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ON THE COASSOCIATED PRIMES OF GENERALIZED LOCAL COHOMOLOGY MODULES*[1]

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Let $\mathfrak{a}$ denote an ideal of a complete Noetherian local ring $(R, \mathfrak{m})$ and M and N two R-modules. For a positive integer t, we show that

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Hom}_R(H^t_{\mathfrak{a}}(M, N), E_R(R/\mathfrak{m})))$$

is finitely generated whenever M is finitely generated of finite projective dimension and

(a) $H^j_{\mathfrak{a}}(M, N)$ is Artinian for all $j > t$; and,

(b) $t > Pd_R(M)$ where $Pd_R(M)$ is the projective dimension of M.

Hence, the set $\mathrm{Coass}H^t_{\mathfrak{a}}(M, N) \cap V(\mathfrak{a})$ is finite where $V(\mathfrak{a})$ denotes the set of all prime ideals of R containing $\mathfrak{a}$. This implies that if d = dim $N > 1$ and N is finitely generated then the set $\mathrm{Coass}H^{d-1}_{\mathfrak{a}}(N) \cap V(\mathfrak{a})$ is finite.

Key Words: Local Cohomology Modules; Generalized Local Cohomology Modules; Associated Primes; Co-Associated Prime

1. INTRODUCTION

Suppose that R is a Noetherian ring, $\mathfrak{a}$ is an ideal of R and N is an R-module. The ith local cohomology module of N with respect to $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(N) = \varinjlim_n \mathrm{Ext}^i_R(R/\mathfrak{a}^n, N).$$

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The reader can refer to [4], for the basic properties of local cohomology modules.

In [11], Huneke asked whether the number of associated prime ideals of a local cohomology module $H^i_{\mathfrak{a}}(R)$ is always finite. If R is regular local containing a field then $H^i_{\mathfrak{a}}(R)$ has only finitely many associated primes for all $i \geq 0$, cf. [12] (in the case of positive characteristic), [17] (in characteristic zero) and [18] (in characteristic free). In [22], Singh has given an example of Noetherian non-local ring R and an ideal $\mathfrak{a}$ such that $H^3_{\mathfrak{a}}(R)$ has infinitely many associated primes. More recently, in [13], Katzman constructed a hypersurface S and an ideal $\mathfrak{a}$ such that $H^2_{\mathfrak{a}}(S)$ has infinitely many associated primes (see also [23]).

On the other hand, Brodmann and Lashgari [3] and the present author with Salarian [15] have shown that the first non finitely generated local cohomology module $H^i_{\mathfrak{a}}(N)$ of finitely generated module N with respect to an ideal $\mathfrak{a}$ has only finitely many associated primes. For some other work on this question, we refer the reader to [20], [21], [8] and [7].

There have been four attempts to dualize the theory of associated prime ideals by Macdonald [19], Chambless [5], Zöschinger [26] and Yassemi [25]. In [25], it is shown that, in the case the ring R is Noetherian, these definitions are equivalent. Let $(R, \mathfrak{m})$ be a local ring and $E = E_R(R/\mathfrak{m})$, the injective hull of $R/\mathfrak{m}$. Following [25], we define a prime $\mathfrak{p}$ to be a coassociated prime of N if $\mathfrak{p}$ is an associated prime of $D(N)$, where $D(\ )$ is the Matlis' dual functor $\mathrm{Hom}_R(\ , E)$. Hence, the natural question concerning with local cohomology theory is "when the set of Coassociated primes of local cohomology module $H^i_{\mathfrak{a}}(N)$ is finite" (cf. [6, Lemma 3], [9] and [14, 2.6]). Delfino and Marley, in [6, Lemma 3], showed that, if $(R, \mathfrak{m})$ is a complete Noetherian local ring, $\mathfrak{a}$ an ideal of R and N a finitely generated R-module of dimension d, then

$$\mathrm{Coass} H^d_{\mathfrak{a}}(N) = \{\mathfrak{p} \in V(\mathrm{Ann} N) | \dim R/\mathfrak{p} = d \text{ and } \sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}\}.$$

In this paper, we show that if R is a complete Noetherian local ring and N has positive Krull dimension such that $H^j_{\mathfrak{a}}(N)$ is Artinian for all $j > t$ and for some positive integer t, then $\mathrm{Hom}_R(R/\mathfrak{a}, D(H^t_{\mathfrak{a}}(N)))$ is finitely generated and so $\mathrm{Coass} H^t\mathfrak{a}(N) \cap V(\mathfrak{a})$ is finite. This result "in some sense" is dual of the main result of [3] and [15] which we mentioned above.

