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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR A CLASS OF FOUR-POINT BOUNDARY VALUE PROBLEMS

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We are concerned with the second-order four-point boundary-value problem

$$x''(t) + h(t)f(t, x(t), x'(t)) = 0, 0 < t < 1,$$
$$x(0) = ax(\xi), x(1) = bx(\eta),$$

where $0 < \xi < \eta < 1, 0 \leq a < 1/(1-\xi), 0 \leq b < 1/\eta$ and $a\xi(1-b) + (1-a)(1-b\eta) > 0$; $h : [0,1] \to [0,\infty), f : [0,1] \times [0,\infty) \times (-\infty,+\infty) \to [0,\infty)$ are nonnegative continuous functions. Sufficient conditions are obtained that guarantee the existence of triple positive solutions by a simple application of a new fixed-point theorem due to Avery and Peterson.

Key Words: Positive Solutions; Four-Point Boundary Value Problem; Fixed-Point Theorem

1. INTRODUCTION

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [8, 9]. Then Gupta [7] studied three-point boundary value problems for nonlinear ordinary differential equations. Since then, the more general nonlinear multi-point boundary value problems have been studied by many authors by using the fixed-point theorems on cone, the Leray-Schauder Continuation Theorem, Nonlinear Alternatives of Leray-Schauder, and coincidence degree theory. To indicate a few, we refer the reader to [5, 6, 7, 11, 12, 13, 14, 15, 16] for some recent results of nonlinear multi-point boundary value problems.

Recently, Ma [14, 15] showed the existence and multiplicity of positive solutions for some three-point, m-point nonlinear ordinary differential equation boundary value problems. Very recently, Bai and Ge [3] considered the existence of positive solutions of the second-order four-point boundary value problem

$$x''(t) + q(t)f(t, x(t)) = 0, 0 < t < 1, \tag{1.1}$$

$$x(0) = ax(\xi), x(1) = bx(\eta), \tag{1.2}$$

However, to my best knowledge, most of literature about multi-point boundary value problems didn't argue the multiplicity of positive solutions with dependence on derivatives.

An interest in triple solutions evolved from the Leggett-Williams multiple fixed-point theorem [10]. And lately, two triple fixed-point theorems due to Avery [1] and Avery and Peterson [2] have been applied to obtain triple solutions of certain boundary value problems for ordinary differential equations as well as for their discrete analogues.

In [2], Avery and Peterson generalize the fixed-point theorem of Leggett-Williams by using theory of fixed-point index and Dugundji extension theorem. In [4], Bai *et al.* give an application of the theorem to prove the existence of three positive solutions to some second-order two-point boundary value problems.

Motivated by all the above works, in this paper, we are concentrate in getting three positive solutions for the following second-order four-point boundary value problem

$$x''(t) + h(t)f(t, x(t), x'(t)) = 0, 0 < t < 1, \tag{1.3}$$

$$x(0) = ax(\xi), x(1) = bx(\eta). \tag{1.4}$$

Throughout, it is assumed that:

(C1) $f \in C([0,1] \times [0,\infty) \times (-\infty, +\infty), [0,\infty))$;

(C2) $0 < \xi < \eta < 1, 0 \leq a < 1/(1-\xi), 0 \leq b < 1/\eta$ and $a\xi(1-b) + (1-a)(1-b\eta) > 0$;

(C3) $h(t)$ is a nonnegative continuous function defined in $(0,1)$, and $h(t)$ does not identically vanish on any subinterval of $(0,1)$. Furthermore, $\int_0^1 h(t)dt < \infty$.

Our main results will depend on an application of a fixed-point theorem due to Avery and Peterson which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis here is the nonlinear term be involved explicitly with the first-order derivative.

2. Background Materials and Definitions

For the convenience of the reader, we present here the necessary definitions from cone theory in Banach spaces; these definitions can be found in the recent literature.

Definition 2.1 — Let E be a real Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(i) $au \in P$ for all $u \in P$ and all $a \geq 0$ and

(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y, \text{ if and only if } y - x \in P.$$

Definition 2.2 — An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3 — The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers $a, b, c.$ and d, we define the following convex sets:

$$\begin{aligned} P(\gamma, d) &= \{x \in P | \gamma(x) < d\} \\ P(\gamma, \alpha, b, d) &= \{x \in P | b \leq \alpha(x), \gamma(x) \leq d\} \\ P(\gamma, \theta, \alpha, b, c, d) &= \{x \in P | b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\} \end{aligned}$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P | a \leq \psi(x), \gamma(x) \leq d\}.$$

