, 0)$, $P_5(\bar{\bar{R}}, 0, \bar{\bar{y}})$, $P_6(0, \tilde{x}, \tilde{y})$ and $P^*(R^*, x^*, y^*)$. The existence of P_0, P_1, P_2 and P_3 are obvious and to show the existence of other equilibria is routine work. So we discuss only the existence of interior equilibrium point.

The non-trivial interior equilibrium $P^*(R^*, x^*, y^*)$ is the positive solution of following algebraic equations:

$$R_0 - aR - \frac{\alpha_1 x}{R + \beta_1} - \frac{\alpha_2 y}{R + \beta_2} = 0, \tag{3.1a}$$

$$x = \frac{K_1(R)[r_2(r_1 - q_1 E) - m(r_2 - q_2 E)K_2(R)]}{r_1 r_2 - mnK_1(R)K_2(R)} = f_1(R). \tag{3.1b}$$

and

$$y = \frac{K_2(R)[r_1(r_2 - q_2 E) - n(r_1 - q_1 E)K_1(R)]}{r_1 r_2 - mnK_1(R)K_2(R)} = f_2(R) \tag{3.1c}$$

Using (3.1b) and (3.1c) in (3.1a), we get

$$R_0 - aR - \frac{\alpha_1 f_1(R)}{R + \beta_1} - \frac{\alpha_2 f_2(R)}{R + \beta_2} = 0.$$

Taking

$$g_2(R) = R_0 - aR - \frac{\alpha_1 f_1(R)}{R + \beta_1} - \frac{\alpha_2 f_2(R)}{R + \beta_2},$$

we have

$$g_2(0) = R_0 - \frac{\alpha_1 f_1(0)}{\beta_1} - \frac{\alpha_2 f_2(0)}{\beta_2}$$

and

$$g_2(R_0/a) = -\frac{\alpha_1 f_1(R_0/a)}{R_0/a + \beta_1} - \frac{\alpha_2 f_2(R_0/a)}{R_0/a + \beta_2} < 0.$$

So, if

$$R_0 - \frac{\alpha_1 f_1(0)}{\beta_1} - \frac{\alpha_2 f_2(0)}{\beta_2} > 0$$

there exists $R^*, 0 < R^* < R_0/a$ which satisfies equation $g_2(R^*) = 0$.

For R^* to be quique, we must have

$$g_2'(R) = -a + \frac{\alpha_1 f_1(R)}{(R+\beta_1)^2} + \frac{\alpha_2 f_2(R)}{(R+\beta_2)^2} - \frac{\alpha_1 f_1'(R)}{(R+\beta_1)} - \frac{\alpha_2 f_2'(R)}{(R+\beta_2)} < 0.$$

Also for the existence of $P^*(R^*, x^*, y^*)$, one of the following inequalities must hold

$$(i) \quad r_2(r_1 - q_1 E) > m(r_2 - q_2 E) K_2(R)$$

and

$$r_1(r_2 - q_2 E) > n(r_1 - q_1 E) K_1(R).$$

$$(ii) \quad r_2(r_1 - q_1 E) < m(r_2 - q_2 E) K_2(R)$$

and

$$r_1(r_2 - q_2 E) < n(r_1 - q_1 E) K_1(R).$$

b. Stability Analysis

The local stability of each equilibrium point can be studied by computing the corresponding variational matrix. From variational matrix analysis, following points regarding the local stability of equilibria can be concluded:

1. P_0 is unstable in $R - x - y$ space.
2. P_1 is a saddle point whose stable manifold is locally stable along R-direction and has an unstable manifold in the $x - y$ plane.
3. When $K_{10} > \max\left\{\frac{R_0\beta_1 r_1}{\alpha_1(r_1 - q_1 E)}, \frac{(r_2 - q_2 E)R_1}{(r_1 - q_1 E)n}\right\}$ then P_2 is a locally asymptotically stable equilibrium. Otherwise it is a saddle point.
4. P_3 is a locally asymptotically stable equilibrium point if equilibrium level of the second species satisfies $\frac{(r_2 - q_2 E)}{r_2} K_{20} > \max\left\{\frac{R_0\beta_2}{\alpha_2}, \frac{r_1 - q_1 E}{m}\right\}$. Otherwise it is also a saddle point.
5. P_4 is a locally asymptotically stable equilibrium if $\bar{x} > \frac{r_2 - q_2 E}{n}$. Otherwise it is saddle point with its stable manifold locally in the $R - x$ plane and unstable along the y-direction.
6. When $\bar{\bar{y}} > \frac{r_1 - qE}{m}$, P_5 is locally asymptotically stable. Otherwise it is a saddle point with its stable manifold locally in the $R - y$ plane and unstable along the x-direction.
7. When $R_0 - \frac{\alpha_1 \bar{x}}{\beta_1} - \frac{\alpha_2 \bar{y}}{\beta_2} < 0$, using Gershgorin's theorem [8], we can see that P_6 is locally asymptotically stable if the following inequalities hold.

