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A CHARACTERIZATION OF UNITARY CONVOLUTION THROUGH ITS NON-TRIVIAL IDENTICAL EQUATION

V. SITARAMAIAH* AND M. V. SUBBARAO**[1]

**Department of Mathematics, Pondicherry Engineering College, Pondicherry* 605 014
E-mail: ramaiahpec@yahoo.co.in
***Department of Mathematical Sciences, University of Alberta, Edmonton, Canada* T6G2G1

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Nicolas and Sitaramaih [4] recently proved a very interesting result characterizing the Dirichlet convolution. They proved that the only ψ-convolution with $\mathbb{Z}^+ \times \mathbb{Z}^+$ as its domain that preserves multiplicativity and satisfies $\psi(x, y) \geq \max\{x, y\}$ for all x, y in the domain and with respect to which the ψ-extension of the Vaidyanathaswamy identity holds is the Dirichlet convolution. Here we prove an analogous result that characterizes the Unitary convolution in terms of the extended Subbarao-Gioia identical equation, for ψ-products.

Key Words : Non-trivial Identical Equation; Unitary Convolution; Lehmer's ψ-Convolution; Multiplicative Function

1. INTRODUCTION

The two most widely studied arithmetical convolutions are, perhaps the Dirichlet convoltion $(f \star g)(n)$ and the Unitary convolution $(fUg)(n)$ defined by

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d),$$

and

$$(fUg)(n) = \sum_{\substack{d|n \\ (d,n/d)=1}} f(d)g(n/d).$$

Here we are dealing with complex valued arithmetic functions f, g whose domain is $\mathbb{Z}^+$, the set of positive integers. Recall that f is multiplicative if it is not identically zero and satisfies $f(mn) =$

[1]Prof M. V. Subbarao, an Emeritus Professor of Mathematics at the University of Alberta, Edmonton, Canada, passed away on 15th February, 2006. By 4th May, 2006, he would have completed 85 years.

$f(m)f(n)$ for all relatively prime integers m, n and that the Dirichlet and Unitary inverses of f, respectively f^{-1} and f_U^{-1} are defined by

$$(f \star f^{-1})(n) = e(n) \; ; \; (fUf_U^{-1})(n) = e(n),$$

where

$$e(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases} \tag{1.1}$$

In 1930, Vaidyanathaswamy [11, 12, 13] established a remarkable identity (identical equation, to use his nomenclature) for multiplicative functions $f(n)$ that involves the Dirichlet inverse $f^{-1}(n)$, namely, for all natural numbers m and n. We have

$$f(mn) = \sum_{\substack{a|m \\ b|n}} f(m/a)f(n/b)f^{-1}(ab)G(a,b) \tag{1.2}$$

where

$$G(a,b) = \begin{cases} (-1)^{\omega(a)}, & \text{if } \gamma(a) = \gamma(b) \\ 0, & \text{otherwise} \end{cases} \tag{1.3}$$

$\omega(a)$ being the number of distinct prime factors of a and $\gamma(a)$ the product of distinct prime factors of a with $\omega(1) = 0$ and $\gamma(1) = 1$.

We now bring in the so-called ψ-convolution, a concept introduced by Lehmer [2] in the same year 1930 as Vaidyanathaswamy's identical equation.

Let $\emptyset \neq T \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ and $\psi : T \longrightarrow \mathbb{Z}^+$ be a mapping satisfying the following conditions :

$$\text{For each } n \in \mathbb{Z}^+, \psi(x,y) = n \text{ has a finite number of solutions.} \tag{1.4}$$

$$\text{If } (x,y) \in T, \text{ then } (y,x) \in T \text{ and } \psi(x,y) = \psi(y,x). \tag{1.5}$$

$$\begin{cases} \text{The Statements } \text{“}(x,y) \in T, (\psi(x,y),z) \in T\text{”} \\ \text{and “}(y,z) \in T, (x,\psi(y,z)) \in T\text{” are equivalent; if one of these} \\ \text{condition hold, we have } \psi(\psi(x,y),z) = \psi(x,\psi(y,z)). \end{cases} \tag{1.6}$$

