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NOTE ON AN EVALUATION OF $\zeta(p)$

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Since the time of Euler, many different evaluations of $\zeta(2p)$ have been presented. Stark also gave a recurrence formula for $\zeta(2p)$ by using Fejér's kernel. The purpose of this note is to show that the above mentioned recurrence formula of Stark can be obtained by making use of Lagrange's trigonometric identity simpler than Fejér's kernel. But his use of Fejér's kernel made it possible to connect $\zeta(2p+1)$ and $I_{2p} := \int_0^{\pi/2} \theta^{2p} / \sin^2 \theta d\theta$. We also reduce the known infinite series representation of I_{2p} to a finite series one.

Key Words: Riemann's ζ-Function; Lagrange's Trigonometric Identity; Mean Value Theorem for Integrals

1. INTRODUCTION

The Riemann Zeta function $\zeta(s)$ is defined by (see [6, p. 96])

$$\zeta(s) := \begin{cases} \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{1-2^{-s}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} & (\Re(s) > 1) \\ (1-2^{1-s})^{-1} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} & (\Re(s) > 0; s \neq 1). \end{cases} \tag{1}$$

Since the time of Euler, there have been many proofs giving the value of

$$\zeta(2k) = r_k \cdot \pi^{2k} \quad (k \in \mathbb{N} := \{1, 2, 3, \ldots\}).$$

where r_k is a rational number to be determined by a suitable recurrence formula. Before going to our main subject, it may be remarked that Rassias and Srivastava [5] presented a systematic investigation of several families of infinite series which are associated with the Riemanns zeta function $\zeta(p)$, the digamma (and polygamma) functions, the harmonic (and generalized harmonic) numbers, and the stirling numbers of the first kind. Stark [7] used Fejér's kernel [3, p. 222]:

$$\frac{1}{2(n+1)}\frac{\sin^2[(n+1)\theta/2]}{\sin^2(\theta/2)} = \frac{1}{2} + \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\cos(k\theta) \quad (n \in \mathbb{N}) \tag{2}$$

to give a recurrence formula for $\zeta(2p)(p \in \mathbb{N} := \{1, 2, 3, \ldots\})$.

Here we also show that the above mentioned Stark's recurrence formula for $\zeta(2p)$ can be obtained, instead of the Fejér's kernel (2), by making use of a simpler Lagrange's trigonometric identity (see [2, p. 18]):

$$L_n(\theta) := \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} = \frac{1}{2} + \sum_{k=1}^{n}\cos(k\theta) \quad (0 < \theta < 2\pi). \tag{3}$$

Recall a known integral formula (see [7]):

$$\int_0^\pi \theta^{2p}\cos(k\theta)\cdot d\theta = (-1)^k \pi^{2p+1}(2p)!\sum_{j=1}^{p}\frac{(-1)^{j+1}}{[2(p-j)+1]!(k\pi)^{2j}} \quad (p \in \mathbb{N}). \tag{4}$$

Using the right most side of $L_n(\theta)$ in (3) with (4), we have

$$\begin{aligned} I(n) : &= \int_0^\pi \theta^{2p} L_n(\theta)\, d\theta \quad (p \in \mathbb{N}) \\ &= \frac{\pi^{2p+1}}{2(2p+1)} + \sum_{k=1}^{n}(-1)^k \pi^{2p+1}(2p)!\sum_{j=1}^{p}\frac{(-1)^{j+1}}{[2(p-j)+1]!(k\pi)^{2j}}. \end{aligned} \tag{5}$$

On the other hand, using the left side of $L_n(\theta)$ in (3), we find that, for fixed $p \in \mathbb{N}$,

$$I(n) = 2^{2p}\int_0^{\pi/2}\frac{\theta^{2p}}{\sin\theta}\sin[(2n+1)\theta]d\theta = O\left(\frac{1}{2n+1}\right) \; (n \to \infty). \tag{6}$$

