Indian J. pure appl. Math., **37**(5): 307-315, October 2006

DISCRETE SPECTRUM OF A GENERAL QUADRATIC PENCIL OF SCHRÖDINGER EQUATIONS

A. AKBULUT* AND E. BAIRAMOV**

**Gazi University, Kýrpehir Faculty of Arts And Science, Department of Mathematics*, 40100, *Kýrpehir, Turkey*
E-mail: aakbulut@gazi.edu.tr
***Ankara University, Faculty of Science, Department of Mathematics*, 06100, *Tandoğan, Ankara, Turkey*
E-mail: bairamov@science.ankara.edu.tr

(*Received* 26 *September* 2003: *Accepted* 11 *May* 2006)

In this paper, using the boundary properties of the analytic functions we investigate the structure of the discrete spectrum of the boundary value problem

$$\begin{aligned} -y'' + [V(x) + 2\lambda U(x) - \lambda^2]y + F &= 0, \; x \in \mathbb{R}_+ = [0, \infty) \\ \alpha y'(0) + \beta y(0) &= 0, \end{aligned}$$

where U, V and F are complex valued functions and $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$.

Key Words: Dicrete Spectrum; Spectral Singularities; Boundary Value Problems

1. INTRODUCTION

The spectral analysis of a non-selfadjoint Sturm-Liouville equation (SLE) with continuons and discrete spectrum was investigated by Naimark [13]. He proved the existence of the spectral singularities in the continuous spectrum of SLE. Lyance showed that the spectral singularities play an important role in the spectral theory of SLE [12]. He also studied the effect of the spectral singularities in the spectral expansion of SLE in terms of the principal functions. Some problems of the spectral analysis of a non-selfadjoint Schrödinger, Dirac and Klein-Gordon differential and difference equations with spectral singularities were studied in [1, 4, 5, 7–10].

Let us consider the boundary value problem (BVP)

$$-y'' + [V(x) + 2\lambda U(x) - \lambda^2]y = 0, \; x \in \mathbb{R}_+ = [0, \infty), \tag{1.1}$$

$$y(0) = 0, \tag{1.2}$$

where U and V are complex valued functions and λ is a spectral parameter. The eq. (1.1) is called a Quadratic Pencil of Schrödinger equation. In [2] and [6] the dependence of the structure of the eigenvalues and the spectral singularities of the BVP (1.1)–(1.2) on the behaviour of U and V at infinity has been considered. An eigenfunction expansion for the BVP (1.1)–(1.2) with spectral singularities has been investigated in [3].

Now let us consider the following BVP

$$-y'' + [V(x) + 2\lambda U(x) - \lambda^2]y + F = 0, \ x \in \mathbb{R}_+, \tag{1.3}$$

$$\alpha y'(0) + \beta y(0) = 0, \tag{1.4}$$

where U, V and F are complex valued functions and U is absolutely continuous in each finite subinterval of $\mathbb{R}_+$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$.

In this paper using the boundary properties of the analytic functions we studied the structure of the eigenvalues and the spectral singularities of the BVP (1.3)–(1.4).

2. The Solution Of (1.3)–(1.4)

Let $f^+(x,\lambda)$ and $f^-(x,\lambda)$ denote the solutious of (1.1) satisfying

$$\lim_{x\to\infty} f^{\pm}(x,\lambda)e^{\mp i\lambda x} = 1, \lambda \in \overline{\mathbb{C}}_{\pm},$$

where $\overline{\mathbb{C}}_{\pm} = \{\lambda : \lambda \in \mathbb{C}, \pm \mathrm{Im}\lambda \geq 0\}$.

The solutions $f^+(x,\lambda)$ and $f^-(x,\lambda)$ are called the Jost solutions of (1.1).