If $f, g \in F$ where F denotes the set of arithmetic functions, then the ψ-product of f and g denoted by $f\psi g$ is defined by

$$(f\psi g)(n) = \sum_{\psi(x,y)=n} f(x)g(y), \tag{1.7}$$

for all $n \in \mathbb{Z}^+$ It is easily seen that $(F, +, \psi)$ is a commutative ring. Clearly, the Dirichlet and Unitary convolutions arise as special cases of the ψ-convolutions. Let $\psi(x,y) = xy$ for all $(x,y) \in T$. If $T = \mathbb{Z}^+ \times \mathbb{Z}^+$ then ψ in (1.7) reduces to the Dirichlet convolution. If $T = \{(x,y)\mathbb{Z}^+ \times \mathbb{Z}^+ :$

$(x, y) = 1\}$, then ψ reduces to the unitary convolution [1]. More generally, if $T = \bigcup_{n=1}^{\infty}\{(d, n/d) : d \in A(n)\}$, where A is Narkiewicz's regular convolution [3], then ψ reduces to the A-convolution. Thus the binary operation in (1.7) is more general than that of Narkiewicz's A-convolution.

In the above we used the same notation viz., (x, y) for the ordered pair and for the greatest common divisor of x and y. However this will not cause any confusion as the context would make it clear.

If ψ is a Lehmer-Narkiewicz convolution (See Nicolas and Sitaramaih [4]), the following ψ-analogue of the identical equation in (1.2) has been established (see [8] and [4]):

If $(m, n) \in T$, where T is the domain of ψ, then for any multiplicative function f,

$$f(\psi(m,n)) = \sum_{\substack{a,x,b,y \\ \psi(a,x)=m \\ \psi(b,y)=n}} f(x)f(y)f^{-1}(\psi(a,b))G(a,b), \tag{1.8}$$

f^{-1} being the inverse of f with respect to ψ (§2 contains undefined notions used in this section).

It may be mentioned here that it has been proved in [4] that if ψ is a multiplicativity preserving convolution with respect to which every multiplicative function is invertible, $\psi(x, y) \geq \max\{x, y\}$ and the identity (1.8) holds for all multiplicative functions f, then ψ is a Lehmer-Narkiewicz convolution.

If ψ is the unitary convolution that is, $T = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : (x, y) = 1\}$ and $\psi(x, y) = xy$ on T, (1.8) reduces to the unitary analogue of the identical equation in (1.2); that is, if m and n are relatively prime and f is any multiplicative function, we have

$$f(mn) = \sum_{\substack{a||m \\ b||n}} f(m/a)f(n/b)f_U^{-1}(ab)G(a,b). \tag{1.9}$$

Subbarao and Gioia [10] noted that the identity in (1.9) is trivial and not much meaningful to which it reduces in view of the simple and almost obvious relation $f_U^{-1}(m) = (-1)^{\omega(m)}f(m)$. In the above, as usual, $a||b$ stands for $a|b$ and $(a, b/a) = 1$.

Evidently, Vaidyanathaswamy did not realize this, because if he did, he would certainly have obtained a non-trivial identical equation for unitary convolution - especially because he himself created the concept of unitary convolution - though he gave it a different name compound of two functions (see [12]).

However a non-trivial identical equation in the case of unitary convolution was given by Subbarao and Gioia [10], namely, the following: For all positive integers m and n and for any multiplicative function f,

$$f(mn) = \sum_{\substack{a||m \\ b||b \\ \gamma(a)|\gamma(b)|\gamma((m,n))}} f(m/a)f(n/b)f_U^{-1}(ab)(-1)^{\omega(a)}. \tag{1.10}$$

For each prime p, let $\theta'_p(\alpha, \beta)$ be a non-negative integer-valued function defined for non-negative integers α, β and satisfying

$$\theta'_p(\alpha, \beta) = 0 \text{ if and only if } \alpha = \beta = 0, \tag{1.11}$$

and

$$\theta'_p(\alpha, 0) = \theta'_p(0, \alpha) = \alpha, \tag{1.12}$$

for every $\alpha \geq 0$. Let $H : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ be defined by

$$H(m,n) = \prod_{i=1}^{r} p_i^{\theta'_{p_i}(\alpha_i, \beta_i)}, \tag{1.13}$$

if $m = \Pi_{i=1}^r p_i^{\alpha_i}$ and $n = \Pi_{i=1}^r p_i^{\beta_i}$ where $p_i's$ are distinct primes, $\alpha_i's$ and $\beta_i's$ are non-negative integers.

