Of concern is the following Cauchy problem for a semilinear evolution equation with a nonlocal initial condition

\[
\begin{align*}
&\frac{du}{dt}(t) = Au(t) + F(t, u(t)), \quad t > 0, \\
&u(0) + H(t_1, \ldots, t_p, u) = u_0,
\end{align*}
\]

where \( A \) is an almost sectorial operator (not necessarily densely defined) and \( F, H \) are given functions. The existence and uniqueness of mild and classical solutions for the Cauchy problem, under various hypotheses, are proved. Then, as an immediate application of this result, we investigate the existence and uniqueness of classical solutions for the following nonlinear nonlocal Cauchy problem

\[
\begin{align*}
&\frac{du}{dt}(t) = \frac{1}{G(u)} Au(t) + F(u(t)), \quad t > 0, \\
&u(0) + H(t_1, \ldots, t_p, u) = u_0,
\end{align*}
\]

where \( G \) is a given functional. This paper is a continuation of the investigation of the earlier articles [22, 23] concerning the theory of almost sectorial operators and its applications.

**Key Words:** Almost sectorial operator; nonlocal conditions; abstract Cauchy problems

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1. Introduction

Let \((X, \| \cdot \|)\) denote a complex Banach space. As usual, \(\sigma(A)\) stands for the spectrum of the linear operator \(A\) with domain \(D(A)\), and \(R(z; A) = (zI - A)^{-1}\), \(z \in \mathbb{C} - \sigma(A)\) of bounded linear operators stands for the resolvent of \(A\). Denote by \(C([0, T]; X)\) the space of continuous functions with uniform norm topology. The following definition is taken from [22] (see also [23]):

**Definition 1.1** — Let \(-1 < \gamma < 0\) and \(0 < \omega < \pi/2\). By \(\Theta_{\gamma, \omega}\) we denote the set of all closed linear operators \(A : D(A) \subseteq X \rightarrow X\) which satisfy

1. \(\sigma(A) \subseteq S_\omega = \{z \in \mathbb{C} - \{0\}; \arg z \leq \omega\} \cup \{0\}\), and
2. for every \(\omega < \mu < \pi\) there exists a constant \(C_\mu\) such that
   \[
   \|R(z; A)\| \leq C_\mu|z|^{-\gamma} \text{ for all } z \in \mathbb{C} - S_\mu.
   \]

A linear operator \(A\) will be called an \(\omega-\)almost sectorial operator if \(A \in \Theta_{\gamma, \omega}\).

**Remark 1.1** : Let \(\gamma = 1\) in Definition 1.1, then, note that \(A\) is a sectorial operator, which appears very often in the applications since many elliptic differential operators are sectorial when they are considered in the \(L^p\)-spaces or in the space of continuous functions (see [13, 21]). However, in some spaces of more regular functions, such as the spaces of Hölder continuous functions, these elliptic operators do no longer satisfy the estimate (2) in Definition 1.1 with \(\gamma = 1\), which was pointed out in [18, 25]. Moreover, for these operators estimate (2) in Definition 1.1 with \(-1 < \gamma < 0\) can be obtained (see [22, Example 2.3 and Example 2.4]).

In this paper, we first consider the following Cauchy problem for the semilinear evolution equation with a nonlocal initial condition:

\[
\begin{cases}
u'(t) = Au(t) + F(t, u(t)), & t \in (0, T], \\
u(0) + H(t_1, \ldots, t_p, u) = u_0,
\end{cases}
\]

where \(0 < t_1 < \cdots < t_{p-1} < t_p < T\) \((p \in \mathbb{N})\), \(A\) is an \(\omega-\)almost sectorial operator (not necessarily densely defined), and \(F : [0, T] \times X \rightarrow X\), \(H : (0, T)^p \times X \rightarrow D(A)\)

\[
(t_1, \ldots, t_p, u) \mapsto H(t_1, \ldots, t_p, u)
\]

are given functions. In many studies of nonlocal Cauchy problems, the mapping \(H\) is given by

\[
H(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) := \sum_{j=1}^{N} C_i u(t_i),
\]
where \( C_i (i = 1, \ldots, p) \) are given constants, which is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube (see [12]).