Indeed, let $f(\theta) := \theta^{2p}/\sin\theta (0 < \theta \leqq \pi/2)$ and $f(0) := f(0+) = 0$. Then it is easy to see that $f'(\theta) \geqq 0$ on $[0, \pi/2]$, and so f is increasing on $[0, \pi/2]$. Now it follows from part (b) of the second mean value theorem for integrals in [1, p. 233] that, for some $c \in [0, \pi/2]$,

$$\begin{aligned} I(n) &= 2^{2p}\left[f(0+)\int_0^c \sin[(2n+1)\theta]d\theta + f(\pi/2)\int_c^{\pi/2}\sin[(2n+1)\theta]d\theta\right] \\ &= O\left(\frac{1}{2n+1}\right) \quad (n \to \infty). \end{aligned}$$

Finally, taking the limit on (5) and (6) as $n \to \infty$ and setting

$$A_{2j} = \sum_{k=1}^{\infty}(-1)^{k+1}k^{-2j} \quad (j \in \mathbb{N}),$$

we obtain the desired recurrence formula (see [7]):

$$A_{2p} = \pi^{2p}\left\{\frac{(-1)^{p+1}}{2(2p+1)!} - \sum_{j=1}^{p-1}\frac{(-1)^{p+j}}{[2(p-j)+1]!\pi^{2j}}A_{2j}\right\}, \tag{7}$$

where the empty sum is (as usual, in what follows) understood to be nil. The case $p = 1$ of (7) gives $A_2 = \pi^2/12$. The greater values of $A_{2p} (p \in \mathbb{N}\backslash\{1\})$ are obtained recursively. It follows from (1) that

$$\zeta(2p) = \frac{2^{2p}}{2^{2p}-2}A_{2p}. \tag{8}$$

Stark [7] gave a connection between $\zeta(2p+1)$ and the integrals

$$I_{2p} := \int_0^{\pi/2}\frac{\theta^{2p}}{\sin^2\theta}d\theta$$

to show that evaluating of $\zeta(2p+1)$ and I_{2p} are equivalent. He also noted that the integrals I_{2p} cannot be evaluated in terms of elementary functions but one has

$$\begin{aligned} I_{2p} &= 2p\int_0^{\pi/2}\theta^{2p-1}\cot\,\theta\,d\theta \\ &= 2p\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k}B_{2k}}{[2(p+k)-1](2k)!}\left(\frac{\pi}{2}\right)^{2(p+k)-1}, \end{aligned} \tag{9}$$

where B_{2k} denote the Bernoulli numbers (see [6, pp. 59-63]).

If the well-known formula (see [6, p. 18, Eq. (18)]):

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k} \quad (k \in \mathbb{N}\cup\{0\}) \tag{10}$$

is used in (9), I_{2p} is expressed as an *infinite* series involving the Riemann zeta function:

$$I_{2p} = -4p\left(\frac{\pi}{2}\right)^{2p-1}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2p-1)2^{2k}}. \tag{11}$$

Setting $a = 1$ at (2.16) in [3], we get

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k+2p-1}t^{2k+2p-1} &= \frac{t^{2p-1}}{2(2p-1)} \quad (p \in \mathbb{N};\ |t| < 1) \\ &+\frac{1}{2}\sum_{k=0}^{2p-1}\binom{2p-1}{k}[\zeta'(-k,1-t)+(-1)^k\zeta'(-k,1+t)]t^{2p-1-k}, \end{aligned} \tag{12}$$

where $\zeta(s,a)$ is the generalized (or Hurwitz) Zeta function defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty}\frac{1}{(k+a)^s} \quad (a \neq 0, -1, -2, \ldots)$$

and $\zeta'(s,a) = \frac{\partial}{\partial s}\zeta(s,a)$.

