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ON EXTREME STEPS OF A RANDOM FUNCTION ON FINITE SETS*

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In this note we determine the possible limiting distributions of extreme steps of a random function on finite sets as numbers of elements of these sets tend to infinity.

Key Words: Extreme Values; Random Functions; Double Arrays; Mixing Conditions; Domains of Attraction

1. Introduction

Double arrays of random variables arise in connection with many combinatorial problems. Limiting distributions of extreme values in such double arrays may belong to the well known types of extreme value distributions, but other distributions can also appear. There are many excellent references of the well established extreme value theory for i.i.d. and stationary random sequences; see Leadbetter, Lindgren and Rootzén [6], Resnick [10], Leadbetter and Rootzén [5]. Many results of this theory can be modified in order to investigate extreme values in double arrays of random variables.

Random functions on finite sets, and particularly random permutations, have been very much studied and many asymptotic results have been obtained. For example, the number of cycles of a random permutation and the logarithm of the order of a random permutation of the set $N_n = \{1, 2, \ldots, n\}$ are asymptotically normally distributed; see for example Kolchin [3].

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2. Random Functions on Finite Sets

Let Ω_n be the set of all functions $f : \mathbf{N}_n \to \mathbf{N}_k$, where $\mathbf{N}_n = 1, 2, \ldots n$ and $k = k(n)$. Suppose that any function $f \in \Omega_n$ has probability k^{-n}. For any $f \in \Omega_n$ and $j \in \mathbf{N}_n$ let

$$X_{nj}(f) = |f(j) - f(l+1)|, \text{where} \quad f(n+1) = f(1).$$

If j is fixed, then every value $f(s)$, where $s \notin \{j, j+1\}$, can be chosen in k ways. The values $f(j)$ and $f(j+1)$ can be chosen in k ways so that $f(j) = f(j+1)$. The values $f(j)$ and $f(j+1)$ can be chosen in $2(k-l)$ ways so that $|f(j) - f(j+1)| = l$, where $l \in \{1, 2, \ldots, k-1\}$. Hence, the number of functions f for which $X_{nj}(f) = 0$ is equal to $k^{n-2}.\ k = k^{n-1}$. The number of functions f for which $X_{nj}(f) = l \in \{1, 2, \ldots, k-1\}$ is equal to $k^{n-2}.\ 2(k-l)$. Therefore, for any $j \in \mathbf{N}_n$, the marginal distribution of random variable X_{nj} is given by

$$P\{X_{nj} = O\} = \frac{1}{k}, P\{X_{nj} = l\} = \frac{2(k-l)}{k^2}, \text{where } l \in \{1, 2, \ldots, k-1\}.$$

Using the equality $X_{nj}(f) = |f(j) - f(j+1)|$, we conclude that random variables X_{ni} and X_{nj} are independent if $1 < |i-j| < n-1$. As usual, we shall say that random variables $X_{n1}, \ldots, X_{nn}$ are 1-dependent (including dependence of X_{nn} and X_{n1}). Let us denote

$$\begin{aligned} M_n &= \max\{X_{n1}, \ldots, X_{nn}\}, \\ m_n &= \min\{X_{n1}, \ldots, X_{nn}\}. \end{aligned}$$

We refer to M_n and m_n as the extreme steps of a random function f. The next two theorems give limiting distributions of the extreme steps in a random function $f : \mathbf{N}_n \to \mathbf{N}_k$, as $n \to \infty$ and $k = k(n) \to \infty$.

Theorem 1 — (a) If $n/k^2 \to 0$ as $n \to \infty$, then the following equality holds:

$$\lim_{n\to\infty} P\left\{M_n \le x \cdot \frac{k}{\sqrt{n}} + k\right\} = \begin{cases} e^{-x^2}, & \text{if } x < 0, \\ 1, & \text{if } x \ge 0. \end{cases}$$

(b) If $n/k^2 \to \alpha$ as $n \to \infty$, where $0 < \alpha < +\infty$, then the following equality holds:

$$\lim_{n\to\infty} P\{M_n \le k - 1 + x\} = \begin{cases} e^{-[x]([x]-1)\alpha}, & x < 0, \\ 1, & x \ge 0. \end{cases}$$

(c) If $n/k^2 \to \infty$ as $n \to \infty$, then it is not possible to determine sequences $a_n > 0$ and $b_n \in \mathbf{R}$, such that the limiting distribution of random variable $(M_n - b_n)/a_n$ is a non-degenerate one.