**Remark 1.2**: In these cases, \( H(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) \) allows the measurements at \( t = 0, t_1, \ldots, t_p, u(t_1), \ldots, u(t_p) \), rather than just at \( t = 0 \). So more information is available.

There has been increasing interest in studying semilinear Cauchy problems for various classes of differential and integrodifferential equations with nonlocal initial conditions in Banach spaces (cf., e.g., [2, 5-8, 15-17, 19, 20]), especially since the first work [5] in 1991. Some recent advances on nonlocal evolution equations can be found in [1, 9] and references therein. As indicated in [2, 5, 12] and references therein, the Cauchy problem with nonlocal initial condition \( u(0) + H(u) = u_0 \) can be applied in physics with better effect than the classical Cauchy problem with initial condition \( u(0) = u_0 \). In [3, 4, 10, 14, 24], one can find more detailed applications of this type of equations, such as in the flow of fluid through fissured rocks, the propagation of long waves of small amplitudes, thermodynamics and shear in second order fluids.

However, much of the previous research on the nonlocal Cauchy problems was done provided the linear operator \( A \) is a “regular operator”, that is, where it was assumed that \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup, an integrated semigroup, an analytic semigroup or a compact semigroup. When \( A \) is an “irregular operator”, especially, \( A \) is an \( \omega \)-almost sectorial operator (not necessarily densely defined), much less is known about the existence of solutions of the Cauchy problems with nonlocal initial conditions. Stimulated by these works, we first investigate, in this paper, the existence and uniqueness of mild and classical solutions to the nonlocal Cauchy problem (1.1). Then, on the basis of this result, we establish the existence and uniqueness of classical solutions to the following nonlocal Cauchy problem

\[
\begin{cases}
    u'(t) = \frac{1}{G(u)} Au(t) + F(u(t)), & t \in (0, T], \\
    u(0) + H(t_1, \ldots, t_p, u) = u_0,
\end{cases}
\]

where the function \( G : X \to (0, +\infty) \) is given. This paper is a continuation of the investigation of the earlier articles [22, 23] concerning the theory of almost sectorial operators and its applications.

**Remark 1.3**: In the study of reaction-diffusion equations that model population dynamics, the diffusion coefficient often depends on a nonlocal quantity related to the total population density in the domain, i.e., the diffusion of individuals is guided by the global state of the population in the medium. Thus, it is natural and important for us to study the abstract Cauchy problems (1.1) and (1.2) with nonlocal conditions (cf. [11]).

**Example 1.1**: In the case of a migration of population, for instance of bacteria in a container, it is obvious that the environment is of prime importance and one will easily imagine that

\[
G = G \left( \int_{\Omega} u(t, x) \right),
\]
where $u$ describes the density of population at time $t$ and at position $x \in \Omega$, i.e., the velocity of the migration depends on the total population.

2. Preliminaries

If $\mathcal{A} \in \Theta_{\gamma, \omega}$, that is, $\mathcal{A}$ is an $\omega$-almost sectorial operator, then it follows from [22] that the part of $-\mathcal{A}$ in the Banach space $(\overline{D(\mathcal{A})}, \| \cdot \|)$ is a complete generator of an analytic semigroup $\{ T(\lambda); \lambda \in \mathbb{C} - \{0\}, |\arg \lambda| < \pi/2 - \omega \}$, which is given by

$$T(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_{\theta}} e^{-\lambda z} \mathcal{R}(z; \mathcal{A}) dz,$$

(2.1)

where $\Gamma_{\theta} = \{ \rho e^{-i\theta}; \rho > 0 \} \cup \{ \rho e^{i\theta}; \rho > 0 \}$ with $\omega < \theta < \mu$, is oriented counter-clockwise. Consequently, it follows from (2.1) that

$$\| T(t) \| \leq M t^{-(\gamma + 1)},$$

for all $t > 0$, where $M > 0$ is a constant.