Let us suppose that the functions U and V satisfy

$$\int_0^\infty |U(x)|dx < \infty, \int_0^\infty x[|V(x)| + |U'(x)|]dx < \infty. \tag{2.1}$$

Jaulent and Jean have obtained [11]: Under the condition (2.1) the Jost solutions of (1.1) has the representation

$$f^{\pm}(x,\lambda) = e^{\pm i\omega(x) \pm i\lambda x} + \int_x^\infty A^{\pm}(x,t)e^{\pm i\lambda t}dt, \lambda \in \overline{\mathbb{C}}_{\pm}, \tag{2.2}$$

where $\omega(x) = \int_x^\infty U(t)dt$ and the kernels $A^{\pm}(x,t)$ are expressed in terms of U and V. $A^{\pm}(x,t)$ are continuously differentiable with respect to their arguments and

$$|A^{\pm}(x,t)| \leq Cp\left(\frac{x+t}{2}\right)\exp[q(x)], \tag{2.3}$$

$$|A^{\pm}_x(x,t)|, |A^{\pm}_t(x,t)| \leq C\left[p^2(\frac{x+t}{2}) + \theta\left(\frac{x+t}{2}\right)\right] \tag{2.4}$$

where

$$p(x) = \int_x^\infty [|V(t)| + |U'(t)|]dt,$$

$$q(x) = \int_x^\infty [t|V(t)| + 2|U(t)|]dt,$$

$$\theta(x) = \frac{1}{4}[|V(x)| + |U(x)|^2 + |U'(x)|],$$

and $C > 0$ is a constant. Therefore $f^\pm(x,\lambda)$ are analytic with respect to λ in $\mathbb{C}_\pm := \{\lambda : \lambda \in \mathbb{C}, \pm\text{Im}\lambda > 0\}$ and continuous on the real axis. The solution $f^\pm(x,\lambda)$ also satisfy

$$f^\pm(x,\lambda) = e^{\pm i\lambda x}[1 + o(1)], \ \lambda \in \overline{\mathbb{C}}_+, \ x \to \infty, \tag{2.5}$$

$$f_x^\pm(x,\lambda) = e^{\pm i\lambda x}[\pm i\lambda + o(1)], \ \lambda \in \overline{\mathbb{C}}_+, \ x \to \infty. \tag{2.6}$$

Let $\widehat{f}^+(x,\lambda)$ and $\widehat{f}^-(x,\lambda)$ denote the solutions of (1.1), subject to the condition

$$\lim_{x\to\infty} \widehat{f}^\pm(x,\lambda)e^{\pm i\lambda x} = 1, \lambda \in \overline{\mathbb{C}}_\pm.$$

The solutions $\widehat{f}^+(x,\lambda)$ and $\widehat{f}^-(x,\lambda)$ are analytic with respect to λ in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively, and continuous on the real axis and

$$\widehat{f}^\pm(x,\lambda) = e^{\mp i\lambda x}[1 + o(1)], \lambda \in \overline{\mathbb{C}}_\pm, x \to \infty, \tag{2.7}$$

$$\widehat{f}_x^\pm(x,\lambda) = e^{\mp i\lambda x}[\mp i\lambda + o(1)], \lambda \in \overline{\mathbb{C}}_\pm, x \to \infty. \tag{2.8}$$

According to (2.5)–(2.8), the Wronskian of the solutions $\widehat{f}^+(x,\lambda)$ and $\widehat{f}^-(x,\lambda)$ is

$$W[f^\pm(x,\lambda), \widehat{f}^\pm(x,\lambda)] = \mp 2i\lambda, \lambda \in \overline{\mathbb{C}}_\pm \setminus \{0\}. \tag{2.9}$$