Adopting the same method of proof of the identity in (1.10) (see [10], Theorem 2) the following simple generalization of (1.10) can be easily established: For all positive integers m, n and for any multiplicative function f, we have

$$f(H(m,n)) = \sum_{\substack{a||m \\ b||n \\ \gamma(a)|\gamma(b)|\gamma((m,n))}} f(m/a) f(n/b) f_U^{-1}(H(a,b))(-1)^{\omega(a)} \tag{1.14}$$

In this paper, we investigate (see §3) (imposing some mild conditions on ψ) whether a "ψ-analogue" of the non-trivial identical equation given in (1.14) holds. We prove (in Theorem 3.1) that among such ψ-convolutions the unitary convolution is the unique convolution with respect to which the ψ-analogue of the identity in (1.14) (see (3.3)) holds for all multiplicative functions. §2 contains the preliminaries and §3 contains the main result (Theorem 3.1).

2. Preliminaries

The following results (Lemmas 2.1 and 2.2) describe necessary and sufficient conditions concerning the existence of unity and inverses in $(F, +, \psi)$:

Lemma 2.1 — (See [6], Theorem 2.2). Let $(F, +, \psi)$ be a commutative ring and $\psi(x, y) \geq \max\{x, y\}$ for all $x, y \in T$. Then $(F, +, \psi)$ possesses the unity if and only if for each $k \in \mathbb{Z}^+, \psi(x, k) = k$ has a solution. In such a case if g stands for the unity, then for each $k \in \mathbb{Z}^+$,

$$g(k) = \begin{cases} 1 - \sum_{\substack{\psi(x,k)=k \\ x<k}} g(x), & \text{if } \psi(k,k) = k \\ 0 & \text{if } \psi(k,k) \neq k \end{cases}$$

Lemma 2.2 — (See [5]; also see [7], Remark 1.1). Let ψ satisfy (1.4) – (1.7) and $\psi(x, y) \geq \max\{x, y\}$ for all $x, y \in T$. For each $k \in \mathbb{Z}^+$, let the equation $\psi(x, k) = k$ have a solution so that the unity exists in $(F, +, \psi)$. Let g denote the unity. Then $f \in F$ is invertible with respect to ψ if and only if

$$S_f(k) \stackrel{def}{=} \sum_{\psi(x,k)=k} f(x) \neq 0,$$

for all $k \in \mathbb{Z}^+$. In such a case, this inverse denoted by $f^{-1}(k)$ can be computed by

$$f^{-1}(1) = \frac{1}{f(1)},$$

and for $k > 1$,

$$f^{-1}(k) = (S_f(k))^{-1}\left[g(k) - \sum_{\substack{\psi(x,y)=k \\ y<k}} f(x)f^{-1}(y)\right].$$

The binary operation ψ in (1.7) is called multiplicativity preserving (See [7]) if $f\psi g$ is multiplicative whenever f and g are.

The following results (Lemmas 2.3 and 2.5) give a characterization of multiplicativity preserving ψ-functions satisfying (i) $\psi(x,y) \geq \max\{x,y\}$ for all $x, y \in T$ and (ii) $\psi(1,k) = k$ for all $k \in \mathbb{Z}^+$:

Lemma 2.3 — (See [7], Theorem 3.1). Let ψ satisfy (1.4) – (1.7), $\psi(x,y) \geq \max\{x,y\}$ for all $x, y \in T$ and $\psi(1,k) = k$ for all $k \in \mathbb{Z}^+$. Suppose that the binary operation ψ in (1.7) is multiplicativity preserving. If $x = \Pi_{i=1}^r P_i^{\alpha_i}$ and $y = \Pi_{i=1}^r p_i^{\beta_i}$ where $p_1, p_2, \cdots, P_r$ are distinct primes, α_i and β_i are non-negative integers, we have

(a) $(x,y) \in T$ if and only if $(p_i^{\alpha_i}, p_i^{\beta_i}) \in T$ for $i = 1, 2, \ldots, r$.