Let $\Sigma$ denote the continuity set of the semigroup $\{ T(t) \}_{t \geq 0}$, that is,

$$\Sigma = \left\{ x \in X; \lim_{\substack{t \to 0; \, t > 0}} T(t)x = x \right\},$$

and for fixed $u_0 \in \Sigma$, set

$$M_0 := \sup_{t \in [0, T]} \| T(t)u_0 \| < +\infty.$$

For an operator $\mathcal{A}$ in the class $\Theta_{\gamma, \omega}$ with $0 < \omega < \frac{\pi}{2}$, it follows that for $\alpha > 1 + \gamma$, $\mathcal{A}^\alpha$ can be defined as a closed linear invertible operator and $\mathcal{A}^{-\alpha}$ is bounded. Denote by $X^\alpha$ the Banach space $D(\mathcal{A}^\alpha)$ endowed with the graph norm $\| u \|_{X^\alpha} = \| \mathcal{A}^\alpha u \|$ for $u \in X^\alpha$ (see [22]).

**Definition 2.1** — A solution $u(t) \in C([0, T]; X)$ of the integral equation

$$u(t) = T(t)u_0 - T(t)H(t_1, \cdots, t_p, u) + \int_0^t T(t-s)F(s, u(s))ds,$$  

(2.2)

for $t \in [0, T]$ and $u_0 \in \Sigma$, is said to be a mild solution of (1.1).

**Definition 2.2** — A classical solution of (1.1) is a function

$$u \in C([0, T]; X) \cap C^1((0, T]; X), \quad u \in D(\mathcal{A}) \quad (0 < t \leq T)$$

satisfying (2.2) and $u(0) + H(t_1, \cdots, t_p, u) = u_0$.

Let $I := [0, T]$ and $B_R := \{ \varphi \in X; \| \varphi \| \leq R \}$, where $R$ is a positive constant. For the sake of convenience we list the assumptions to be used in the paper as follows:
\((H_1)\) \(F : I \times X \to X\) is continuous in \(t\) on \(I\) and there exist constants \(L_1 > 0\) and \(M_1 > 0\) such that for all \(t \in I\) and \(u, v \in \overline{B}_R\),

\[
\|F(t, u) - F(t, v)\| \leq L_1\|u - v\| \tag{2.3}
\]

and

\[
M_1 = \max_{t \in I} \|F(t, 0)\|.
\]

\((H_2)\) \(H : (0, T)^p \times X \to X^1\) and there exists a nonnegative function \(\Psi\) satisfying

\[
\Psi(\lambda \tau) \leq \lambda \Psi(\tau), \quad \forall \lambda > 0, \tau \in C(I, [0, \infty)), \text{ and}
\]

\[
\Psi(\tau_1) \leq \Psi(\tau_2), \quad \forall \tau_1, \tau_2 \in C(I, [0, \infty)) \text{ with } \tau_1(t) \leq \tau_2(t) \quad (t \in I), \tag{2.4}
\]

such that

\[
\|H(t_1, \ldots, t_p, u) - H(t_1, \ldots, t_p, v)\|_{X^1} \leq \Psi(\|u - v\|),
\]

for all \(u, v \in C(I; \overline{B}_R)\).

\((H_3)\) There exists a constant \(\delta \in (0, 1)\) such that

\[
M_0 + K_0 + M(L_1 R + M_1) \frac{T^{-\gamma}}{-\gamma} < \delta R
\]

where

\[
K_0 := \sup_{t \in I, u \in \overline{B}_R} \|T(t)H(t_1, \ldots, t_p, u)\|.
\]

\((H_4)\) \(G : X \to (0, +\infty)\) and there exists a constant \(K_1\) such that

\[
|G(u) - G(v)| \leq K_1\|u - v\|, \quad u, v \in \overline{B}_R.
\]

We set

\[
\Theta_A := \{A H(t_1, \ldots, t_p, u); u \in C(I; X)\},
\]

\[
K := M^2 L_1^2 \frac{\Gamma(-2\gamma - 1)}{4^{-\gamma - 1}}, \quad k_0 := \max\{2, KT e^{(K+2)T}\}.\]
Now, we are in a position to present and prove our first result.