The following fixed-point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

***Theorem* 2.1** — [2] Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d,

$$\alpha(x) \leq \psi(x) \text{ and } ||x|| \leq M\gamma(x), \tag{2.1}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose

$$T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$$

is completely continuous and there exist positive numbers $a, b,$ and c with $a < b$ such that

(S1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) | a(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;

(S2) $\alpha(Tx) > b$ for $x \in P(\gamma, a, b, d)$ with $\theta(Tx) > c$;
(S3) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.
Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$, such that

$$\begin{aligned} \gamma(x_i) \le d, &\quad \text{for} \quad i = 1, 2, 3; \\ b < &\ \alpha(x_1); \\ a < \psi(x_2), &\quad \text{with} \quad \alpha(x_2) < b; \end{aligned}$$

and

$$\psi(x_3) < a.$$

3. Existence of Triple Positive Solutions

In this section, we impose growth conditions on f which allow us to apply Theorem 2.1 to establish the existence of triple positive solutions of Problem (1.3)–(1.4).

Let $X = C^1[0, 1]$ be endowed with the ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in [0, 1]$, and the maximum norm, $||x|| = \max\left\{\max_{0\le t\le 1} |x(t)|, \max_{0\le t\le 1} x'(t)\right\}$. From the fact $x''(t) = -h(t)f(t, x(t), x'(t)) \le 0$, we know that x is concave on [0, 1]. So, define the cone $P \subset X$ by

$$P = \{x \in X | x(t) \ge 0, x(0) = ax(\xi), x(1) = bx(\eta), x \text{ is concave on } [0, 1]\}.$$

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals θ, γ, and the nonnegative continuous functional ψ be denned on the cone P by

$$\gamma(x) = \max_{0\le t\le 1} |x'(t)|, \psi(x) = \theta(x) = \max_{0\le t\le 1} |x(t)|, \alpha(x) = \max_{\xi\le t\le \eta} |x(t)|. \tag{3.1}$$

Lemma 3.1 — If $x \in P$, then

$$\max_{0\le t\le 1} |x(t)| \le M \max_{0\le t\le 1} |x'(t)|,$$

where

$$M = \begin{cases} 1 + \frac{a\xi}{1-a}, & \text{if } a < 1, b \ge 1; \\ 1 + \frac{b(1-\eta)}{1-b}, & \text{if } b < 1, a \ge 1; \\ \min\left\{1 + \frac{a\xi}{1-a}, 1 + \frac{b(1-\eta)}{1-b}\right\} & \text{if } a < 1, b < 1; \end{cases} \tag{3.2}$$

PROOF: Firstly, we note that under the condition (C2), there are three cases:
Case 1 — $0 \le a < 1, 1 \le b < 1/\eta$;
Case 2 — $0 \le b < 1, 1 \le a < 1/(1 - \xi)$;
Case 3 — $0 \le a < 1, 0 \le b < 1$.
We only consider Case 1. the others are similar. By the concavity of x, there is

$$x(t) - x(0) \le tx'(0) \le \max_{0\le t\le 1} |x'(t)|, t \in [0, 1]. \tag{3.3}$$

If $a = 0$, then $x(0) = 0$. Clearly, we can choose $M = 1$. If $0 < a < 1$, by (3.3), there is

$$x'(0)\xi \geq x(\xi) - x(0) = \left(\frac{1}{a} - 1\right) x(0), \tag{3.4}$$

i.e.,

$$x(0) \leq \frac{a\xi}{1-a} x'(0) \leq \frac{a\xi}{1-a} \max_{0 \leq t \leq 1} |x'(t)|.$$

Let $M = 1 + \frac{a\xi}{1-a}$, the proof is complete. □

Denote $G(t, s)$ is Green's function for boundary value problem

$$\begin{aligned} &-x''(t) = 0, \quad 0 < t < 1, \\ &x(0) = ax(\xi), x(1) = bx(\eta), \end{aligned}$$

then $G(t, s)$ is given by

$$G(t,s) = \begin{cases} s \in [0,\xi]: & \begin{cases} \frac{s}{\delta}[(1-b\eta) + (b-1)t], & s \leq t; \\ \frac{t}{\delta}[(1-b\eta) + (b-1)s] + \frac{a(1-\xi+b\xi-b\eta)}{\delta}, (s-t) & t \leq s; \end{cases} \\ s \in [\xi,\eta]: & \begin{cases} \frac{1}{\delta}[(1-b\eta) + (b-1)t](a\xi - as + s) & s \leq t; \\ \frac{1}{\delta}[(1-b\eta) + (b-1)s](a\xi - at + t) & t \leq s; \end{cases} \\ s \in [\eta,1]: & \begin{cases} \frac{1-s}{\delta}(t - at + a\xi) + (s-t), & s \leq t; \\ \frac{1-s}{\delta}(a\xi - at + t), & t \leq s. \end{cases} \end{cases}$$