Theorem 2 — (a) If $n/k \to 0$ as $n \to \infty$, then the following equality holds:

$$\lim_{n\to\infty} P\left\{m_n \le x \cdot \frac{k}{2n}\right\} = \begin{cases} 0, & \text{if } x \le 0, \\ 1 - e^{-x}, & \text{if } x > 0. \end{cases}$$

(b) If $n/k \to \beta$ as $n \to \infty$, where $0 < \beta < +\infty$, then the limiting distribution of random variable m_n is given by

$$\lim_{n\to\infty} P\{m_n \le x\} = \begin{cases} 0, & \text{if } x < 0, \\ 1 - e^{-(2[x]+1)\beta}, & \text{if } x \ge 0. \end{cases}$$

(c) If $n/k \to \infty$ as $n \to \infty$, then it is not possible to determine sequences $a_n > 0$ and $b_n \in \mathbf{R}$, such that the limiting distribution of random variable $(m_n - b_n)/a_n$ is a non-degenerate one.

3. On Extreme Values in Double Arrays of Random Variables

As one considers extreme values of stationary random sequences and extreme values in double arrays of random variables that are stationary in each row, it can be useful to introduce the associated i.i.d. sequence and the associated double array with i.i.d members in each row. In connection with i.i.d. settings, let us recall the definition of the domain of attraction of a non-degenerate distribution function G.

Definition 1 — A distribution function F belongs to the domain of attraction for maxima of a non-degenerate distribution function G if there exist real constants $a_n > 0$ and b_n, $n \in N$, such that
$F^n(a_n x + b_n) \to G(x)$, weakly as $n \to \infty$. In that case we use notation $F \in D(G)$.

Remark 1 : A classical result of Gnedenko [1] states that only three types of distribution functions have non-empty domains of attraction for maxima. See de Haan [2] and Leadbetter, Lindgren and Rootzén [6] for details. The following Frechet, Weibull and Gumbel distribution functions determine these three types:

$$\Phi_\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ \exp(-x^{-\alpha}), & \text{if } x \ge 0. \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} \exp(-x^{-\alpha}), & \text{if } x < 0, \\ 1, & \text{if } x \ge 0. \end{cases}$$

$$\Lambda(x) = \exp(-e^{-x}), \quad -\infty < x < +\infty;$$

where $\alpha > 0$. We refer to Φ_α, Ψ_α, and Λ as the extreme value distributions. The three possible types of limiting distributions for minima in i.i.d. sequences of random variables are determined by the following distribution functions:

$$\tilde{\Phi}_\alpha(x) = \begin{cases} 1 - \exp\{-(-x)^{-\alpha}\}, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

$$\tilde{\Psi}_\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - \exp(-x^\alpha), & \text{if } x \geq 0. \end{cases}$$

$$\tilde{\Lambda}(x) = 1 - \exp(-e^x), \quad -\infty < x < +\infty;$$

where $\alpha > 0$.

Remark 2: Limiting distributions obtained in Theorem 1(a) and Theorem 2(a) are the Weibull type extreme value distributions $\Psi_2(x)$ and $\tilde{\Psi}_1(x)$ (for maxima and minima respectively). Note also that some discrete distributions appeared in Theorem 1(b) and Theorem 2(b) as limiting distributions of extreme values M_n and m_n as $n \to \infty$.

Remark 3: Let F_n be the distribution function of random variables $X_{nj}, 1 \leq j \leq n$, that were introduced in section 2. Distribution functions $F_n, n \in \mathbf{N}$, have jumps at the right end point, and consequently, none of them belongs to domains of attraction of extreme value distributions.

Definition 2 — Let $X_{n1}, X_{n2}, \ldots, X_{nk_n}, n = 1, 2, \ldots$ be a double array of random variables such that the following conditions are satisfied:

(a) For any n random variables $X_{n1}, X_{n2}, \ldots, X_{nk_n}$ are independent with the common distribution function F_n;

(b) $\lim_{n\to\infty} k_n = +\infty$.

The sequence (F_n) belongs to the domain of attraction of a non-degenerate distribution function G if there exist real constants $a_n > 0$ and b_n, $n \in \mathbf{N}$, such that

$$F_n^{kn}(a_n x + b_n) \to G(x)$$

weakly as $n \to \infty$. In that case we shall use notation $(F_n) \in \tilde{D}(G)$.