**Theorem 3.1** — Let \( \mathcal{A} \in \Theta_{\gamma, \omega} \) with \( 0 < \omega < \frac{\pi}{2} \) and \( -1 < \gamma < -\frac{1}{2} \), and let the hypotheses \((H_1) - (H_3)\) hold, \( \Theta_{\mathcal{A}} \subset \Sigma \), and \( \sqrt{2k_0} \Psi \left( \| \mathcal{A}^{-1} \| + M \frac{T-\gamma}{-\gamma} \right) < 1 \). Then for every \( u_0 \in \Sigma \), (1.1) has a unique mild solution on \( I \).

**Proof:** Set

\[
Z := \{ u \in C(I; X); u \in \Sigma \}, \quad Z_R := \{ u \in Z; u(t) \in B_R, t \in I \}.
\]

It is evident that \( Z_R \) is a bounded closed subset of \( Z \). Let \( u_1 \in Z_R \) and set \( u_{0,1} := u_0 + H(t_1, \ldots, t_p, u_1) \). Thus, we have \( u_{0,1} \in \Sigma \) under the hypothesis \((H_2)\) since \( u_0 \in \Sigma \). On \( Z_R \) we define a mapping \( \Phi \) by

\[
(\Phi u)(t) = T(t)u_{0,1} + \int_0^t T(t-s)F(s, u(s))ds,
\]

for all \( t \in I \), then \( \Phi(Z) \subset C(I; X) \) and a straightforward calculation yields

\[
\| (\Phi u)(t) \| \leq \| T(t)u_0 \| + \| T(t)H(t_1, \ldots, t_p, u_1) \|
\]

\[
+ \int_0^t \| T(t-s)F(s, u(s)) \| ds
\]

\[
\leq M_0 + K_0 + \int_0^t M(t-s)^{-(\gamma+1)}\| F(s, u(s)) - F(s, 0) \| ds
\]

\[
+ \int_0^t M(t-s)^{-(\gamma+1)}\| F(s, 0) \| ds
\]

\[
\leq M_0 + K_0 + M(L_1 R + M_1) \frac{t-\gamma}{-\gamma}
\]

\[
\leq R,
\]

by the hypotheses \((H_1), (H_2)\) and \((H_3)\). Next we prove that \( (\Phi u)(t) \in \Sigma \) for all \( t \in I \). Fix \( t_0 \in (0, T] \). Then for every \( t > 0 \) we have that

\[
T(t)(\Phi u)(t_0) = T(t)T(t_0)u_{0,1} + \int_0^{t_0} T(t)T(t_0-s)F(s, u(s))ds
\]

\[
= T(t+t_0)u_{0,1} + \int_0^{t_0} T(t+t_0-s)F(s, u(s))ds.
\]
It follows from the analyticity of the semigroup \( \{T(\lambda); \lambda \in \mathbb{C}\} \) that

\[
\lim_{t \to 0^+} T(t + t_0)u_{0,1} = T(t_0)u_{0,1}.
\]

On the other hand, since

\[
\|T(t + t_0 - s)F(s, u(s))\| \leq M(M_1 + L_1 R) t^{-(\gamma + 1)},
\]

for \( s \in [0, t_0] \), by using the dominated convergence theorem we obtain

\[
\lim_{t \to 0^+} \int_0^t T(t + t_0 - s)F(s, u(s))ds = \int_0^{t_0} T(t_0 - s)F(s, u(s))ds.
\]