$$\frac{r_1\tilde{x}^2K_1'(0)}{K_{10}^2}+\frac{r_2\tilde{y}^2K_2'(0)}{K_{20}^2}<\frac{\alpha_1\tilde{x}}{\beta_1}+\frac{\alpha_2\tilde{y}}{\beta_2}-R_0$$

$$n\tilde{y}<\frac{r_1\tilde{x}}{K_{10}}$$

and

$$m\tilde{x}<\frac{r_2\tilde{y}}{K_{20}}.$$

The variational matrix corresponding to (R^*, x^*, y^*) is computed as

$$M^*=\begin{bmatrix} -R^*\left(a-\frac{\alpha_1x^*}{(R^*+\beta_1)^2}-\frac{\alpha_2y^*}{(R^*+\beta_2)^2}\right) & -\frac{\alpha_1R^*}{R^*+\beta_1} & -\frac{\alpha_2R^*}{R^*+\beta_2}\\ \frac{r_1{x^*}^2K_1'(R^*)}{K_1^2(R^*)} & -\frac{r_1x^*}{K_1(R^*)} & -mx^*\\ \frac{r_2{y^*}^2K_2'(R^*)}{K_2^2(R^*)} & -ny^* & -\frac{r_2y^*}{K_2(R^*)}\end{bmatrix}$$

It can be observed that if the inequalities

$$\frac{r_1{x^*}^2K_1'(R)}{K_1^2(R^*)}+\frac{r_2{y^*}^2K_2'(R^*)}{K_2^2(R^*)}<\left(a-\frac{\alpha_1x^*}{(R^*+\beta_1)^2}-\frac{\alpha_2y^*}{(R^*+\beta_2)^2}\right)R^*$$

$$\frac{\alpha_1R^*}{R^*+\beta_1}+ny^*<\frac{r_1x^*}{K_1(R^*)} \tag{3.2}$$

and

$$\frac{\alpha_2R^*}{R^*+\beta_2}+mx^*<\frac{r_2y^*}{K_2(R^*)},$$

then by Gershgorin's theorem [8], all eigenvalues of M^* have negative real parts and the interior equilibrium (R^*, x^*, y^*) is locally asymptotically stable.

Theorem **3.1** — In addition to equations (2.2), if the carrying capacities of the species satisfy

$$0<K_1'(R)<\rho_1,\quad 0<K_2'(R)<\rho_2$$

for some positive constants ρ_1, ρ_2 with inequalities.

$$\left(\frac{r_1K_1(R_0/a)\rho_1}{K_{10}^2}+\frac{\alpha_1}{\beta_1}\right)^2<\frac{r_1}{K_1(R^*)}\left(a-\frac{\alpha_1x^*}{\beta_1(R^*+\beta_1)}-\frac{\alpha_2y^*}{\beta_2(R^*+\beta_2)}\right)$$

$$\left(\frac{r_2K_2(R_0/a)\rho_2}{K_{20}^2}+\frac{\alpha_2}{\beta_2}\right)^2<\frac{r_2}{K_2(R^*)}\left(a-\frac{\alpha_1x^*}{\beta_1(R^*+\beta_1)}-\frac{\alpha_2y^*}{\beta_2(R^*+\beta_2)}\right)$$

and

$$(m+n)^2<\frac{r_1r_2}{K_1(R^*)K_2(R^*)}.$$

Then (R^*, x^*, y^*) is globally asymptotically stable with respect to all solutions initiating in the positive orthant.

