Indian J. pure appl. Math., **37**(5): 265-272, October 2006

ON BANACH FRAMES

P. K. JAIN*, S. K. KAUSHIK** AND L. K. VASHISHT*

**Department of Mathematics, University of Delhi, Delhi* 110 007, *India*
E-mail : pawanujain@vsnl.net; lalitkvashisth@hotmail.com
***Department of Mathematics, Kirorimal College*
University of Delhi Delhi 110007, *India*
E-mail : shikk2003@yahoo.co.in

(*Received* 13 *July* 2004; *after final revision* 25 *August* 2006; *accepted* 8 *September* 2006)

Examples and counter examples to distinguish various types of Banach frames have been given. It has been proved that if a Banach space has a Banach frame, then it also has a normalized tight as well as a normalized tight and exact Banach frame. In particular, it has been proved that a Banach space with a weak* separable dual has a normalized tight and exact Banach frame. A condition for the stability of a Banach frame has been given. Finally, we consider finite sum of Banach frames and give a condition under which finite sum of Banach frames is a Banach frame.

Key Words: Banach frames; frames; stability

1. INTRODUCTION

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] in 1952, while addressing some difficult problems from the theory of non-harmonic Fourier series. Infact they abstracted Gabor's method to define a frame for a Hilbert space. Later, in 1986, Daubechies, Grossmann and Meyer [4] found a fundamental new application to wavelet and Gabor transforms in which frames continue to play an important role.

In 1991, Gröchenig [8] generalized frames for Banach spaces and called them *atomic decompositions*. He also introduced a more general concept for Banach spaces called a *Banach frame*. Banach frames were further studied in [1–3, 7].

In the present paper, our aim is to provide examples and counter examples for various types of Banach frames (§3). In Section 4, a necessary and sufficient condition for a Banach frame to be an exact Banach frame has been given; it has been proved that if E is a Banach space such that E^* is weak*-separable, then E has a normalized tight and exact Banach frame. Conditions for the stability of Banach frames have been given in Section 5. Finally, in Section 6, a condition under which finite sum of Banach frames is a Banach frame has been given.

2. Basic Notations and Definitions

Throughout the paper E will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, E^* and E^{**}, respectively, the first and second conjugate spaces of E, $L(E, E)$ the Banach space of all continuous linear mappings of E into E, E_d an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$, $[f_n]$ the closed linear span of $\{f_n\}$ and $\widetilde{[f_n]}$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*, E)$-topology.

A sequence $\{f_n\} \subset E^*$ is said to be complete if $[f_n] = E^*$ and *total* if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. For $x = \{x_n\}$, $y = \{y_n\}$ in E and $\alpha \in \mathbb{K}$, we define $x \pm y = \{x_n \pm y_n\}$ and $\alpha x = \{\alpha x_n\}$.

Definition — (Gröchenig [8]). Let E be a Banach space and E_d be an associated Banach space of scalar valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame for E with respect to E_d*, if

(i) $\{f_n(x)\} \in E_d$, for each $x \in E$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in E.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.1) is called the *frame inequality*.

The Banach frame $(\{f_n\}, S)$ is called *tight* if it is possible to choose A, B satisfying (2.1) with $A = B$ and is *normalized tight* if $A = B = 1$. The Banach frame $(\{f_n\}, S)$ is said to be *exact* if there exists no reconstruction operator S_0 such that $(\{f_n\}_{n \neq i}, S_0)$ $(i \in \mathbb{N})$ is a Banach frame for E.

3. Examples

Let $E = c_0$ and let $\{f_n\}$ be the sequence of elements in E^* defined by

$$f_n(x) = \xi_n, \quad n \in \mathbb{N},\ x = \{\xi_j\} \in E.$$

Then, by Remark 7.1 in [9], there exists an associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$ of scalar-valued sequences with norm given by

$$||\{f_n(x)\}||_{E_d} = ||x||_E\,, \quad x \in E\,.$$

Define $S : E_d \to E$ by $S(\{f_n(x)\}) = x$, $x \in E$.