Thus \( \Phi \) maps \( Z_R \) into itself. Furthermore, for all \( u, v \in Z_R \) we get

\[
\| (\Phi u)(t) - (\Phi v)(t) \| \leq \int_0^t \| T(t - s) \left( F(s, u(s)) - F(s, v(s)) \right) \| ds
\]

\[
\leq \int_0^t L_1 \| u(s) - v(s) \| ds
\]

\[
\leq ML_1 \frac{t^{-\gamma}}{-\gamma} \max_{t \in I} \| u(t) - v(t) \|
\]

\[
\leq \delta \max_{t \in I} \| u(t) - v(t) \|
\]

in view of hypothesis \((H_3)\), which implies that \( \Phi \) is a contractive mapping on \( Z_R \). Thus \( \Phi \) has a unique fixed point \( u_2 \in Z_R \). Therefore, by induction we can define a sequence \( \{u_n\}_{n=2}^\infty \) by

\[
u_n(t) = T(t)u_{0,n-1} + \int_0^t T(t - s)F(s, u_n(s))ds, \tag{3.2}
\]

where \( u_{0,n-1} = u_0 - H(u_{n-1}) \). We deduce from (3.2), \((H_1)\) and \((H_2)\) that

\[
\| u_3(t) - u_2(t) \| \leq \| T(t) \left( H(t_1, \cdots, t_p, u_2) - H(t_1, \cdots, t_p, u_1) \right) - \left( H(t_1, \cdots, t_p, u_2) - H(t_1, \cdots, t_p, u_1) \right) \|
\]

\[
+ \| H(t_1, \cdots, t_p, u_2) - H(t_1, \cdots, t_p, u_1) \|
\]

\[
+ \| H(t_1, \cdots, t_p, u_2) - H(t_1, \cdots, t_p, u_1) \|
\]

\[
+ \| H(t_1, \cdots, t_p, u_2) - H(t_1, \cdots, t_p, u_1) \|
\]
\[\begin{align*}
&+ \int_0^t \|T(t-s)(F(s,u_3(s)) - F(s,u_2(s)))\| ds \\
&\leq \int_0^t \|T(s)A(H(t_1, \cdots, t_p, u_2) - H(t_1, \cdots, t_p, u_1))\| ds \\
&+ \|A^{-1}\| \cdot \|H(t_1, \cdots, t_p, u_2) - H(t_1, \cdots, t_p, u_1)\|_{X^1} \\
&+ \int_0^t L_1 \|T(t-s)\| \cdot \|u_3(s) - u_2(s)\| ds \\
&\leq \left(\|A^{-1}\| + M \frac{T^{-\gamma}}{-\gamma}\right) \Psi(\|u_2(t) - u_1(t)\|) \\
&+ \int_0^t M L_1 (t-s)^{-(\gamma+1)} \|u_3(s) - u_2(s)\| ds.
\end{align*}\]

Set
\[w(t) := \|u_3(t) - u_2(t)\|,\]
\[a(\cdot) := \left(\|A^{-1}\| + M \frac{T^{-\gamma}}{-\gamma}\right) \Psi(\|u_2(\cdot) - u_1(\cdot)\|).\]

Then, by Hölder’s inequality, we get
\[w(t) \leq a(\cdot) + \int_0^t \left(M L_1 (t-s)^{-(\gamma+1)} e^s\right) e^{-s} w(s) ds\]
\[\leq a(\cdot) + \int_0^t \left(M^2 L_1^2 (t-s)^{-(2\gamma+2)} e^{2s} ds\right) \left(\int_0^t e^{-2s} w^2(s) ds\right)^{\frac{1}{2}} , \quad t \in I.\]

Thus,
\[w^2(t) \leq 2a^2(\cdot) + 2 \left(\int_0^t M^2 L_1^2 (t-s)^{-(2\gamma+2)} e^{2s} ds\right) \left(\int_0^t e^{-2s} w^2(s) ds\right)\]
\[\leq 2a^2(\cdot) + M^2 L_1 \frac{\Gamma(-2\gamma - 1)}{4^{-\gamma - 1}} e^{2t} \int_0^t e^{-2s} w^2(s) ds , \quad t \in I,\]
in view of \(\gamma < -\frac{1}{2}\). Now, it follows from the Bellman-Gronwall inequality that
\[w^2(t) \leq \left(2 + KT e^{(K+2)T}\right) a^2(\cdot) , \quad t \in I,\]
namely,

\[ w(t) \leq \sqrt{2k_0} a(\cdot), \quad t \in I \]