PROOF: Let us consider the positive definite function about (R^*, x^*, y^*)

$$\begin{aligned} V(R, x, y) &= R - R^* - R^* \text{In}\left(\frac{R}{R^*}\right) + x - x^* \\ &-x^* \text{In}\left(\frac{x}{x^*}\right) + y - y^* - y^* \text{In}\left(\frac{y}{y^*}\right) \end{aligned}$$

After some algebraic manipulations and utilizing functions.

$$\eta_i(R) = \begin{cases} \left(\frac{1}{K_i(R)} - \frac{1}{K_i(R^*)}/(R - R^*)\right) & R \neq R^* \\ & i = 1, 2. \\ -\frac{K_i'(R^*)}{K_i^2(R^*)} & R = R^* \end{cases}$$

The derivative of V i.e. $\dot{V}$ can be written as

$$\begin{aligned} \dot{V} &= -\frac{1}{2}b_{11}(R - R^*)^2 + b_{12}(R - R^*)(x - x^*) - \frac{1}{2}b_{22}(x - x^*)^2 \\ &-\frac{1}{2}b_{11}(R - R^*)^2 + b_{13}(R - R^*)(y - y^*) - \frac{1}{2}b_{33}(y - y^*)^2 \\ &-\frac{1}{2}b_{22}(x - x^*)^2 + b_{23}(x - x^*)(y - y^*) - \frac{1}{2}b_{33}(y - y^*)^2 \end{aligned}$$

where

$$\begin{aligned} b_{11} &= a - \frac{\alpha_1 x^*}{(R + \beta_1)(R^* + \beta_1)} - \frac{\alpha_2 y^*}{(R + \beta_2)(R^* + \beta_2)}, \\ b_{22} &= \frac{r_1}{K_1(R^*)}, b_{33} = \frac{r_2}{K_2(R^*)}, b_{23} = -(m + n), \\ b_{12} &= -r_1 x \eta_1(R) - \frac{\alpha_1}{R + \beta_1}, b_{13} = -r_2 y \eta_2(R) - \frac{\alpha_2}{R + \beta_2}. \end{aligned}$$

The conditions for $\dot{V}$ to be negative definite are given by

$$b_{12}^2 < b_{11}b_{22},$$
$$b_{13}^2 < b_{11}b_{33},$$

and

$$b_{23}^2 < b_{22}b_{33}.$$

It is clear that the inequalities in the theorem imply negative definiteness of $\dot{V}$. Also the subset of Ω for $\dot{V} = 0$, contains only the equilibrium point (R^*, x^*, y^*), and hence system (2.1) is globally asymptotically stable about (R^*, x^*, y^*).

4. Optimal Harvesting Policy

In this section, we discuss the optimal harvesting policy that maximize net revenue at any time that is given by

$$\pi(R, x, y, E, t) = (p_1 q_1 x + p_2 q_2 y - c)E$$

where c is the harvesting cost per unit effort, p_1 and p_2 are the price per unit biomass of x and y respectively. Our problem is to optimize the function

$$\rho = \int_0^\infty e^{-\delta t}(p_1 q_1 x + p_2 q_2 y - c)E\, dt$$

subject to state equation (2.1) by using Pontryagin's maximum principle with δ as the instantaneous annual rate of discount.

The corresponding Hamiltonian function with $\lambda_1, \lambda_2, \lambda_3$ as adjoint variables is given by

$$\begin{aligned} H \;=\; & e^{-\delta t}(p_1 q_1 x + p_2 q_2 y - c)E + \lambda_1(t)\left[R(R_0 - aR) - \frac{\alpha_1 R x}{R+\beta_1} - \frac{\alpha_2 R y}{R+\beta_2}\right] \\ & +\lambda_2(t)\left[r_1 x\left(1 - \frac{x}{K_1(R)}\right) - q_1 E x - mxy\right] \\ & +\lambda_3(t)\left[r_2 y\left(1 - \frac{y}{K_2(R)}\right) - q_2 E y - nxy\right]. \end{aligned}$$

As per the maximum principle, we must have

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{R}, \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x}, \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial y}.$$