Setting $t = 1/2$ at (12) and using various identities for $\zeta(s)$ and $\zeta(s,a)$ (see [6, Chapter 2]), for example,

$$\zeta'(-2p) = \lim_{\epsilon\to 0}\frac{\zeta(-2p+\epsilon)}{\epsilon} = (-1)^p\frac{(2p)!}{2(2\pi)^{2p}}\zeta(2p+1) \quad (p\in\mathbb{N}), \tag{13}$$

we obtain

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k+2p-1)2^{2k}} = \frac{1}{2(2p-1)} - \frac{1}{2}\log 2$$
$$+\frac{1}{2}\sum_{k=1}^{p-1}(-1)^k\binom{2p-1}{2k}\frac{(2k)!}{(2\pi)^{2k}}(1-2^{2k})\zeta(2k+1) \quad (p\in\mathbb{N}) \tag{14}$$

the special case $p=1$ of which is recorded in [6, p. 212, Eq. (467)].

If (14) is applied to (11), I_{2p} is expressed as a *finite* series involving the Riemann zeta function:

$$\begin{aligned} I_{2p} &= \int_0^{\pi/2}\frac{\theta^{2p}}{\sin^2\theta}d\theta = 2p\int_0^{\pi/2}\theta^{2p-1}\cot\,\theta\,d\theta \\ &= 2p\left(\frac{\pi}{2}\right)^{2p-1}\left[\log 2 + \sum_{k=1}^{p-1}(-1)^{k+1}\binom{2p-1}{2k}\frac{(2k)!}{(2\pi)^{2k}}(1-2^{2k})\,\zeta(2k+1)\right]. \end{aligned} \tag{15}$$

In addition, we have (cf. [4, p. 428, Entry 3.748.2]):

$$\int_0^{\pi/2}\theta^{2p}\cot\,\theta\,d\theta = \left(\frac{\pi}{2}\right)^{2p}\left(\frac{1}{2p} - \sum_{k=1}^{\infty}\frac{\zeta(2k)}{(k+p)2^{2k}}\right) \quad (p\in\mathbb{N}). \tag{16}$$

Setting $a=1$ and $t=1/2$ at (2.13) in [3], similarly as in getting (14), we obtain

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(k+p)2^{2k}} = \frac{1}{2p} - \log 2 + (-1)^p(2p)!\frac{1-2^{2p+1}}{(2\pi)^{2p}}\zeta(2p+1)$$
$$+\sum_{k=1}^{p-1}(-1)^k\binom{2p}{2k}\frac{(2k)!}{(2\pi)^{2k}}(1-2^{2k})\,\zeta(2k+1) \quad (p\in\mathbb{N}). \tag{17}$$

From (16) and (17), the integral in (16) can be evaluated as a finite series involving the Riemann Zeta function as follows:

$$\int_0^{\pi/2}\theta^{2p}\cot\theta\,d\theta = \left(\frac{\pi}{2}\right)^{2p}[\log 2 + (-1)^p(2p)!\frac{2^{2p+1}-1}{(2\pi)^{2p}}\zeta(2p+1)$$
$$+\sum_{k=1}^{p-1}(-1)^{k+1}\binom{2p}{2k}\frac{(2k)!}{(2\pi)^{2k}}(1-2^{2k})\zeta(2k+1)] \; (p\in\mathbb{N}). \tag{18}$$

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REFERENCES

1. R. G. Bartle, *The Elements of Real Analysis*, 2nd Edition, John Wiley & Sons, Inc., 1976.
2. J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 6th Edition, McGraw-Hill International Editions, 1996.
3. J. Choi, Y. J. Cho, and H. M. Srivastava, Series involving the Zeta function and multiple Gamma functions, *Appl. Math. Comput.*, **159** (2004), 509–537.
4. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Corrected and Enlarged Ed. prepared by A. Jeffrey and D. Zwillinger), Sixth Ed., Academic Press, New York, 2000.
5. Th. M. Rassias and H. M. Srivastava, Some Classes of infinite series associated with Reiemann zeta and polygamma functions and generalized harmonic numbers, *Appl. Math. Comput.*, **131** (2002), 593–605.
6. H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston, and London, 2001.
7. E. L. Stark, The series $\sum_{k=1}^{\infty} k^{-s}, s = 2, 3, 4, \cdots$, once more, *Math. Mag.*, **47** (1974), 197–202.