So $f^\pm(x,\lambda)$ and $\widehat{f}^\pm(x,\lambda)$ provide the fundamental solutions of (1.1) for $\lambda \in \overline{\mathbb{C}}_\pm \setminus \{0\}$. Using (2.9) we find that the functions

$$E^\pm(x,\lambda) = \mp\frac{1}{2i\lambda}\{f^\pm(x,\lambda)\int_x^\infty F(t)\widehat{f}^\pm(t,\lambda)dt - \widehat{f}^\pm(x,\lambda)\int_x^\infty F(t)f^\pm(t,\lambda)dt.$$

$$-H^\pm(\lambda)\widehat{f}^\pm(x,\lambda) - \widehat{H}^\pm(\lambda)f^\pm(x,\lambda)\}, \ \lambda \in \overline{\mathbb{C}}_\pm\setminus\{0\} \tag{2.10}$$

are the solutions of the BVP (1.3)–(1.4), where

$$H^{\pm}(\lambda) = \int_0^{\infty} F(x)f^{\pm}(x,\lambda)dx + \alpha f_x^{\pm}(0,\lambda) + \beta f^{\pm}(0,\lambda),$$
$$\widehat{H}^{\pm}(\lambda) = \int_0^{\infty} F(x)\widehat{f}^{\pm}(x,\lambda)dx + \alpha \widehat{f}_x^{\pm}(0,\lambda) + \beta \widehat{f}(0,\lambda). \tag{2.11}$$

3. Eigenvalues and Spectral Singularities of (1.3)–(1.4)

We also denote the sets of eigenvalues and spectral singularities of the BVP (1.3)–(1.4) by σ_d and σ_{ss}, respectively.

Lemma 3.1 — If (2.1) and

$$\sup_{x\in\mathbb{R}_+} \{|F(x)| \exp(\varepsilon x^{1+\delta})\} < \infty, \varepsilon > 0, \delta > 0 \tag{3.1}$$

hold, then

$$\sigma_d = \{\lambda : \ \lambda \in \mathbb{C}_+, \ H^+(\lambda) = 0\} \cup \{\lambda : \ \lambda \in \mathbb{C}_-, \ H^-(\lambda) = 0\}, \tag{3.2}$$

where the functions $H^{\pm}$ are defined by (2.11).

Proof: Let $\lambda_0 \in \mathbb{C}_+$. From (2.5) we get that $f^+(x,\lambda_0) \in L_2(\mathbb{R}_+)$ and $\widehat{f}^+(x,\lambda_0) \notin L_2(\mathbb{R}_+)$. Since

$$\widehat{f}^+(x,\lambda_0) \int_x^{\infty} F(t) f^+(t,\lambda_0)dt = O(e^{-\frac{\varepsilon}{2}x^{1+\delta}}), \ \ x \to \infty$$

and

$$f^+(x,\lambda_0) \int_x^{\infty} F(t) \widehat{f}^+(t,\lambda_0)dt = O(e^{-\frac{\varepsilon}{2}x^{1+\delta}}), \ \ x \to \infty,$$

it follows from (2.10) that $E^+(x,\lambda_0)$ belong to $L_2(\mathbb{R}_+)$ if and only if $H^+(\lambda_0) = 0$. ■

Analogously to the BVP (1.3)–(1.4), we have

$$\sigma_{ss} = \{\lambda : \lambda \in \mathbb{R}^*, \, H^+(\lambda) = 0\,\} \cup \{\lambda : \lambda \in \mathbb{R}^*, \, H^-(\lambda) = 0\,\}, \tag{3.3}$$

([2]).

To investigate the quantitative properties of the eigenvalues and the spectral singularities of the BVP (1.3)–(1.4) we need to discuss the quantitative properties of the zeros of H^+ and H^-in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively.For the sake of simplicity, we will consider only the zeros of H^+ in $\overline{\mathbb{C}}_+$. Similar procedure may also be employed for the zeros of H^-in $\overline{\mathbb{C}}_-$.