At the equilibrium (R^*, x^*, y^*), the above equations reduces to

$$\left\{D - \left(aR^* - \frac{\alpha_1 R^* x^*}{(R^*+\beta_1)^2} - \frac{\alpha_2 R^* y^*}{(R^*+\beta_2)^2}\right)\right\}\lambda_1$$

$$+\frac{r_1 {x^*}^2 K_1'(R^*)}{K_1^2(R^*)}\lambda_2 + \frac{r_2 {y^*}^2 K_2'(R^*)}{K_2^2(R^*)}\lambda_3 = 0,$$

$$-\frac{\alpha_1 R^*}{R^*+\beta_1}\lambda_1 + \left(D - \frac{r_1 x^*}{K_1(R^*)}\right)\lambda_2 - my^*\lambda_3 = -e^{-\delta t}p_1 q_1 E$$

and

$$-\frac{\alpha_2 R^*}{R^*+\beta_2}\lambda_1 - nx^*\lambda_2 + \left(D - \frac{r_2 y^*}{K_2(R^*)}\right)\lambda_3 = -e^{-\delta t}p_2 q_2 E$$

where $D = \frac{d}{dt}$

On eliminating λ_2 and λ_3 from the above equations, the reduced differential equation in λ_1 can be written as

$$(D^3 + c_2 D^2 + c_1 D + c_0)\lambda_1 = M_1 e^{-\delta t} \tag{4.1}$$

where

$$c_2 = -\left[aR^* - \frac{\alpha_1 R^* x^*}{(R^*+\beta_1)^2} - \frac{\alpha_2 R^* y^*}{(R^*+\beta_2)^2} + \frac{r_1 x^*}{K_1(R^*)} + \frac{r_2 y^*}{K_2(R^*)}\right],$$

$$c_1 = \left[aR^* - \frac{\alpha_1 R^* x^*}{(R^*+\beta_1)^2} - \frac{\alpha_2 R^* y^*}{(R^*+\beta_2)^2}\right]\left[\frac{r_1 x^*}{K_1(R^*)} + \frac{r_2 y^*}{K_2(R^*)}\right]$$

$$+x^*y^*\left(\frac{r_1 r_2}{K_1(R^*)K_2(R^*)} - mn\right) + \frac{R^*\alpha_1 r_1 {x^*}^2 K_1'(R^*)}{(R^*+\beta_1)K_1^2(R^*)}$$

$$+\frac{R^*\alpha_2 r_2 {y^*}^2 K_1'(R^*)}{(R^*+\beta_2)K_2^2(R^*)},$$

$$c_0 = -x^*y^*\left[aR^* - \frac{\alpha_1 R^* x^*}{(R^*+\beta_1)^2} - \frac{\alpha_2 R^* y^*}{(R^*+\beta_2)^2}\right]\left(\frac{r_1 r_2}{K_1(R^*)K_2(R^*)} - mn\right)$$

$$+\frac{R^*\alpha_2}{(R^*+\beta_2)}\frac{r_1 x^*}{K_1(R^*)}\left[\frac{nx^*y^*K_1'(R^*)}{K_1(R^*)} - \frac{r_2 {y^*}^2 K_2'(R^*)}{K_2^2(R^*)}\right]$$

$$+\frac{R^*\alpha_1}{(R^*+\beta_1)}\frac{r_2 y^*}{K_2(R^*)}\left[\frac{mx^*y^*K_2'(R^*)}{K_2(R^*)} - \frac{r_1 {x^*}^2 K_1'(R^*)}{K_1^2(R^*)}\right]$$

$$M_1 = E\left[\frac{p_2 q_2 r_2 {y^*}^2 K_2'(R^*)}{K_2^2(R^*)}\left(\delta + \frac{r_1 x^*}{K_1(R^*)}\right)\right.$$

$$+\frac{p_1 q_1 r_1 {x^*}^2 K_1'(R^*)}{K_1^2(R^*)}\left(\delta + \frac{r_2 y^*}{K_2(R^*)}\right)$$

$$\left.-x^*y^*\left\{\frac{p_2 q_2 n r_1 x^* K_1'(R^*)}{K_1^2(R^*)} + \frac{p_1 q_1 m r_2 y^* K_2'(R^*)}{K_2^2(R^*)}\right\}\right].$$