Remark 4: Examples of sequences of distribution functions (F_n), such that $(F_n) \in \tilde{D}(G)$ for extreme value distribution function G although none of distribution functions F_n belongs to $D(G)$, were given in Mladenović [8, 9]. The limiting distribution of the maximal step of a random permutation of the set $\mathbf{N}_n$ is determined in Mladenović [8]. Necessary and sufficient conditions under which a sequence of distribution functions (F_n) belongs to $\tilde{D}(\Lambda)$ are given in Mladenović [9]. The appearance of discrete distributions in Theorems 1 and 2 should not be considered as a surprise because of every non-degenerate distribution function can determine the limiting behavior of extreme values in double arrays of random variables that are independent in each row.

Example 1 — Let G be a non-degenerate distribution function, and $X_{n1}, \ldots, X_{nn}, n \in \mathbf{N}$, a double array of random variables such that the following condition is satisfied: for any n, random variables $X_{n1}, \ldots, X_{nn}$, are independent with the common distribution function

$$F_n(x) = \begin{cases} 1 + \frac{\ln G(x)}{n}, & \text{if } G(x) \geq e^{-n}, \\ 0, & \text{if } G(x) < e^{-n}. \end{cases}$$

Let $M_n = \max\{X_{n1}, \ldots, X_{nn}\}$. If $G(x) > 0$, than $F_n(x) = 1 + \frac{\ln G(x)}{n}$ for sufficiently large n, and consequently we get as $n \to \infty$:

$$\begin{aligned} P\{M_n \leq x\} = (F_n(x))^n &= \left(1 + \frac{\ln G(x)}{n}\right)^n \\ \to e^{\ln G(x)} &= G(x). \end{aligned}$$

If $G(x) = 0$, then $P\{M_n \leq x\} = (F_n(x))^n = 0$. Hence, the limiting distribution of M_n is determined by distribution function G.

Definition 3 — Let (k_n) be a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$. A double array of random variables $X_{nj}, 1 \leq j \leq k_n$, where $n \in \mathbf{N}$, is strict stationary if it is strict stationary in each row.

In the next part of this section we shall assume that $k_n = n$. In that case a double array will be called a triangular array.

Definition 4 — [Leadbetter (1974)] Let $X_{nj}, 1 \leq j \leq n, n \in \mathbf{N}$, be a strict stationary triangular array, and (u_n) a sequence of real numbers. The condition $D(u_n)$ is satisfied if for all $1 \leq j_1 < \cdots < j_k < j_{k+1} < \cdots < j_{k+1} \leq n$, where $j_{k+1} - j_k \geq l$, the following inequality holds

$$\left| P\left(\bigcap_{s=1}^{k}\{X_{nj_s} \leq u_n\}\right) \; P\left(\bigcap_{s=k+1}^{k+l}\{X_{nj_s} \leq u_n\}\right) - P\left(\bigcap_{s=1}^{k+l}\{X_{nj_s} \leq u_n\}\right)\right| \leq \alpha_{n,l_n},$$

and $\alpha_n, l_n \to 0$ as $n \to \infty$ for some $l_n = o(n)$.

Definition 5 — ([Loynes (1965)] Let $X_{nj}, 1 \leq j \leq n, n \in \mathbf{N}$, be a strict stationary triangular array, and (u_n) a sequence of real numbers. The condition $D'(u_n)$ is satisfied if

$$\lim_{r \to \infty} \limsup_{n \to \infty} n \cdot \sum_{j=2}^{[n/r]} P\{X_{n1} > u_n, X_{nj} > u_n\} = 0.$$

Remark 5: Note that conditions $D(u_n)$ and $D'(u_n)$ were originally introduced for stationary random sequences.

Definition 6 — Let $X_{nj}, 1 \leq j \leq n, n \in \mathbf{N}$, be a strict stationary triangular array of random variables and $F_n(x) = P\{X_{nj} \leq x\}$ the common marginal distribution function of random variables from the nth row. The associated triangular array of random variables that are independent in each row is the triangular array $X^*_{nj}, 1 \leq j \leq n, n \in \mathbf{N}$, such that for any n, random variables in the nth row are independent with the common distribution function $F_n(x)$.