(b) For each prime p and non-negative integers α, β such that $(p^\alpha, p^\beta) \in T$, there is a unique non-negative integer $\theta_p(\alpha,\beta) \geq \max\{\alpha,\beta\}$ such that $\psi(p^\alpha, p^\beta) = p^{\theta_p(\alpha,\beta)}$.

(c) If $(x,y) \in T$, then

$$\psi(x,y) = \prod_{i=1}^{r} p_i^{\theta_{p_i}(\alpha_i,\beta_i)} \tag{2.1}$$

Lemma 2.4 (*See* 7, *Theorem* 3.2). Let $T \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ be such that

(a) $(1,x) \in T$ for every $x \in \mathbb{Z}^+$.

(b) $(x,y) \in T$ if and only if $(y,x) \in T$.

(c) If x and y are as given in Lemma 2.3, then $(x,y) \in T$ if and only if $(p_i^{\alpha_i}, p_i^{\beta_i}) \in T$ for $i = 1, 2, \ldots, r$.

Further, for each prime p and non-negative integers α, β such that $(p^\alpha, p^\beta) \in T$, let $\theta_p(\alpha,\beta)$ be a non-negative integer satisfying

(d) $\theta_p(\alpha,\beta) \geq \max\{\alpha,\beta\}$

(e) $\theta_p(\alpha,\beta) = 0$ if and only if $\alpha = \beta = 0$

(f) $\theta_p(0,\alpha) = \alpha$ for every $\alpha \geq 0$

(g) $\theta_p(\alpha,\beta) = \theta_p(\beta,\alpha)$

(h) For non-negative integers α, β, γ and for any prime p, the statements "$(p^\beta, p^\gamma) \in T$, $(p^\alpha, p^{\theta_p(\beta,\gamma)}) \in T$" and "$(p^\alpha, p^\beta) \in T$ and $(p^{\theta_p(\alpha,\beta)}, p^\gamma) \in T$" are equivalent; when one of these conditions holds, we have

$$\theta_p(\alpha, \theta_p(\beta,\gamma)) = \theta_p(\theta_p(\alpha,\beta),\gamma).$$

If for $(x,y) \in T$, $\psi(x,y)$ is defined by (2.1), then $(F, +, \psi)$ is a commutative ring with unity e, where e is given by (1.1). Also, $f\psi g$ is multiplicative whenever f and g are.

Lemma 2.5 (See [4], Lemma 2.17). Let ψ be as in Lemma 2.2. Then the following statements are equivalent :

(a) Every multiplicative function is invertible with respect to ψ.

(b) For each $k \in \mathbb{Z}^+, \psi(x,k) = k$ if and only if $x = 1$.

(Thus, if (a) holds e is the unity in $(F, +, \psi)$).

Remark 2.6 : If ψ is as in Lemma 2.2 and each multiplicative function is invertible with respect to ψ, by Lemma 2.5, we have $\psi(x,k) = k$ if and only if $x = 1$, for each positive integer k. In addition to the preceding hypothesis if ψ is multiplicativity preserving, we can take ψ, T and θ_p as in Lemma 2.4. We fix a prime p and write $\theta = \theta_p$. Then, by (2.1) and Lemma 2.4, we have

$$\theta(x,y) = x \text{ if and only if } y = 0. \tag{2.2}$$

We also recall property (e) of Lemma 2.4 :

$$\theta(x,y) = 0 \text{ if and only if } x = y = 0. \tag{2.3}$$

Finally, by Lemma 2.2 for any prime power $p^t, t > 0$, we have

$$\begin{aligned} f^{-1}(p^t) &= -\sum_{\substack{\theta(x,y)=t \\ y<t}} f(p^x)f^{-1}(p^y) \\ &= -f(p^t) - \sum_{\substack{\theta(x,y)=t \\ 0<y<t}} f(p^x)f^{-1}(p^y) \end{aligned} \tag{2.4}$$