Lemma 3.2 — [3]Suppose $0 \leq a < 1/(1-\xi), 0 \leq b < 1/\eta$, and $a\xi(1-b) + (1-a)(1-b\eta) > 0$, the Green's function $G(t, s)$ satisfy

$$0 \leq G(t,s) \leq G(\tau(s), s), 0 \leq s \leq 1, \tag{3.5}$$

$$G(t,s) \geq kG(\tau(s), s), \text{ for } \xi \leq t \leq s \leq 1, \tag{3.6}$$

where τ is defined such that

$$G(\tau(s), s) = \max_{0 \leq t \leq 1} G(t,s) \text{ for } 0 \leq s \leq 1$$

and

$$k = \min\left\{\frac{1-\eta}{1-b\eta}, \frac{a\xi + (1+a)\eta}{a\xi}, \frac{1-b\eta}{b(1-\eta)}, \frac{\xi}{1-a+a\xi}\right\}.$$

By Lemma 3.1, 3.2, the following conclusion is clear.

Lemma 3.3 — If $x \in P \cap C^2(0,1)$, then

$$\min_{\xi \le t \le \eta} |x(t)| \ge k \max_{0 \le t \le 1} |x(t)|.$$

Combining with the definitions of the functional, we have

$$k\theta(x) \le \alpha(x) \le \theta(x) = \psi(x), ||x|| = \max\{\theta(x), \gamma(x)\} \le M\gamma(x), \tag{3.7}$$

for all $x \in \overline{P(\gamma, d)} \cap C^2(0,1) \subset P$, and the condition (2.1) is satisfied.

Let

$$\begin{aligned} \delta &= \min\left\{\int_\xi^\eta G(\xi,s)h(s)ds, \int_\xi^\eta G(\eta,s)h(s)ds\right\}, \\ \overline{M} &= \max\left\{\int_0^1 \left|\frac{\partial G(t,s)}{\partial t}|_{t=0}\right| h(s)ds, \int_0^1 \left|\frac{\partial G(t,s)}{\partial t}|_{t=1}\right| h(s)ds\right\}, \\ \overline{N} &= \max_{0 \le t \le 1} \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

We now present our main result of the paper.

***Theorem* 3.1** — Suppose there exist constants $0 < a < b \le Mkd$ such that the following assumptions hold

(A1) $f(t,u,v) \le \frac{d}{M}$, for $(t,u,v) \in [0,1] \times [0, Md] \times [-d, d]$;

(A2) $f(t,u,v) > \frac{b}{\delta}$, for$(t,u,v) \in [\xi,\eta] \times [b, b/k] \times [-d,d]$;

(A3) $f(t,u,v) < \frac{a}{N}$ for $(t,u,v) \in [0,1] \times [0,a] \times [-d,d]$.

Then, the boundary value problem (1.3)–(1.4) has at least three positive solutions x_1, x_2, and x_3 satisfying

$$\begin{aligned} \max_{0 \le t \le 1} |x_i'(t)| &\le d, \text{ for } i = 1,2,3; \\ b < \min_{\xi \le t \le \eta} |x_1(t)| &\le \max_{0 \le t \le 1} |x_1(t)| \le M\ d; \\ a < \max_{0 \le t \le 1} |x_2(t)| &\le \frac{b}{k}, \text{ with } \min_{\xi \le t \le \eta} |x_2(t)| < b; \end{aligned}$$

and

$$\max_{0 \le t \le 1} |x_3(t)| < a.$$

PROOF : Problem (1.3)–(1.4) has a solution $x = x(t)$ if and only if x solves the operator equation

$$x(t) = Tx(t) := \int_0^1 G(t,s)h(s)f(s, x(s), x'(s))ds.$$

With Lemma 3.2, it is well know that $T : P \to P$ is completely continuous.

We now show that all the conditions of Theorem 2.1 are satisfied.