1. ***Tight and Exact***. The pair $(\{f_n\}, S)$ is a Banach frame for E with respect to E_d and with bounds $A = B = 1$. Also, there exists no reconstruction operator S_0 such that $(\{f_n\}_{n\neq j}, S_0)$ $(j \in \mathbb{N})$ is Banach frame for E.

2. ***Non-tight and Exact***. Let $\{\alpha_n\} \subset \mathbb{R}$ be the sequence of scalars defined by

$$\alpha_n = \begin{cases} \frac{1}{2}, & \text{if } n = 1 \\ 1, & \text{if } n > 1,\ n \in \mathbb{N}. \end{cases}$$

Put $\phi_n = \alpha_n f_n$, $n \in \mathbb{N}$. Then $\{\phi_n(x)\} \in E_d$, $x \in E$ and

$$\frac{1}{2}||x||_E \leq ||\{\phi_n(x)\}||_{E_d} \leq ||x||_E, \quad x \in E\,.$$

Let $T : E \to E_d$ be the coefficient mapping given by $T(x) = \{f_n(x)\}$, $x \in E$. Then, $U = T^{-1} : E_d \to E$ is a bounded linear operator such that $(\{\phi_n\}, U)$ is a Banach frame for E with respect to E_d and with bounds $A = \frac{1}{2}$, $B = 1$. Exactness of $(\{\phi_n\}, U)$ is obvious.

3. ***Tight and Non-exact***. Define the sequence $\{g_n\} \subset E^*$ by $g_1 = f_1$, $g_n = f_{n-1}$, $n \geq 2$. Then, there exists a reconstruction operator $S_0 : E_{d_0} = \{\{g_n(x)\} : x \in E\}$ such that $(\{g_n\}, S_0)$ is a Banach frame for E with respect to E_d and with bounds $A = B = 1$. Also, note that $\{g_n\}_{n\neq 1}$ is total over E, therefore, there exists an associated Banach space $E_{d_1} = \{\{g_n(x)\}_{n\neq 1} : x \in E\}$ and a reconstruction operator $S_1 : E_{d_1} \to E$ such that $(\{g_n\}_{n\neq 1}, S_1)$ is Banach frame for E with respect to E_{d_1}. Hence $(\{g_n\}, S_0)$ is tight and non-exact Banach frame for E.

4. ***Non-tight and Non-exact***. Let $\{\alpha_n\} \subset \mathbb{R}$ be the sequence defined by

$$\alpha_n = \begin{cases} 1, & \text{if } n \in \mathbb{N},\ n \neq 3 \\ \frac{1}{2}, & \text{if } n = 3 \end{cases}$$

Put $\phi_1 = \alpha_1 f_1$, $\phi_n = \alpha_n f_{n-1}$, $n \geq 2$.

Then, $\{\phi_n(x)\} \in E_d$, $x \in E$ and

$$\frac{1}{2}||x||_E \leq ||\{\phi_n(x)\}||_{E_d} \leq ||x||_E, \quad x \in E\,.$$

Therefore, one may find a reconstruction operator $S_0 : E_d \to E$ such that $(\{\phi_n\}, S_0)$ is a Banach frame for E with respect to E_d. Further, there exists a reconstruction operator $S_1 : E_{d_1} = \{\{\phi_n(x)\}_{n\neq 1} : x \in E\}$ such that $(\{\phi_n\}_{n\neq 1}, S_1)$ is a Banach frame for E with respect to E_{d_1}. Hence $(\{\phi_n\}, S_0)$ is non-tight and non-exact Banach frame for E with respect to E_d.

5. ***Not a Banach frame***. Let $\{g_n\} \in E^*$ be the sequence defined by $g_1 = f_2$, $g_n = f_n$, $n \geq 2$. Then there exists no reconstruction operator S such that $(\{g_n\}, S)$ is a Banach frame for E.

Regarding existence of Banach frames, we observe that there exist Banach spaces which do not have Banach frames. Infact, the Banach space ℓ^∞/c_0 has no Banach frame.