3. Main Result

In this section we prove the following :

***Theorem* 3.1** — Let ψ be as in Lemma 2.2, ψ be multiplicativity preserving and each multiplicative function be invertible with respect to ψ. Let

$$H(m,n) = \psi(m,n), \text{for all } (m,n) \in T, \tag{3.1}$$

where $H(m,n)$ is as given in (1.13) and assume that the following identical equation

$$f(H(m,n)) = \sum_{\substack{\psi(a,x)=m \\ \psi(b,y)=n \\ \gamma(a)|\gamma(b)|\gamma((m,n))}} f(x)f(y)f^{-1}(H(a,b))(-1)^{\omega(a)} \tag{3.2}$$

holds for all positive integers m, n and all multiplicative functions f, where f^{-1} is the inverse of f with respect to the convolution ψ. Then ψ is the unitary convolution.

PROOF: From the hypothesis given in the Theorem and by Lemma 2.5, we can assume that ψ is as given in Lemma 2.4. Let T and θ_p be as in Lemma 2.4. We fix a prime and write $\theta = \theta_p$. In virtue of Lemma 2.4, to prove that ψ is the unitary convolution it is enough to prove that $(p^\lambda, p^\mu) \notin T$ for any positive integers λ and μ. On the contrary, we assume that $(p^\lambda, p^\mu) \in T$, for some positive integers λ and μ and obtain a contradiction. Since $(p^\lambda, p^\mu) \in T$, if $\theta(a,x) = \lambda$ and $\theta(b,y) = \mu$, it follows from (h) of Lemma 2.4 that $(p^a, p^b) \in T$ so that $\theta(a,b)$ is well-defined. Now, taking $m = p^\lambda$ and $n = p^\mu$ in (3.2), by (3.1), (1.13) and (2.1), we obtain

$$f(p^{\theta(\lambda,\mu)}) = \sum_{\substack{\theta(a,x)=\lambda \\ \theta(b,y)=\mu \\ \gamma(p^a)|\gamma(p^b)|\gamma((p^\lambda,p^\mu))}} f(p^x)f(p^y)f^{-1}(p^{\theta(a,b)})(-1)^{\omega(p^a)} \tag{3.3}$$

If we denote the sum on the right-hand side of (3.3) by Σ, then we can write

$$\sum = \sum_{\substack{b=0 \\ a=0}} + \sum_{\substack{b>0 \\ a=0}} + \sum_{\substack{b>0 \\ a>0}} \tag{3.4}$$

From (2.4), we have

$$\sum_{\substack{b=0 \\ a=0}} = f(p^\lambda)f(p^\mu) \tag{3.5}$$

and

$$\begin{aligned} \sum_{\substack{b>0 \\ a=0}} &= f(p^\lambda) \sum_{\substack{\theta(b,y)=\mu \\ b>0}} f^{-1}(p^b)f(p^y) = f(p^\lambda)((f^{-1}\psi f)(p^\mu) - f(p^\mu)) \\ &= f(p^\lambda)(e(p^\mu) - f(p^\mu)) = -f(p^\lambda)f(p^\mu), \end{aligned} \tag{3.6}$$

since $e(p^\mu) = 0$. Substituting (3.5) and (3.6) into (3.4), we obtain

$$\sum = \sum_{\substack{b>0 \\ a>0}} = -\sum_{\substack{\theta(a,x)=\lambda \\ \theta(b,y)=\mu \\ a>0, b>0}} f(p^x)f(p^y)f^{-1}(p^{\theta(a,b)})$$