The complete solution of equation (4.1) is

$$\lambda_1(t) = A_1 e^{a_1 t} + A_2 e^{a_2 t} A_3 e^{a_3 t} + \frac{M_1}{N} e^{-\delta t}$$

where the A_i,s are arbitary constants and the a_i,s are roots of equation

$$m^3 + c_2 m^3 + c_1 m + c_0 = 0, \quad i = 1, 2, 3$$

and

$$N = -\delta^3 + c_2\delta^2 - c_1\delta + c_0 \neq 0.$$

Taking $A_i = 0$ $(i = 1,\ 2,\ 3)$, we have $(e^{\delta t})\lambda_1 = \frac{M_1}{N}$.
Similarly, we get $e^{\delta t}\lambda_2 = \frac{M_2}{N}$ and $e^{\delta t}\lambda_3 = \frac{M_3}{N}$,
where

$$M_2 = E\left[\left\{aR^* - \frac{\alpha_1 R^* x^*}{(R^*+\beta_1)^2} - \frac{\alpha_2 R^* y^*}{(R^*+\beta_2)^2} + \delta\right\}\right.$$

$$\left\{p_1 q_1 \left(\delta + \frac{r_2 y^*}{K_2(R^*)}\right) - p_2 q_2 n y^*\right\}$$

$$-\frac{r_2 R^* {y^*}^2 K_2'(R^*)}{K_2^2(R^*)} \left\{\frac{p_2 q_2 \alpha_1}{R^* + \beta_1} - \frac{p_1 q_1 \alpha_2}{R^* + \beta_2}\right\}\Bigg]$$

and

$$.M_3 = E\left[\left\{aR^* - \frac{\alpha_1 R^* x^*}{(R^* + \beta_1)^2} - \frac{\alpha_2 R^* y^*}{(R^* + \beta_2)^2} + \delta\right\}\right.$$

$$.\quad \left\{p_2 q_2 \left(\delta + \frac{r_1 x^*}{K_1(R^*)}\right) - p_1 q_1 m x^*\right\}$$

$$\left. -\frac{r_1 R^* {x^*}^2 K_2'(R^*)}{K_1^2(R^*)} \left\{\frac{p_2 q_2 \alpha_1}{R^* + \beta_1} - \frac{p_1 q_1 \alpha_2}{R^* + \beta_2}\right\}\right].$$

Again, the condition that Hamiltonian H must be maximum for $E \in [0, E_{\max}]$ is

$$\frac{\partial H}{\partial E} = e^{-\delta t}(p_1 q_1 x^* + p_2 q_2 y^* - c) - \lambda_2 q_1 x^* - \lambda_3 q_2 y^* = 0. \tag{4.2}$$

Substituting the values of λ_2 and λ_3 in (4.2), we have

$$q_1 x^* \left(p_1 - \frac{M_2}{N}\right) + q_2 y^* \left(p_2 - \frac{M_3}{N}\right) = c. \tag{4.3}$$

The above equation with the value of E at the interior equilibrium given as,

$$E = \frac{1}{q_1}\left[r_1\left(1 - \frac{x^*}{K_1(R^*)}\right) - my^*\right] = \frac{1}{q_2}\left[r_2\left(1 - \frac{y^*}{K_2(R^*)}\right) - nx^*\right] \tag{4.4}$$

which gives the optimal equilibrium level of the competitive species i.e. x_δ and y_δ. The optimal value of E can be obtained from (4.4). Here, it can be noted that optimal values of x_δ, y_δ and E are defined on R^*, the equilibrium level of the resource biomass.

From the analysis carried out in this section, we observe the following facts:

(*i*) Since $e^{\delta t}\lambda_i (i = 1, 2, 3)$ are independent of time, i.e. remain constant over the time in the optimal equilibrium, they satisfy the transversality condition.

(*ii*) From (4.2), we have

$$\begin{aligned} \lambda_2 q_1 x^* + \lambda_3 q_2 y^* &= e^{-\delta t}(p_1 q_1 x^* + p_2 q_2^* - c) \\ &= e^{-\delta t}\frac{\partial \pi}{\partial E}. \end{aligned}$$

Thus, the total user's cost of harvest per unit effort is equal to the discounted value of the future price at the steady state effort level.

(*iii*) Equation (4.3) can be rewritten as

$$\pi = (p_1 q_1 x^* + p_2 q_2 y^* - c)E = \frac{(M_2 q_1 x^* + M_3 q_2 y^*)E}{N}.$$

From the expressions for M_2, M_3 and N, we note that π is a decreasing function of δ and when $\delta \to \infty$, we have

$$p_1 q_1 x^* + p_2 q_2 y^* - c \to 0 \text{ since } \frac{M_2}{N} \text{ and } \frac{M_3}{N} \to 0.$$

Thus, the net economic revenue to the society becomes zero when the discount rate is infinite.