Remark 3.1 : Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d and with frame bounds A and B. Then, it is easy to observe that the coefficient mapping $T : E \to E_d$ defined by $T(x) = \{f_n(x)\}, x \in E$ is a topological isomorphism onto a closed subspace of E_d. In this case $\|T\| \leq B$. Further note that T^{-1} exists on the range of T (i.e. $R(T)$) and $||T^{-1}|| \leq \frac{1}{A}$.

Moreover, we observe that for a sequence $\{f_n\} \subset E^*$, if the coefficient mapping $T : E \to E_d$ defined by $T(x) = \{f_n(x)\}$, $x \in E$ is a topological isomorphism onto E_d, then there exists a reconstruction operator $S : E_d \to E$ such that $(\{f_n\}, S)$ is a Banach frame for E and with frame bounds $\|T^{-1}\|^{-1}$ and $\|T\|$. Indeed, for each $x \in E$

$$\|\{f_n(x)\}\|_{E_d} = \|T(x)\|_{E_d} \leq \|T\| \, \|x\|_E$$

and

$$\|x\|_E = \|T^{-1}(\{f_n(x)\})\|_E \leq \|T^{-1}\| \, \|\{f_n(x)\}\|_{E_d}$$

Put $S = T^{-1}$. Then $(\{f_n\}, S)$ is a Banach frame for E.

4. Existence of Banach frames

Lemma 4.1. — Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Then $(\{f_n\}, S)$ is exact if and only if $f_n \notin \widetilde{[f_i]}_{i \neq n}$, for all n.

Proof: Suppose first that the Banach frame $(\{f_n\}, S)$ is exact. Then, there exists no reconstruction operator S_0 such that $(\{f_i\}_{i\neq n}, S_0)$ $(n \in \mathbb{N})$ is a Banach frame for E. Therefore, $\widetilde{[f_i]}_{i\neq n} \neq E^*$. Hence $f_n \notin \widetilde{[f_i]}_{i\neq n}$. Conversely, let $f_n \notin \widetilde{[f_i]}_{i\neq n}$, for all $n \in \mathbb{N}$, and let $(\{f_n\}, S)$ be not exact. Then there exists a positive integer m and a reconstruction operator S_1 defined by $S_1(\{f_i(x)\}_{i\neq m}) = x, x \in E$ such that $(\{f_i\}_{i\neq m}, S_1)$ is a Banach frame for E. Therefore, by frame inequality, $\widetilde{[f_i]}_{i\neq m} = E^*$. This gives $f_m \in \widetilde{[f_i]}_{i\neq m}$, which is a contradiction. □

Theorem 4.2. — Let E be a Banach space having a Banach frame. Then E has a normalized tight Banach frame as well as a normalized tight and exact Banach frame.

Proof: Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Then, by frame inequality, $\{f_n\}$ is total over E. Therefore, by Remark 7.1 in [9], there exists an associated Banach space $\{\{f_n(x)\} : x \in E\}$ with norm given by $\|\{f_n(x)\}\| = \|x\|_E$, $x \in E$. Define $S_0 : \{\{f_n(x)\} : x \in E\} \to E$ by $S_0(\{f_n(x)\}) = x$, $x \in E$. Then, S_0 is a bounded linear operator such that $(\{f_n\}, S_0)$ is a normalized tight Banach frame.

Further, we may assume, without loss of generality that $\{f_n\}$ is finitely linearly independent. (In case $\{f_n\}$ is not finitely linearly independent, we can derive a subsequence $\{g_j\} \subset \{f_n\}$ which is finitely linearly independent and total over E). Then, for each $n \in \mathbb{N}$, there exists an $x_n \in E$

such that $f_i(x_n) = 0$, $i = 1, 2, \ldots, n-1$ and $f_n(x_n) = 1$. Define $\{h_n\} \subset E^*$ by

$$h_1 = f_1, \quad h_n = f_n - \sum_{i=1}^{n-1} f_n(x_i) h_i, \quad n = 2, 3, \ldots.$$