A generalization of local cohomology functors has been given by Herzog in [10] (see also [24] and [2]). For each $i \geq 0$, the functor $H^i_{\mathfrak{a}}(\ ,\ )$ defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N),$$

for all R-modules M and N. Clearly, this notion is a generalization of usual local cohomology functor. There are not many results concerning the finiteness of associated primes of this generalized local cohomology modules (cf. [1] and [16]). In this paper, we prove the following theorem.

Theorem 1.1—Suppose that $\mathfrak{a}$ is an ideal of a complete Noetherian local ring R, M a non-zero finitely generated R-module of finite projective dimension, and N an R-module. Also, suppose that t is a positive integer such that

(a) $H^j_{\mathfrak{a}}(M, N)$ is Artinian for all $j > t$; and,

(b) $t > \mathrm{Pd}_R(M)$ where $\mathrm{Pd}_R(M)$ is the projective dimension of M.
Then $\mathrm{Hom}_R(R/\mathfrak{a}, D(H^t_{\mathfrak{a}}(M,N)))$ is finitely generated and so, the set

$$\mathrm{Coass} H^t_{\mathfrak{a}}(M,N) \cup V(\mathfrak{a})$$

is finite where $V(\mathfrak{a})$ denotes the set of all prime ideals of R containing $\mathfrak{a}$.

Throughout this paper, all rings are non trivial commutative rings unless an additional condition is considered. For an R-module $X, \mathrm{Pd}_R(X)$ stands for projective dimension of X. We use $\mathbb{N}$ (respectively $\mathbb{N}_0$) to denote the set of positive (non-negative) integers. All other notations are standard.

2. Coassociated Primes of Generalized Local Cohomology Modules

We now briefly recall some basic properties of generalized local cohomology modules.

(1) Let M and N be finitely generated R-modules, $\mathfrak{a}$ an ideal of R and $I^\bullet_N$ be an injective resolution of N. According to [24] one has

$$H^i_{\mathfrak{a}}(M,N) \cong H^i(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M, I^{\mathfrak{a}}_N))) \cong H^i(\mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(I^\bullet_N))) \text{ for all } i.$$

(2) Let $a_1, \ldots, a_n$ be a generating set of $\mathfrak{a}$, and let $K^t_\bullet$ denote the Koszul complex of R with respect to $a^t_1, \ldots, a^t_n$,. Let $P_\bullet$ be a projective resolution for M. If $C^t_\bullet$ denotes the total complex associated to the double complex $K^t_\bullet \otimes_R P_\bullet$, then by [10, Satz 1.1.6] we have

$$H^i_{\mathfrak{a}}(M,N) \cong \varinjlim_{t\mathbb{N}_0} \mathrm{Hom}_R(C^t_\bullet N)) \text{ for all } i.$$

(3) From the definition of generalized local cohomology and (2) it follows easily that for any exact sequence $0 \longrightarrow W \longrightarrow X \longrightarrow Y \longrightarrow 0$, and any finitely generated R-modules N and M we have the following long exact sequences

$$0 \longrightarrow H^0_{\mathfrak{a}}(M,W) \longrightarrow H^0_{\mathfrak{a}}(M,X) \longrightarrow H^0_{\mathfrak{a}}(M,Y) \longrightarrow H^1_{\mathfrak{a}}(M,W) \longrightarrow \ldots,$$

and

$$0 \longrightarrow H^0_{\mathfrak{a}}(Y,N) \longrightarrow H^0_{\mathfrak{a}}(W,N) \longrightarrow H^0_{\mathfrak{a}}(W,N) \longrightarrow H^1_{\mathfrak{a}}(Y,N) \longrightarrow \ldots.$$

Definition 2.1—(See [25].) Let $(R,\mathfrak{m})$ be a local ring, M an R-module and $E = E_R(R/\mathfrak{m})$, the injective hull of $R/\mathfrak{m}$. We define a prime ideal $\mathfrak{p}$ of R to be a coassociated prime of M if $\mathfrak{p}$ is an associated prime of $D(M)$ where $D(\ )$ is the Matlis' dual functor $\mathrm{Hom}_R(\ , E)$. We denote the set of coassociated prime of M by $\mathrm{Coass}_R M$(or simply $\mathrm{Coass}_R M$ if there is no ambiguity about the under ring).