Let $X_{nj}, 1 \leq j \leq n, n \in \mathbf{N}$, be a strict stationary triangular array of random variables, and $X^*_{nj}, 1 \leq j \leq n, n \in \mathbf{N}$, the associated triangular array of independent random variables in each row. Let us denote $M_n = \max\{X_{n1}, \dots, X_{nn}\}$ and $M^*_n = \max\{X^*_{n1}, \dots, X^*_{nn}\}$. The next theorem gives conditions under which maxima M_n and M^*_n have the same asymptotic distribution as $n \to \infty$ with the same normalizing constants.

Theorem 3 — If $X_{n1}, \dots, X_{nn}$ are stationary and m-dependent, $u_n = a_n x + b_n, a_n > 0, b_n \in R$, satisfies the usual tail normalizing condition, and for $i \neq j$, the indicator functions $1_{\{X_{ni} > u_n\}}$ and $1_{\{X_{nj} > u_n\}}$ are negatively correlated or have correlation 0, then Leadbetter conditions $D(u_n)$ and $D'(u_n)$ hold, and hence maxima have the same asymptotic distribution as if the sequence were i.i.d.

PROOF: Condition $D(u_n)$ is an obvious consequence. Denote $F_n(x) = P\{X_{nj} \leq x\}$. Suppose that $n(1 - F_n(u_n)) \sim \tau \in R$ as $n \to \infty$. For $j \in \{3, \dots, n-1\}$ we get

$$\begin{aligned} P\{X_{n1} > u_n, X_{n2} > u_n\} &\leq P\{X_{n1} > u_n, X_{nj} > u_n\} \\ &= (1 - F_n(u_n))^2 \sim \frac{\tau^2}{n^2}, \text{as } n \to \infty. \end{aligned}$$

Consequently

$$n \cdot \sum_{j=2}^{[n/r]} P\{X_{n1} > u_n, X_{nj} > u_n\} \sim \frac{\tau^2}{r},$$

as $n \to \infty$, and condition $D'(u_n)$ follows. Application of Theorem 3.5.2 from Leadbetter, Lindgren and Rootzén [6] ends the proof.

4. Proof of Theorems 1 and 2

Let $X^*_{n1}, \dots, X^*_{nn}$ be a sequence of n independent random variables which have the same distribution as random variables $X_{n1}, \dots, X_{nn}$ that were introduced in section 2, i.e.

$$P\{X^*_{nj} = 0\} = \frac{1}{k}, P\{X^*_{nj} = l\} = \frac{2(k-l)}{k^2},$$

where $l \in \{1, 2, \ldots, k-1\}$. Throughout this section we shall use the following notations: F_n – the common distribution function of random variables X_{nj} and $X^*_{nj}, j \in \mathbf{N}_n$, and

$$M^*_n = \max\{X^*_{n1}, \ldots, X^*_{nn}\},$$

$$m^*_n = \min\{X^*_{n1}, \ldots, X^*_{nn}\}.$$

It is easy to verify that for any $m \in \{1, \ldots, k-1\}$, the following equalities hold:

$$F_n(m) = \frac{2m+1}{k} - \frac{m(m+1)}{k^2},$$

$$1 - F_n(m) = \frac{k^2 - 2mk - k + m^2 + m}{k^2}.$$

Lemma 1 — (a) If $n/k^2 \to 0$ as $n \to \infty$, then the following equality holds:

$$\lim_{n\to\infty} P\left\{M^*_n \le x \cdot \frac{k}{\sqrt{n}} + k\right\} = \begin{cases} e^{-x^2}, & \text{if } x < 0, \\ 1, & \text{if } x \ge 0. \end{cases}$$

(b) If $n/k^2 \to \alpha$ as $n \to \infty$, where $0 < \alpha < +\infty$, then the limiting distribution of M^*_n is given by the following equality:

$$\lim_{n\to\infty} P\{M^*_n \le k - 1 + x\} = \begin{cases} e^{-[x]([x]-1)\alpha}, & x < 0, \\ 1, & x \ge 0. \end{cases}$$

(c) If $n/k^2 \to \infty$ as $n \to \infty$, then it is not possible to determine sequences $a_n > 0$ and $b_n \in \mathbf{R}$, such limiting distribution of random variable $(M^*_n - b_n)/a_n$ is a non-degenerate one.