Then $\{h_n\}$ is total over E such that $h_i(x_j) = \delta_{ij}$, $i, j \in \mathbb{N}$. Therefore, there exists an associated Banach space $E_{d_1} = \{\{h_n(x)\} : x \in E\}$ equipped with norm $||\{h_n(x)\}||_{E_{d_1}} = ||x||_E$, $x \in E$ and a bounded linear operator $U : E_{d_1} \to E$ defined by $U(\{h_n(x)\}) = x, x \in E$ such that $(\{h_n\}, U)$ is a normalized tight Banach frame for E. Further, $h_n \notin \widetilde{[h_i]}_{i \neq n}$, for all $n \in \mathbb{N}$.

Hence, by Lemma 4.1, $(\{h_n\}, U)$ is an exact Banach frame for E. □

Corollary 4.3 — If E is a Banach space such that E^* is weak* separable, then E has a normalized tight and exact Banach frame. In particular, every separable Banach space has a normalized tight and exact Banach frame.

Remark 4.4 : The result in the corollary improves considerably Proposition 2.4 in [2].

***Theorem* 4.5** — A Banach frame $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ is exact if there exists a sequence $\{x_n\} \subset E$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i = f \quad \Longrightarrow \quad \lim_{n\to\infty} \alpha_i^{(n)} = f(x_i), \quad (i \in \mathbb{N}).$$

PROOF: Note that, by assumption, $\{f_n\}$ is finitely linearly independent. Define

$$\psi_j \left(\sum_{i=1}^{n} \alpha_i f_i \right) = \alpha_j, \quad \left(\sum_{i=1}^{n} \alpha_i f_i \in \text{span}\{f_n\} \right), \quad j = 1, 2, \ldots, n.$$

Then, for each j, ψ_j is a well defined continuous linear functional on span$\{f_n\}$. Since $f \in [f_n]$ has the form $f = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i$, one can extend ψ_j to $[f_n]$ by $\psi_j(f) = \lim_{n\to\infty} \alpha_j^{(n)} = f(x_j), j \in \mathbb{N}$. Thus $\{\psi_n\} \subset [f_n]^*$ is a sequence such that $\psi_j(f_i) = f_i(x_j) = \delta_{ij}$, $(i, j = 1, 2, \ldots)$. Therefore $f_n \notin \widetilde{[f_i]}_{i \neq n}$, for each n. Hence, by Lemma 4.1 again, $(\{f_n\}, S)$ is exact. □

5. A STABILITY RESULT

Stability theorems for Banach frames were studied by Christensen [3]. We shall now prove a stability theorem for Banach frames which in some sense is more general than a similar result given by Christensen in [3].

***Theorem* 5.1** — Let $(\{f_n\}, S)$, $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Let $\{g_n\} \subset E^*$ be such that $\{g_n(x)\} \in E_d$, $x \in E$ and let $V : E \to E_d$ be the coefficient mapping given by $V(x) = \{g_n(x)\}$, $x \in E$. If there exist constants $\alpha, \beta, \gamma \geq 0$ such that

(i) $\max\left\{ \frac{\alpha||U|| + \beta||V|| + \gamma}{(||S||)^{-1}}, \beta \right\} < 1$,

and

(ii) $||\{(f_n - g_n)(x)\}||_{E_d} \le \alpha||\{f_n(x)\}||_{E_d} + \beta||\{g_n(x)\}||_{E_d} + \gamma||x||_E$,

then there is a reconstruction operator $T : E_d \to E$ such that $(\{g_n\}, T)$ is a Banach frame for E with respect to E_d and with frame bounds $\left(\frac{(||S||)^{-1}-(\alpha||U||+\gamma)}{1+\beta}\right)$ and $\left(\frac{||U||+(\alpha||U||+\gamma)}{1-\beta}\right)$, where $U : E \to E_d$ is the coefficient mapping given by $Ux = \{f_n(x)\}$, $x \in E$.