Note that $\mathrm{Coass} M = \phi$ if and only if $M = 0$.

Now, we prove the main result of this paper which is a dual of the main result of [3] "in some sense".

Theorem 2.2—Suppose that $\mathfrak{a}$ is an ideal of a complete Noetherian local ring R, M a finitely generated R-module of finite projective dimension, and N an R-module. Also, suppose that t is a positive integer such that

(a) $H^i_{\mathfrak{a}}(M, N)$ is Artinian for all $j > t$; and,

(b) $t > \mathrm{Pd}_R(M)$.

Then $\mathrm{Hom}_R(R/\mathfrak{a}, D(H^t_{\mathfrak{a}}(M, N)))$ is finitely generated and so, the set

$$\mathrm{Coass} H^t_{\mathfrak{a}}(M, N) \cap V(\mathfrak{a})$$

is finite.

PROOF: Set $d := \dim N$. We use induction on d. In the case $d = 0$, by [2, 5.2], $H^i_{\mathfrak{a}}(M, N) = 0$ for all $i > \mathrm{Pd}_R(M)$. If M is projective, then $H^t_{\mathfrak{a}}M, N = 0$, because $t > 0$. Now suppose, inductively, that $\mathrm{Pd}_R(M) > 0$ and consider the exact sequence $0 \longrightarrow M' \longrightarrow P \longrightarrow M \longrightarrow 0$ in which P is projective and $\mathrm{Pd}_R(M') = \mathrm{Pd}_R(M) - 1$. Thus, by [2, 5.2], we get $H^i_{\mathfrak{a}}(M', N) \cong H^{i+1}_{\mathfrak{a}}(M, N)$ for all $i \in \mathfrak{a}$. Now, by inductive hypothesis, $\mathrm{Hom}_R(R/\mathfrak{a}, D(H^{t-1}_{\mathfrak{a}}(M', N)))$ is finitely generated and so, $\mathrm{Hom}_R(R/\mathfrak{a}, D(H^t_{\mathfrak{a}}(M, N)))$ is finitely generated too. This complete our inductive step in the case $d = 0$.

Suppose, inductively, that $d \geq 1$ and the assertion is true for every finitely generated R-module with Krull dimension less than d and N is a finitely generated R- module of Krull dimension d. Set $L := N/\Gamma_{\mathfrak{a}}(N)$ and consider the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow L \longrightarrow 0$$

to deduce the exact sequence

$$H^i_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)) \longrightarrow H^i_{\mathfrak{a}}(M, N) \longrightarrow H^i_{\mathfrak{a}}(M, L) \longrightarrow H^{i+1}_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)).$$

Since $t > \mathrm{Pd}_R(M)$, by [16, 2.2], $H^i_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)) \cong \mathrm{Ext}^i_R(M, \Gamma_{\mathfrak{a}}(N)) = 0$ for all $i \geq t$ and so $H^i_{\mathfrak{a}}(M, N) \cong H^i_{\mathfrak{a}}(M, L)$ for all $i \geq t$. Hence, we can (and do) assume that N is an $\mathfrak{a}$-torsion-free R-module. So $\mathfrak{a}$ contains an element x which is a non-zerodivisor on N. Now, by applying the functor $H^i_{\mathfrak{a}}(M, \ )$ on the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$, we conclude that the generalized local cohomology module $H^i_{\mathfrak{a}}(M, N/xN)$ is Artinian for all $i > t$. Since $\dim N/xN < d$, by inductive hypothesis, $\mathrm{Hom}_R(R/\mathfrak{a}, D(H^t_{\mathfrak{a}}(M, N/xN)))$ is finitely generated. Also, the short exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$ provides an exact sequence

$$H^t_{\mathfrak{a}}(M, N) \xrightarrow{x} H^t_{\mathfrak{a}}(M, N) \longrightarrow H^t_{\mathfrak{a}}(M, N/xN) \longrightarrow H^{t+1}_{\mathfrak{a}}(M, N)$$

which, in turn, yields the following exact sequence

$$D(H^{t+1}_{\mathfrak{a}}(M, N) \xrightarrow{g} D(H^t_{\mathfrak{a}}(M, N/xN)) \xrightarrow{f} D(H^t_{\mathfrak{a}}(M, N)) \xrightarrow{x} D(H^t_{\mathfrak{a}}(M, N)).$$