Let us define

$$N_1^+ = \{\lambda : \ \lambda \in \mathbb{C}_+, \ H^+(\lambda) = 0\}, \ N_2^+ = \{\lambda : \ \lambda \in \mathbb{R}, \, H^+(\lambda) = 0\,\}.$$

Lemma 3.2 — If (2.1) and (3.1) hold, then

(i) The set N_1^+ *is* bounded and has at most a countable number of elements, and the limit points can lie only in a bounded subinterval of the real axis.

(ii) The set N_2^+ is compact and $\mu(N_2^+) = 0$, where $\mu(N_2^+)$ denotes the linear Lebesgue measure of N_2^+.

PROOF: Using (2.2) and (2.3) we get the function H^+ is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and

$$H^+(\lambda) = \kappa + \rho\lambda + \int_0^\infty \varphi(t)e^{i\lambda t}dt, \tag{3.4}$$

where

$$\begin{aligned} \kappa &= -i\alpha U(0)e^{i\omega(0)} - \alpha A^+(0,0) + \beta e^{i\omega(0)}, \\ \rho &= i\alpha e^{i\omega(0)} \end{aligned}$$

and

$$\varphi(t) = \alpha A_x^+(0,t) + \beta A^+(0,t) + e^{i\omega(t)}F(t) + \int_0^t F(\xi)A^+(\xi,t)d\xi. \tag{3.5}$$

Since $\varphi \in L_1(\mathbb{R}_+)$ by (3.5), consequently (3.4) implies that

$$H^+(\lambda) = \kappa + \rho\lambda + o(1), \ \lambda \in \overline{\mathbb{C}}_+, \ |\lambda| \to \infty. \tag{3.6}$$

Equation (3.11) shows the boundedness of the sets N_1^+ and N_2^+. From the analyticity of the function H^+ in $\mathbb{C}_+$ we get that N_1^+ has at most a countable number of elements and its limit points can lie only in a bounded subinterval of the real axis. By the boundary value uniqueness theorem of analytic functions we obtain that the set N_2^+ is closed and $\mu(N_2^+) = 0$. ■

From (3.2), (3.3) and Lemma 3.2 we get the following

***Theorem* 3.1** — Under conditions (2.1) and (3.1)

(i) The set of eigenvalues of the BVP (1.3) –(1.4) is bounded, is no more than countable and its limit points can lie only in a bounded subinterval of the real axis.

(ii) The set of spectral singularities of the BVP (1.3)–(1.4) is bounded and its linear Lebesgue measure is zero.

Definition 3.1 — ([14]) The multiplicity of zero of the function H^+ (or H^-) in $\overline{\mathbb{C}}_+$ (or $\overline{\mathbb{C}}_-$) is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.3) –(1.4).

Now, let us assume that

$$\lim_{x\to\infty} U(x) = 0, \ \sup_{x\in\mathbb{R}_+} [[|V(x)| + |U'(x)|] \exp(\varepsilon x)] < \infty, \varepsilon > 0. \tag{3.7}$$

***Theorem* 3.2** — Under conditions (3.1) and (3.7) the BVP (1.3)–(1.4) has a finite number of eigenvalues and spectral singularities and each of them is of a finite multiplicity.

PROOF: From (2.3)–(2.4) we find that

$$|A^+(0,t)|, |A_x^+(0,t)| \le C \exp\left(-\frac{\varepsilon}{2}t\right), \tag{3.8}$$

$$\int_0^t |F(\xi)||A^+(\xi,t)|d\xi \le C \exp\left(-\frac{\varepsilon}{2}t\right), \tag{3.9}$$

where $C > 0$ is a constant. (3.5), (3.8) and (3.9) imply that

$$|\varphi(t)| \le C \exp\left(-\frac{\varepsilon}{2}t\right). \tag{3.10}$$

It follows from (3.4) and (3.10) that the function H^+ has an analytic continuation from the real axis to the half plane $\operatorname{Im} \lambda > -\frac{\varepsilon}{2}$. So the limit points of the sets N_1^+ and N_2^+ can not lie in $\mathbb{R}$, i.e., the bounded sets N_1^+ and N_2^+ have no limit points (see Lemma 3.2). Therefore, we have the finiteness of the zeros of H^+ in $\overline{\mathbb{C}}_+$. Moreover all zeros of H^+ in $\overline{\mathbb{C}}_+$ has a finite multiplicity. Using (3.2) and (3.3) we obtain that the BVP (1.3) –(1.4) has a finite number of eigenvalues and spectral singularities and each of them is of a finite multiplicity. ■