PROOF : Note that $U : E \to E_d$ is bounded and $SU : E \to E$ is the identity mapping on E. Therefore

$$||x||_E = ||SU(x)||_E \le ||S||\ ||\{f_n(x)\}||_{E_d}, \quad x \in E\,.$$

Thus

$$(||S||)^{-1}||x||_E \le ||\{f_n(x)\}||_{E_d} \le ||U||\ ||x||_E, \quad x \in E\,. \tag{5.1}$$

Now, for each $x \in E$, $\{g_n(x)\} \in E_d$ and

$$\begin{aligned} ||\{g_n(x)\}||_{E_d} &= ||\{(f_n - (f_n - g_n))(x)\}||_{E_d} \\ &\le ||\{f_n(x)\}||_{E_d} + ||\{(f_n - g_n)(x)\}||_{E_d} \\ &\le ||\{f_n(x)\}||_{E_d} + \alpha||\{f_n(x)\}||_{E_d} + \beta||\{g_n(x)\}||_{E_d} + \gamma||x||_E\,. \end{aligned}$$

Therefore, by (5.1), we have

$$(1-\beta)||\{g_n(x)\}||_{E_d} \le ((1+\alpha)||U|| + \gamma)||x||_E, \quad x \in E.$$

Similarly, we obtain

$$(1+\beta)||\{g_n(x)\}||_{E_d} \ge \left((||S||)^{-1} - (\alpha||U|| + \gamma)\right)||x||_E, \quad x \in E\,.$$

Since $SU = I$, by hypotheses

$$\begin{aligned} ||I - SV|| &\le ||S||\,||U - V|| \\ &\le ||S||(\alpha||U|| + \beta||V|| + \gamma) < 1. \end{aligned}$$

Thus SV is invertible. Put $T = (SV)^{-1}S$. Then $T : E_d \to E$ is a bounded linear operator such that $T(\{g(x)\}) = x$, $x \in E$. Hence $(\{g_n\}, T)$ is a Banach frame for E with respect to E_d and with desired frame bounds. □

Note. In the particular case when $\beta = 0$, Theorem 5.1 reduces to Theorem 2.2 in [3].

6. Finite Sum of Banach Frames

Let $(\{f_{1,n}\}, S_1)$ and $(\{f_{2,n}\}, S_2)$, where $\{f_{1,n}\} \subset E^*$ and $f_{2,1} = -f_{1,1}$, $f_{2,2} = -(f_{1,1} + f_{1,2})$ and $f_{2,n} = f_{1,n}$, $\forall\ n \ge 3$ be Banach frames for E. Then there exists, in general, no resconstruction operator U such that $\left(\left\{\sum_{i=1}^{2} f_{i,n}\right\}, U\right)$ is a Banach frame for E. So it is natural to ask the question that under what conditions this finite sum of Banach frames is a Banach frame.

The following Theorem gives a condition under which the finite sum of Banach frames is again a Banach frame.

***Theorem* 6.1** — Let $(\{f_{i,n}\}, S_i)$ $(\{f_{i,n}\} \subset E^*, S_i : E_d \to E)$, $i \in \{1, 2, \ldots, k\}$, be Banach frames for E w.r.t. to E_d. Then there exists a reconstruction operator U such that $\left(\left\{\sum_{i=1}^{k} f_{i,n}\right\}, U\right)$ is a tight Banach frame for E, provided

$$||\{f_{j,n}(x)\}||_{E_d} \leq \left\|\left\{\left(\sum_{i=1}^{k} f_{i,n}(x)\right)\right\}\right\|_{E_d}, \quad x \in E, \text{ forsome } j \in \{1, 2, \ldots, k\}.$$

PROOF : By hypothesis

$$\begin{aligned} ||x||_E &= ||S_j(\{f_{j,n}(x)\})||_E, \\ &\leq ||S_j||\ ||\{f_{j,n}(x)\}||_{E_d} \\ &\leq ||S_j|| \left\|\left\{\left(\sum_{i=1}^{k} f_{i,n}\right)(x)\right\}\right\|_{E_d}, \quad x \in E. \end{aligned}$$