If $x \in \overline{P(\gamma, d)}$, then there is $\gamma(x) = \max_{0 \le t \le 1} |x'(t)| \le d$. With Lemma 3.1, $\max_{0 \le t \le 1} |x(t)| \le Md$ then assumption (A1) implies $f(t, x(t), x'(t)) \le \frac{d}{\overline{M}}$. On the other hand, for $x \in P$, there is $Tx \in P$, then Tx is concave on [0, 1], and $\max_{t \in [0,1]} |(Tx)'(t)| = \max\{|(Tx)'(0)|, |(Tx)'(1)|\}$, so

$$\begin{aligned} \gamma(Tx) &= \max_{t \in [0,1]} |(Tx)'(t)| \\ &= \max\{|(Tx)'(0)|, |(Tx)'(1)|\} \\ &\le \frac{d}{\overline{M}}.\overline{M} = d. \end{aligned}$$

Hence, $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

To check condition $(S1)$ of Theorem 2.1, we choose $x(t) = b/k, 0 \le t \le 1$. It is easy to see that $x(t) = b/k \in P(\gamma, \theta, \alpha, b, b/k, d)$ and $\alpha(x) = \alpha(b/k) > b$, and so $\{x \in P(\gamma, \theta, \alpha, b, b/k, d) | \alpha(x) > b\} \neq \emptyset$. Hence, if $x \in P(\gamma, \theta, \alpha, b, b/k, d)$, then $b \le x(t) \le b/k, |x'(t)| \le d$ for $\xi \le t \le \eta$. From assumption (A2), we have $f(t, x(t), x'(t)) > \frac{b}{\delta}$ for $\xi \le t \le \eta$, and by the conditions of α and the cone P, we have to distinguish two cases, (i) $\alpha(Tx) = (Tx)(\xi)$ and (ii) $\alpha(Tx) = (Tx)(\eta)$.

In case (i), we have

$$\begin{aligned} \alpha(Tx) &= (Tx)(\xi) \\ &= \int_0^1 G(\xi, s)h(s)f(s, x(s), x'(s))ds \\ &> \frac{b}{\delta} \cdot \int_\xi^\eta G(\xi, s)h(s)ds \\ &\ge b. \end{aligned}$$

In case (ii), we have

$$\begin{aligned} \alpha(Tx) &= (Tx)(\eta) \\ &= \int_0^1 G(\eta, s)h(s)f(s, x(s), x'(s))ds \\ &> \frac{b}{\delta} \cdot \int_\xi^\eta G(\eta, s)h(s)ds \\ &\ge b; \end{aligned}$$

i.e.,

$$\alpha(Tx) > b, \text{ for all } x \in P(\gamma, \theta, \alpha, b, b/N, d).$$

This shows that condition (S1) of Theorem 2.1 is satisfied.

Secondly, by (3.7) and $Tx \in P$, we have

$$\alpha(Tx) \ge k\theta(Tx) > k \cdot \frac{b}{k} = b, \text{ for all } x \in P(\gamma, \alpha, b, d) \text{ with } \theta(Tx) > \frac{b}{k}.$$

Thus, condition (S2) of Theorem 2.1 is satisfied.

We finally show that (S3) of Theorem 2.1 also holds. Clearly, as $\psi(0) = 0 < a$, there holds $0 \notin R(\gamma, \psi, a, d)$. Suppose that $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$. Then, by the assumption (A3),

$$\begin{aligned}\psi(Tx) &= \max_{0\le t\le 1} |(Tx)(t)| \\ &= \max_{0\le t\le 1} \int_0^1 G(t,s)h(s)f(s,x(s),x'(s))ds \\ &< \frac{a}{N}\cdot \max_{0\le t\le 1} \int_0^1 G(t,s)h(s)ds = a.\end{aligned}$$

So, the condition (S3) of Theorem 2.1 is satisfied. Therefore, an application of Theorem 2.1 implies the boundary value problem (1.3)–(1.4) has at least three positive solutions x_1, x_2, and x_3 satisfying

$$\begin{aligned}\max_{0\le t\le 1} |x_i'(t)| &\le d, \text{ for } i = 1,2,3 \\ b &< \min_{\xi\le t\le \eta} |x_1(t)|;\end{aligned}$$

$$a < \min_{0\le t\le 1} |x_2(t)|, \text{ with } \min_{\xi\le t\le \eta} |x_2(t)| < b;$$

and

$$\max_{0\le t\le 1} |x_3(t)| < a.$$

Combining with (3.5) and Lemma 3.1, we have

$$a < \max_{0\le t\le 1} |x_2(t)| \le \frac{b}{k}, \tag{3.8}$$

$$b < \min_{\xi\le t\le \eta} |x_1(t)| \le \max_{0\le t\le 1} |x_1(t)| \le Md. \tag{3.9}$$

The proof is complete. □

Remark 3.1 : To apply Theorem 2.1, it is only required that $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$. Therefore, condition (C1) can be substituted with a weaker condition

$(C1)' f \in C([0,1] \times [0, Md] \times [-d, d], [0, \infty))$.

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