5. Numerical Analysis

In this section, we study the applicability of the model (2.1) by choosing the following functions and values of parameters for our numerical calculations

$$\begin{aligned}
&K_1(R) = 100 + 0.2\,R, \\
&K_2(R) = 75 + 0.2\,R, \\
&r_1 = 10, \alpha_1 = 0.2, \qquad \beta_1 = 1, \quad m = 0.02, \\
&r_2 = 9, \alpha_2 = 0.2, \qquad \beta_2 = 1, \quad n = 0.05, \\
&R_0 = 100, a = 1, \qquad q_1 = 1, \quad q_2 = 1.
\end{aligned}$$

We perform analysis in the absence and presence of resource biomass for different values of harvesting effort.

when $R = 0$,

E	$\tilde{x}$	$\tilde{y}$
0	92.7256	36.3636
2	74.5441	27.2727
4	56.3626	18.1818

when $R \neq 0$,

E	R^*	x^*	y^*
0	99.7071	111.2416	36.2667
2	99.7694	89.5893	26.5963
4	99.8110	67.9163	16.9263

It can be checked that conditions for local stability are satisfied and hence equilibrium points obtained above are locally asymptotically stable for chosen values of parameters.

By choosing $\rho_1 = \rho_2 = 0.2, K_{10} = 100$ and $K_{20} = 75$, it can be verified that conditions in theorem 3.1 are satisfied showing that P^* is globally asymptotically stable.

From the above tables, we see that the presence of resource increases the equilibrium level of first population x^* while decreases the corresponding equilibrium level of second population y^*, this is due to the high value of competitive coefficient for the second population. Also, increase in harvesting effort increases the equilibrium level of resource biomass while decreases the equilibrium level of both competitive species.

6. Discussion

In the present paper, we have discussed the combined harvesting of a system consisting of two competitive species having resource dependent carrying capacities. The existence of non-negative equilibria and the stability of each equilibrium point have been discussed. We have obtained the conditions for local and global stability of the system. The local stability of the system implies that there exists a neighbourhood of interior equilibrium point such that any solution of the model, which is inside the neighbourhood at some time $t > 0$, remains inside the neighbourhood and tends to equilibrium point as $t \to \infty$. Global stability of the system implies that all solutions initiating in the interior of the positive orthant settle down to their respective equilibrium level for larger time.

In section 4, using Pontryagin's Maximum Principle, optimal equilibrium levels of competitive species and the effort have been obtained. It has been observed that optimal level of effort also increases with an increase in the equilibrium level of resource biomass concentration. It has been noted that at the steady state, the total user's cost of harvest per unit effort must be equal to the discounted value of the future price. It has also been concluded that zero discount rate gives the maximum value of revenue and in the case of an infinite discount rate, we have zero economic rent.

Acknowledgement

The author (PP) is thankful to CSIR, New Delhi for financial support under JRF scheme.

References

1. C. W. Clark, *Mathematical Bioeconomics : The Optimal Management of Renewable Resources*, John Wiley and Sons, New York (1976).
2. D. K. Bhattacharya and S. Begum, Bionomic equilibrium of two species system, *Math. Biosci.*, **135**(2) (1996), 111-127.
3. T. Pradhan and K. S. Chaudhuri, A dynamic reaction model of a two species fishery with taxation as a control instrument: A capital theoretic analysis, *Ecol. Model.*, **121** (1996), 1-16.
4. T. Pradhan and K. S. Chaudhuri, Bioeconomic harvesting of schooling fish species : A dynamic reaction model, *Korean J. Comput. and Appl. Math.*, **6**(1) (1999) 127-141.
5. M. Mesterton - Gibbson, A technique for finding optimal two species harvesting policies, *Ecol. Model.*, **92** (1996), 235-244.
6. J. Chattopadhyay, A. Mukhopadhyaya and P. K. Tapaswai, A resource based competitive system in three species fishery, *Non-linear Studies*, **3** (1996), 73-83.
7. B. Dubey, B. Das and J. Hussain, A model for two competiting species withself and cross diffusion, *J. of Pure and Applied Math.*, **33**(6) (2002), 847-860.
8. P. L. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Second Edition, Academic Press, New York (1985), 387.