It is clear that the conditions (3.1) and (3.7) guarantees of the analytic continuation of the functions $H^+$and H^- from the real axis to lower and upper half-planes, respectively. So the finiteness of eigenvalues and spectral singularities of the BVP (1.3)–(1.4) are obtained as a result of this analytic continuation.

Now let us suppose that, for some $\varepsilon > 0$ and $\frac{1}{2} \le \gamma < 1$,

$$\lim_{x\to\infty} U(x) = 0, \ \sup_{x\in\mathbb{R}_+} [[|V(x)| + |U'(x)|] \exp(\varepsilon x^\gamma)] < \infty, \tag{3.11}$$

holds. Note that the condition (3.11) is weaker than (3.7). It is obvious that under conditions (3.1) and (3.7), the function H^+ is analytic in $\mathbb{C}_+$ and infinitely many differentiable on the real axis. But H^+ does not have an analytic continuation from the real axis to lower half-plane. Similarly, H^- does not have an analytic continuation from the real axis to the upper half-plane. Consequently, under conditions (3.1) and (3.11) the finiteness of eigenvalues and spectral singularities of the BVP (1.3)–(1.4) can not be shown in a similar to Theorem 3.2.

Let us denote the sets of all limit points of N_1^+ and N_2^+ by N_3^+ and N_4^+, respectively, the set of all zeros of H^+ with infinite multiplicity in $\overline{\mathbb{C}}_+$ by N_5^+. Analogously define the sets N_3^-, N_4^- and N_5^-.

It is evident from the boundary uniqueness theorem of analytic functions that

$$N_1^\pm \cap N_5^\pm = \phi, \ \ N_3^\pm \subset N_2^\pm, \ \ N_4^\pm \subset N_{2,}^\pm \ N_5^\pm \subset N_{2,}^\pm$$

and

$$\mu\left(N_3^\pm\right) = \mu\left(N_4^\pm\right) = \mu(N_5^\pm) = 0.$$

Since all derivatives of the H^+ and H^- are continuous up to the real axis, we get

$$N_3^{\pm} \subset N_5^{\pm}, N_4^{\pm} \subset N_5^{\pm}. \tag{3.12}$$

We will use the following uniqueness theorem for the analytic functions on the upper half-plane, to prove the next result.

***Theorem* 3.3** — ([2]). Let us assume that the function f is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis and there exists $T > 0$ such that

$$|f^{(n)}(z)| \leq C_n, \ n = 0, 1, \ldots, \lambda \in \mathbb{C}_+, \ |z| < 2T, \tag{3.13}$$

and

$$\left| \int_{-\infty}^{-T} \frac{\ln |f(x)|}{1+x^2} dx \right| < \infty, \quad \left| \int_{T}^{\infty} \frac{\ln |f(x)|}{1+x^2} dx \right| < \infty. \tag{3.14}$$

If the set Q with linear Lebesgue measure zero, is the set of all zeros of the function f with infinite multiplicity and if

$$\int_0^a \ln G(s) d\mu(Q_s) = -\infty,$$

then $f(z) \equiv 0$, where $G(s) = \inf_n \frac{C_n s^n}{n!}$, $n = 0, 1, \ldots, \mu(Q_s)$ is the linear Lebesgue measure of s-neighborhood of Q and a is an arbitrary positive constant.

Lemma 3.3 — If (3.1) and (3.11) hold, then $N_5^+ = N_5^- = \phi$.