so that from (3.3),

$$f(p^{\theta(\lambda,\mu)}) = -\sum_{\substack{\theta(a,x)=\lambda \\ \theta(b,y)=\mu \\ a>0, b>0}} f(p^x)f(p^y)f^{-1}(p^{\theta(a,b)}) \tag{3.7}$$

we can assume that λ and μ are such that $(p^\lambda, p^\mu) \in T$ and $\lambda + \mu$ is least. The conditions under the sum on the right-hand side of (3.7) imply that $(p^a, p^b) \in T$ (from (h) of Lemma 2.4) ; also, by (d) of Lemma 2.4, $\theta(a,x) = \lambda$ and $\theta(b,y) = \mu$ imply that $a \leq \lambda$ and $b \leq \mu$, so that $0 < a + b \leq \lambda + \mu$. From the least property of $\lambda + \mu$, we must have $a = \lambda$ and $b = \mu$; and by (2.3), $x = y = 0$. From (3.7) it now follows at once that

$$f(p^{\theta(\lambda,\mu)}) = -f^{-1}(p^{\theta(\lambda,\mu)}),$$

or

$$f(p^{\theta(\lambda,\mu)}) + f^{-1}(p^{\theta(\lambda,\mu)}) = 0, \tag{3.8}$$

for all multiplicative functions f.

We now distinguish the following cases :

Case 1 — Let $\lambda = \mu$, and $k = \theta(\lambda, \lambda)$. We now let f be the multiplicative function defined by

$$f(p^x) = \begin{cases} 1, & \text{if } x = 0 \text{ or } \lambda \\ 0, & \text{if } x \notin \{0, \lambda\} \end{cases} \tag{3.9}$$

By (2.4) with f as in (3.9), we have

$$\begin{aligned} f^{-1}(p^\lambda) &= -f(p^\lambda) - \sum_{\substack{\theta(x,y)=\lambda \\ 0<y<\lambda}} f(p^x) f^{-1}(p^y) \\ &= -1 - \sum_{\substack{\theta(\lambda,y)=\lambda \\ 0<y<\lambda}} f^{-1}(p^y) \end{aligned} \tag{3.10}$$

By (2.2), $\theta(\lambda, y) = \lambda$ and $y > 0$ can not hold simultaneously. Hence the sum on the right-hand side of (3.10) is an empty sum so that its value is zero. Hence

$$f^{-1}(p^\lambda) = -1. \tag{3.11}$$

Now from (2.9) with f as in (3.9), we obtain

$$\begin{aligned} f^{-1}(p^k) &= -f(p^k) - \sum_{\substack{\theta(x,y)=k \\ 0<y<k}} f(p^x) f^{-1}(p^y) \\ &= -f(p^k) - f(p^\lambda) \sum_{\substack{\theta(\lambda,y)=k \\ 0<y<k}} f^{-1}(p^y) \\ &= -f(p^k) - \sum_{\substack{\theta(\lambda,y)=\lambda \\ 0<y<k}} f^{-1}(p^y) \end{aligned} \tag{3.12}$$

From (3.8)($\lambda = \mu, k = \theta(\lambda, \lambda)$), (3.12) and (3.11), we have

$$\begin{aligned} 0 = f^{-1}(p^k) + f(p^k) &= -\sum_{\substack{\theta(\lambda,y)=k \\ 0<y<k}} f^{-1}(p^y) = -f^{-1}(p^\lambda) - \sum_{\substack{\theta(\lambda,y)=k \\ 0<y<k \\ y \neq \lambda}} f^{-1}(p^y) \\ &= 1 - \sum_{\substack{\theta(\lambda,y)=k \\ 0<y<k \\ y \neq \lambda}} f^{-1}(p^y) \end{aligned}$$

so that

$$1 = \sum_{\substack{\theta(\lambda,y)=k \\ 0<y<k \\ y \neq \lambda}} f^{-1}(p^y). \tag{3.13}$$

In particular, it follows from (3.13) that $\exists_Y \neq \lambda$ with $\theta(\lambda, y) = k$ which also satisfies $f^{-1}(p^y) \neq 0$. From (3.9), $f(p^y) = 0$. From this and (2.4), we obtain,

$$0 \neq f^{-1}(p^y) = -f(p^y) - \sum_{\substack{\theta(a,b)=\lambda \\ 0<b<\lambda}} f(p^a) f^{-1}(p^b) = -\sum_{\substack{\theta(\lambda,b)=\lambda \\ 0<b<\lambda}} f^{-1}(p^b)$$

$$= \text{An empty sum} = 0,$$

a contradiction. Thus in Case 1, we have proved that $(p^\alpha, p^\alpha) \notin T$ for any positive integer α.