Thus $\left\{\left(\sum_{i=1}^{k} f_{i,n}\right)\right\}$ is total over E. Therefore, by Remark 7.1 in [9], there exists an associated Banach space $E_{d_1} = \left\{\left\{\left(\sum_{i=1}^{k} f_{i,n}\right)(x)\right\} : x \in E\right\}$ equipped with norm $\left\|\left\{\left(\sum_{i=1}^{k} f_{i,n}\right)(x)\right\}\right\|_{E_{d_1}} = ||x||_E$, $x \in E$ and a bounded linear operator $U : E_{d_1} \to E$ defined by $U\left(\left\{\left(\sum_{i=1}^{k} f_{i,n}\right)(x)\right\}\right) = x$, $x \in E$ such that $\left(\left\{\sum_{i=1}^{k} f_{i,n}\right\}, U\right)$ is a tight Banach frame for E with respect to E_{d_1}. □

Towards the converse, we observe that if $\left(\left\{\sum_{i=1}^{k} f_{i,n}\right\}, U\right)$ $(U : E_d \to E)$ is a Banach frame for E with respect to E_d, where $\{f_{i,n}\} \subset E^*$ $(i = 1, 2, \ldots, k)$, then there exists, in general, no reconstruction operator S_i, for $i = 1, 2, \ldots, n$, such that $(\{f_{i,n}\}, S_i)$ is a Banach frame for E.

Example 6.2 — Let $E = \ell^p$ $(1 \leq p \leq \infty)$. Define $\{f_{1,n}\}$, $\{f_{2,n}\}$ in E^* by

$$f_{1,1}(x) = 0\ ; \quad f_{1,n}(x) = \xi_n, n > 1; \quad x = \{\xi_n\} \in E$$

$$f_{2,1}(x) = \xi_1; \quad f_{2,n}(x) = 0, n > 1; \quad x = \{\xi_n\} \in E.$$

Then there exists a reconstruction operator $U : E_d = \left\{\left\{\left(\sum_{i=1}^{2} f_{i,n}\right)(x)\right\} : x \in E\right\} \to E$ such that $\left(\left\{\sum_{i=1}^{2} f_{i,n}\right\}, U\right)$ is a Banach frame for E with respect to E_d, but there exist no reconstruction operators S_1, S_2 such that $(\{f_{1,n}\}, S_1)$ and $(\{f_{2,n}\}, S_2)$ are Banach frames for E.

ACKNOWLEDGEMENT

The authors thank the referees for their useful comments and suggestions towards the improvement of the paper. The research of the second author is partially supported by the UGC (India) (Letter No. F.13-3/2002 (SR-I)), and the research of the third author is supported by the UGC (India) (Ref. No. Sch/JRF/AA/41/2002-03/).

REFERENCES

1. P. G. Casazza and O. Christensen, Perturbation of operators and applications to frame theory, *J. Fourier Anal. Appl.*, **3** (1997), 543–557.

2. P. G. Casazza, D. Han and D.R. Larson, Frames for Banach spaces, *Contemporary Mathematics*, **247** (1999), 149–182.

3. O. Christensen and C. Heil, Perturbation of Banach frames and atomic decompositions, *Math. Nachr.*, **185** (1997), 33–47.

4. I. Daubechies, A. Grossmann and Y. Meyer, Painless non-orthogonal expansions, *J. Math. Physics*, **27** (1986), 1271–1283.

5. R. J. Duffin and A.C. Schaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.*, **72** (1952), 341–366.

6. N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience Publ., New York, 1958.

7. S. J. Favier and R.A. Zalik, On the stability of frames and Riesz bases, *Appl. Comp. Harm. Anal.*, **2** (1995), 160–173.

8. K. Gröchenig, Describing functions : Atomic decompositions versus frames, *Monatsh. Math.*, **112** (1991), 1–41.

9. I. Singer, *Bases in Banach Spaces-II*, Springer Verlag, Berlin-Heidelberg-New York, 1981.