PROOF: We prove that $N_5^+ = \phi$. The case $N_5^- = \phi$ is similar. It follows from (3.4) – (3.5) that the function H^+ is analytic in $\mathbb{C}_+$ and all of its derivatives are continuous up to the real axis. Moreover, by Lemma 3.2 for sufficiently large $T > 0$ we have

$$\left| \int_{-\infty}^{-T} \frac{\ln |H^+(\lambda)|}{1+\lambda^2} d\lambda \right| < \infty, \quad \left| \int_{T}^{\infty} \frac{\ln |H^+(\lambda)|}{1+\lambda^2} d\lambda \right| < \infty. \tag{3.15}$$

From (3.4) we obtain that

$$|H^+(\lambda) - i\lambda| < \infty, \ \lambda \in \overline{\mathbb{C}}_+, \tag{3.16}$$

and

$$\left| \frac{d^n}{d\lambda^n} H^+(\lambda) \right| \leq B_n, \ \lambda \in \overline{\mathbb{C}}_+, n = 1, 2, \ldots, \tag{3.17}$$

where

$$B_1 = 1 + \int_0^\infty t|\varphi(t)|dt, \ B_n = \int_0^\infty t^n |\varphi(t)| dt, \ n = 2, 3, \ldots \ . \tag{3.18}$$

Using (2.3) and (2.4) we find that

$$|A^+(x,t)|, |A_x^+(x,t)| \leq C \exp\left[-\frac{\varepsilon}{2} (\frac{x+t}{2})^\gamma \right],$$

and consequently

$$|\varphi(t)| \leq C \exp\left[-\frac{\varepsilon}{2}(\frac{t}{2})^{\gamma}\right], \tag{3.19}$$

by (3.5), where $C > 0$ is a constant. From (3.16)–(3.19) we have that

$$\left|\frac{d^n}{d\lambda^n} H^+(\lambda)\right| \leq K_n,\ n = 0, 1, \ldots,\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| < 2T, \tag{3.20}$$

where

$$K_n = C \int_0^{\infty} t^n \exp\left[-\frac{\varepsilon}{2}\left(\frac{t}{2}\right)^{\gamma}\right] dt. \tag{3.21}$$

It is easy to see from (3.15) and (3.20) that H^+ satisfies (3.13) and (3.14). Since the function H^+ is not equal to zero identically, then by Theorem 3.3, N_5^+ satisfies

$$\int_0^a \ln G(s) d\mu(N_{5,s}^+) > -\infty, \tag{3.22}$$

where $G(s) = \inf\limits_n \frac{K_n s^n}{n!}$, $\mu(N_{5,s}^+)$ is the linear Lebesgue measure of s-neighborhood of N_5^+, and the constant K_n is defined by (3.21).

Now we will obtain the following estimates for K_n,

$$K_n \leq D d^n n! n^{n\frac{1-\gamma}{\gamma}}, \tag{3.23}$$

where D and d are constants depending on C, ε and γ. Substituting (3.23) in the definition of $G(s)$ we arrive at

$$G(s) \leq D \exp\left\{-\frac{1-\gamma}{\gamma} e^{-\frac{1}{1-\gamma}} d^{-\frac{\gamma}{1-\gamma}} s^{-\frac{\gamma}{1-\gamma}}\right\}. \tag{3.24}$$

(3.22) and (3.24) imply that

$$\int_0^a s^{-\frac{\gamma}{1-\gamma}} d\mu(N_{5,s}^+) < \infty. \tag{3.25}$$

Since $\frac{\gamma}{1-\gamma} \geq 1$, (3.25) holds for arbitrary s if and only if $\mu(N_{5,s}^+) = 0$ or $N_5^+ = \phi$. ■

***Theorem* 3.4** — Under conditions (3.1) and (3.11) the BVP (1.3)–(1.4) has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.