Case 2. Let $\lambda \neq \mu$ and $k = \theta(\lambda, \mu)$. We now let f denote the multiplicative function defined by

$$f(p^x) = \begin{cases} 1, & \text{if } x \in \{0, \lambda, \mu\} \\ 0, & \text{otherwise} \end{cases} \tag{3.14}$$

By (2.4) and (3.8) (with f as in (3.14)), we obtain

$$0 = f(p^k) + f^{-1}(p^k) = -\sum_{\substack{\theta(a,b)=k \\ 0<b<k}} f(p^a) f^{-1}(p^b) = -\sum_{\substack{\theta(\lambda,b)=k \\ 0<b<k}} f^{-1}(p^b) - \sum_{\substack{\theta(\lambda,b)=k \\ 0<b<k}} f^{-1}(p^b)$$

$$= -\sum_1 - \sum_2, \tag{3.15}$$

say. By (2.4) and (3.14), we have

$$\sum_1 = \sum_{\substack{\theta(\lambda,b)=k \\ 0<b<k}} \left(-f(p^b) - \sum_{\substack{\theta(x,y)=b \\ 0<y<b}} f(p^x) f^{-1}(p^y) \right)$$

$$= -\sum_{\substack{\theta(\lambda,b)=k \\ 0<b<k}} f(p^b) - \sum_{\substack{\theta(\lambda,b)=k \\ 0<b<k}} \sum_{\substack{\theta(\lambda,y)=b \\ 0<y<b}} f^{-1}(p^y) - \sum_{\substack{\theta(\lambda,b)=k \\ 0<b<k}} \sum_{\substack{\theta(\mu,y)=b \\ 0<y<b}} f^{-1}(p^y)$$

$$= -\sum_3 - \sum_4 - \sum_5, \tag{3.16}$$

say. We have by (3.14)

$$\sum_3 = \sum_{\substack{\theta(\lambda,b)=k \\ 0<b<k}} f(p^b) = \sum_{\substack{\theta(\lambda,b)=k \\ 0<b<k \\ b=\lambda \text{ or } \mu}} 1 = 1, \tag{3.17}$$

since by Case 1, $(p\lambda, p^\lambda) \notin T, b = \lambda$ is forbidden. If Σ_4 is non-empty, we have $\theta(\lambda, b) = k, \theta(\lambda, y) = b$ for some $0 < b < k$ and $0 < y < b$. Hence

$$k = \theta(\lambda, b) = \theta(\lambda, \theta(\lambda, y)) = \theta(\theta(\lambda, \lambda), y).$$

In particular it follows that $(p^\lambda, p^\lambda) \in T$ and this is not true by Case 1. Hence Σ_4 is an empty sum and its value is zero. Similarly, if Σ_5 is non-empty, we have $\theta(\lambda, b) = k, \theta(\mu, y) = b$ for some $0 < b < k$ and $0 < y < b$, so that

$$k = \theta(\lambda, b) = \theta(\lambda, \theta(\mu, y)) = \theta(\theta(\lambda, \mu), y) = \theta(k, y).$$

By (2.2), this is not possible since $y > 0$. Hence Σ_5 is also an empty sum and its value equals zero. Substituting $\Sigma_4 = \Sigma_5 = 0$ and $\Sigma_3 = 1$ (from (3.17)), it follows from (3.16) that $\Sigma_1 = -1$. In a similar way we can show that $\Sigma_2 = -1$. Using these values in (3.15) we obtain the absurdity $0 = 2$.

The proof of Theorem 3.1 is complete.

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