Proof: (a) Let $k = k(n)$ be a sequence of positive integers such that $n/k^2 \to 0$ as $n \to \infty$. Let us denote $u_n = k + xk/\sqrt{n}$, where $x < 0$ and $r_n = u_n - [u_n]$. Then $[u_n] = u_n - r_n = k + xk/\sqrt{n} - r_n$ and consequently we get

$$n(1 - F_n(u_n)) = n(1 - F_n([u_n]))$$

$$= x^2 + \frac{n}{k^2}\left\{-\frac{2xkr_n}{\sqrt{n}} + \frac{xk}{\sqrt{n}} + r_n^2 - r_n\right\}$$

$$\to x^2, \text{as } n \to \infty.$$

Now. it is easy to obtain that $\lim_{n\to\infty} P\{M_n^* \le u_n\} = e^{-x^2}$ for $x < 0$. Obviously; $P\{M_n^* \le u_n\} = 1$ for all $n \in \mathbf{N}$ and $x \ge 0$.

(b) Let $k = k(n)$ be a sequence of positive integers such that $n/k^2 \to \alpha$, where $0 < \alpha < +\infty$. For any $x < 0$, we get

$$\begin{aligned} P\{M_n^* \le k-1+x\} &= \{F_n(k-1+[x])\}^n \\ &= \left(1 - \frac{[x]([x]-1)}{k^2}\right)^{k^2\cdot(n/k^2)} \\ &\to e^{-[x]([x]-1)\alpha}, \text{ as } n \to \infty. \end{aligned}$$

For $x \ge 0$ and every $n \in \mathbf{N}$, we get $P\{M_n^* \le k-1+x\} = 1$.

(c) If $k(n)$ is a sequence of positive integers such that $n/k^2 \to \infty$ as $n \to \infty$, then the statement of Lemma follows because of the following relations:

$$\begin{aligned} F_n(k-1) &= 1, F_n(k-2) = 1 - 2/k^2, \\ \{F_n(k-2)\}^n &= \left(1 - \frac{2}{k^2}\right)^{k^2\cdot(n/k^2)} \to 0, n \to \infty. \end{aligned}$$

Proof of Theorem 1 — (a)–(b) Using Theorem 3, it is enough to note the following: If $n/k^2 \to 0$ as $n \to \infty$ and $u_n = k + xk/\sqrt{n}$, where $x < 0$, then $1 - F_n(u_n) \sim x^2/n$ as $n \to \infty$. If $n/k^2 \to \alpha, 0 < \alpha < +\infty$, and $u_n = k-1+x$, where $x < 0$, then $1 - F_n(u_n) = [x]([x]-1)/k^2$ as $n \to \infty$.

(c) The statement is a consequence of Lemma 1 and 1-dependence. □

Lemma 2 — (a) If $n/k \to 0$ as $n \to \infty$, then the following equality holds:

$$\lim_{n\to\infty} P\left\{m_n^* \le x \cdot \frac{k}{2n}\right\} = \begin{cases} 0, & \text{if } x \le 0, \\ 1 - e^{-x}, & \text{if } x > 0. \end{cases}$$

(b) If $n/k \to \beta$ as $n \to \infty$, where $0 < \beta < +\infty$, then the limiting distribution of random variable m_n is given by

$$\lim_{n\to\infty} P\{m_n^* \le x\} = \begin{cases} 0, & \text{if } x < 0, \\ 1 - e^{-(2[x]+1)\beta}, & \text{if } x \ge 0. \end{cases}$$

(c) If $n/k \to \infty$ as $n \to \infty$, then it is not possible to determine sequences $a_n > 0$ and $b_n \in \mathbf{R}$, such that the limiting distribution of random variable $(m_n^* - b_n)/a_n$ is a non-degenerate one.

PROOF : (a) Let $k = k(n)$ be a sequence of positive integers such that $n/k \to 0$ as $n \to \infty$. For every $x > 0$ let us denote $u_n = u_n(x) = x \cdot \frac{k}{n}, [u_n] = u_n - r_n = x \cdot \frac{k}{n} - r_n$, where $0 \le r_n < 1$. Then we get

$$\begin{aligned} P\{m_n^* \le u_n\} &= P\{m_n^* \le [u_n]\} \\ &= 1 - \left(1 - \frac{2[u_n]+1}{k} + \frac{[u_n]^2 + [u_n]}{k^2}\right)^n \\ &= 1 - \left(1 - \frac{x}{n} + o\left(\frac{1}{n}\right)\right)^n \to 1 - e^{-x}, \end{aligned}$$

as $n \to \infty$. Since $P\{m_n^* \le 0\} = (1 - 1/k)^n \to 0$ as $n \to \infty$, and $P\{m_n^* \le x\} = 0$ for every $x < 0$ and $n \in \mathbf{N}$, the statement of lemma follows.