PROOF: To prove the theorem we have to show that the function H^+ and H^- have a finite number of zeros with a finite multiplicities in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. From (3.12) and Lemma 3.3 we get that $N_3^{\pm} = N_4^{\pm} = \phi$. So the bounded sets $N_1^{\pm}$ and $N_2^{\pm}$ have no limit points, i.e., the function H^+ and H^- have only a finite number of zeros in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. Since $N_5^{\pm} = \phi$, these zeros are of finite multiplicity. ■

From Theorem 3.4 it is seen that the weakest conditions which guaranties the finiteness of eigenvalues and spectral singularities of the BVP (1.3) –(1.4) are

$$\lim_{x\to\infty} U(x) = 0,\ \sup_{x\in\mathbb{R}_+} [[|V(x)| + |U'(x)|] \exp(\varepsilon x^{1\backslash 2})] < \infty,\ \sup_{x\in\mathbb{R}_+} [|F(x)| \exp(\varepsilon x^{1+\delta})] < \infty, \tag{3.26}$$

for some $\varepsilon > 0, \delta > 0$.

REFERENCES

1. M. Adıvar and E. Bairamov, Spectral properties of non-selfadjoint difference operators, *J. Math. Anal. Appl.*, **261** (2001), 461–478.

2. E. Bairamov, Ö. Çakar and A. M. Krall, Spectrum and spectral singularities of a quadratic pencil of a Schrödinger operator with a general boundary condition, *J. Differential Equations,* **151** (1999), 252–267.

3. E. Bairamov, Ö. Çakar and A. M. Krall, An eigenfunction expansion for a quadratic pencil of a Schrödinger operator with spectral singularities, *J. Differential Equations*, **151** (1999), 268–289.

4. E. Bairamov, Ö. Çakar and A. M. Krall, Non-selfadjoint difference operators and Jacobi matrices with spectral singularities, *Math. Nachr.*, **229** (2001), 5–14.

5. E. Bairamov, Ö. Çakar and A. M. Krall, Spectral analysis of non-selfadjoint discrete Schrödinger operators with spectral singularities, *Math. Nachr.*, **231** (2001), 89–104.

6. E. Bairamov, Ö. Çakar and A. M. Krall, Spectral Properties including spectral singularities of a quadratic pencil of Schrödinger operators on the whole real axis, *Quastiones Mathematicae,* **26** (2003), 15–31.

7. E. Bairamov and A.O.Çelebi, Spectrum and spectral expansion for the non-selfadjoint discrete Dirac operators, *Quart. J. Math. Oxford* **50** (2), (1999), 371–384.

8. E. Bairamov, Ö. Karaman, Spectral singularities of Klein-Gordon s-wave equations with an integral boundary condition, *Acta Math. Hungar.*, **97** (1–2) (2002), 121–131.

9. E. Bairamov,A. M. Krall, Dissipative operators generated by the Sturm-Liouville differential expression, *J. Math. Anal. Appl.*, **254** (2001), 178–190.

10. M. G. Gasymov, Expansion in terms of the solutions of a scattering theory problem for the non-selfadjoint Schrödinger equation, *Soviet Math. Dokl.*, **9** (1968), 390–393.

11. M. Jaulent and C. Jean, The inverse s-wave scattering problem for a class of potentials depending on energy, *Comm. Math. Phys.*, 28 (1972), 177–220.

12. V. E. Lyance, A differential operator with spectral singularities I, II, *Amer. Math. Soc. Trans. Ser.*, **60**(2) (1967), 185–225, 227–283, *Amer. Math. Soc., Providence,* 1967.

13. M. A. Naimark, Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint operator of second order on a semi-axis, *Amer. Math. Soc. Trans. Ser.*, **16** (2) (1960), 103–193, *Amer. Math. Soc., Providence,* 1960.

14. M. A. Naimark, Linear Differential Operators II, Ungar, New York, 1968.