Since R is complete and $H^{t+1}_{\mathfrak{a}}(M, N)$ is Artinian, $\mathrm{Im} g$ is finitely generated. By breaking the above exact sequence in two exact sequences

$$0 \longrightarrow \mathrm{Im} g \longrightarrow D(H^t_{\mathfrak{a}}(M, N/xN)) \longrightarrow \mathrm{Im} f \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im} f \longrightarrow D(H^t_{\mathfrak{a}}(M,N)) \xrightarrow{x} D(H^t_{\mathfrak{a}}(M,N)),$$

and applying the left exact functor $\mathrm{Hom}_R(R/\mathfrak{a}, \ )$ on them, we deduce the exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{a}, D(H^t_{\mathfrak{a}}(M, N/xN))) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Im} f) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{Im} g)$$

and an isomorphism $\mathrm{Hom}_R(R/\mathfrak{a}, D(H^t_{\mathfrak{a}}(M,N))) \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Im} f)$. Therefore $\mathrm{Hom}_R(R/\mathfrak{a}, D (H^t_{\mathfrak{a}}(M,N)))$ is finitely generated. Hence

$$\begin{aligned} \mathrm{Ass}\ \mathrm{Hom}_R(R/\mathfrak{a}, D(H^t_{\mathfrak{a}}(M,N))) &= \mathrm{Ass} D(H^t_{\mathfrak{a}}(M,N)) \cap \mathrm{Supp}(R/\mathfrak{a}) \\ &= \mathrm{Ass} D(H^t_{\mathfrak{a}}(M,N)) \cap V(\mathfrak{a}) \\ &= \mathrm{Coass} H^t_{\mathfrak{a}}(M,N) \cap V(\mathfrak{a}) \end{aligned}$$

is finite. This complete the proof of theorem. □

Now, the following corollary is immediately consequence from 2.2.

Corollary 2.3—Suppose that $\mathfrak{a}$ is an ideal of a complete Noetherian local ring R, and N an R-module. Also, suppose that t is a positive integer such that $H^j_{\mathfrak{a}}(N)$ is Artinian for all $j > t$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, D(H^t_{\mathfrak{a}}(N)))$ is finitely generated and so, the set $\mathrm{Coass} H^t_{\mathfrak{a}}(N) \cap V(\mathfrak{a})$ is finite.

Corollary 2.4—Suppose that $\mathfrak{a}$ is an ideal of a complete Noetherian local ring R, and N a finitely generated R-module such that $d = \dim N > 1$. Then

$$\mathrm{Hom}_R(R/\mathfrak{a}, D(H^{d-1}_{\mathfrak{a}}(N)))$$

is finitely generated and so, the set $\mathrm{Coass} H^{d-1}_{\mathfrak{a}}(N) \cap V(\mathfrak{a})$ is finite.

PROOF: It follows from [4, 7.1.7] and Theorem 2.2. □

For an R-module N, the cohomological dimension of N with respect to $\mathfrak{a}$ is defined as

$$cd(\mathfrak{a}, N) := \max\{i \in \mathbb{Z} | H^t_{\mathfrak{a}}(N) \neq 0\}.$$

Corollary 2.5—Suppose that $\mathfrak{a}$ is an ideal of a complete Noetherian local ring R, and N a finitely generated R-module. Set $c := cd(\mathfrak{a}, N)$. Then

$$\mathrm{Hom}_R(R/\mathfrak{a}, D(H^c_{\mathfrak{a}}(N)))$$

is finitely generated and so, the set $\mathrm{Coass} H^c_{\mathfrak{a}}(N) \cap V(\mathfrak{a})$ is finite.

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