(b) Let $k = k(n)$ be a sequence of positive integers such that $n/k \to \beta$ as $n \to \infty$, where $0 < \beta < +\infty$. For any $x \ge 1$ and $k - 1 \ge x$, the following equalities hold

$$\begin{aligned} P\{m_n^* \le x\} &= 1 - (1 - F_n([x]))^n \\ &= 1 - \left(1 - \frac{2[x]+1}{k} + \frac{[x]([x]+1)}{k^2}\right)^{k(n/k)} \end{aligned}$$

and we get $P\{m_n^* \le x\} \to 1 - e^{-(2[x]+1)\beta}$, as $n \to \infty$. For $0 \le x < 1$, we obtain that

$$\begin{aligned} P\{m_n^* \le x\} &= P\{m_n^* \le 0\} = 1 - (1 - F_n(0))^n \\ &= 1 - \left(1 - \frac{1}{k}\right)^{k\cdot(n/k)} \to 1 - e^{-\beta}, \text{as } n \to \infty. \end{aligned}$$

Obviously, $P\{m_n \le x\} = 0$ for any $x < 0$.

(c) Let $k = k(n)$ be a sequence of positive integers such that $n/k \to \infty$ and $k \to \infty$ as $n \to \infty$. Then, for every $x \ge 0$ the following relations hold:

$$\begin{aligned} P\{m_n^* \le x\} &\ge P\{m_n^* = 0\} = 1 - (1 - F_n(0))^n \\ &= 1 - (1 - 1/k)^{k\cdot(n/k)} \to 1, \text{ as } n \to \infty. \end{aligned}$$

Since $P\{m_n^* \le x\} = 0$ for $x < 0$ and any $n \in \mathbf{N}$, the statement of this part of Lemma follows. □

PROOF OF THEOREM 2 — If $n/k \to 0$ as $n \to \infty$, and $u_n = xk/(2n)$ where $x > 0$, then $F_n(u_n-) \sim x/n$ as $n \to \infty$. If $n/k \to \beta, 0 < \beta < +\infty$, as $n \to \infty$, and $u_n = x$ where $x > 0$, then $F_n(u_n-) \sim C(x)/n$ as $n \to \infty$ for some constant $C(x)$. Statements (a) and (b) follow from Lemma 2, and modification of Theorem 3 for minima. The statement (c) is a consequence of Lemma 2 and 1-dependence. □

References

1. B. V. Gnedenko. Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.*. **44** (1943), 423–453.

2. L. de Haan. *On Regular Variation and its Application to the Weak Convergence of Sample Extremes.* Mathematical Centre Tracts, **32** (1970), Amsterdam.

3. V. F. Kolchin. *Random functions*, Nauka, Moscow, In Russian (1984).

4. M. R. Leadbetter, On extreme values in stationary sequences. *Z. Wahrsch. verw. Gebiete*, **28** (1974), 289–303.

5. M. R. Leadbetter and H. Rootzèn, Extremal theory for stochastic processes, *Ann. Probab.*, **16** (1988), 431–478.

6. M. R. Leadbetter, G. Lindgren and H. Rootzèn, *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, New York-Heidelberg-Berlin (1983).

7. R. M. Loynes. Extreme values in uniformly mixing stationary stochastic processes. *Ann. Math. Statist.*. **36** (1965). 993–999.

8. P. Mladenović, A note on random permutations and extreme value distributions. *Proc. Japan. Acad.*, **78** (2002), Ser. A, 157–160.

9. P. Mladenović, A generalization of the Mejzler-de Haan theorem. *Theory of Probability and its Applications*, **50**(1) (2006), 141–153.

10. S. I. Resnick. *Extremes Values, Regular Variation and Point Processes*, Springer-Verlag, New York-Berlin-Heidelberg-London-Tokyo (1983).

11. G. S. Watson, Extreme values in samples from m-dependent stationary stochastic processes. *Annn. Math. Statist.*, **25** (1954), 798–800.