Therefore, we have

\[ \|u_3(t) - u_2(t)\| \leq \sqrt{2k_0} \left( \|A^{-1}\| + M \frac{T^{-\gamma}}{-\gamma} \right) \Psi(\|u_2(\cdot) - u_1(\cdot)\|), \quad t \in I. \]

Again using induction we get

\[ \|u_{n+1}(t) - u_n(t)\| \]

\[ \leq \sqrt{2k_0} \left( \|A^{-1}\| + M \frac{T^{-\gamma}}{-\gamma} \right) \left( \sqrt{2k_0} \Psi \left( \left( \|A^{-1}\| + M \frac{T^{-\gamma}}{-\gamma} \right) n^{-3} \right) \right) \]

\[ \times \Psi(\|u_2(\cdot) - u_1(\cdot)\|). \]

Now, by a standard argument, we know that \( \{u_n\}_{n=2}^\infty \) is a Cauchy sequence in \( C(I; X) \) provided

\[ \sqrt{2k_0} \Psi \left( \|A^{-1}\| + M \frac{T^{-\gamma}}{-\gamma} \right) < 1. \]

Therefore, there exists a function \( u \) in \( C(I; X) \) satisfying

\[ u(t) = \lim_{n \to \infty} u_n(t) \]

uniformly for \( t \in I \), which is a mild solution of (1.1) due to (3.1) and (3.2).

In the sequel, we prove the uniqueness. Let \( v \) be also a mild solution of (1.1). Then, by (2.2), we obtain

\[ \|u - v\| \leq \left( \|A^{-1}\| + M \frac{T^{-\gamma}}{-\gamma} \right) \Psi(\|u(\cdot) - v(\cdot)\|) \]

\[ + ML_1 \int_0^t (t - s)^{-(\gamma+1)}\|u(s) - v(s)\| ds, \quad t \in I. \]

Applying the generalization of Gronwall’s inequality and (2.4), we get

\[ \|u(t) - v(t)\| \leq \sqrt{2k_0} \left( \|A^{-1}\| + M \frac{T^{-\gamma}}{-\gamma} \right) \left( \sqrt{2k_0} \Psi \left( \left( \|A^{-1}\| + M \frac{T^{-\gamma}}{-\gamma} \right)^n \right) \right) \]

\[ \times \Psi(\|u(t) - v(t)\|) \]
for \( t \in I \) and \( n \geq 1 \). Letting \( n \to \infty \) in (3.3) it follows that \( u \) is a unique mild solution of (1.1). \( \square \)

**Remark 3.1:** Note that, in Theorem 3.1, we make no assumption on the density of the domain \( D(A) \) of \( A \).

In the following theorem, we establish the existence and uniqueness of classical solutions of (1.1).

**Theorem 3.2** — Let the hypotheses in Theorem 3.1 hold, with (2.3) replaced by the following conditions

1. \( F : I \times X \to X \) is continuous in \( t \) on \( I \) and there exists constants \( L_2 > 0 \) and \( \nu > \gamma + 1 \) such that
\[
\| F(t, u) - F(s, v) \| \leq L_2 (\| t - s \|^\nu + \| u - v \|), \quad t, s \in I, \ u, v \in \overline{B}_R.
\]

Suppose in addition that

2. \( F(t, u(t)) \in D(A) \) for \( t \in (0, T) \) and \( AF(\cdot, u(\cdot)) \in L^\infty((0, T); X) \);
3. \( u_0 \in D(A), Au_0 \in \Sigma \) and \( \Theta_A \subset \Sigma \).

Then (1.1) has a unique classical solution defined on \( I \).

**Proof:** We first note that \( -\gamma > 1 + \gamma \). Moreover, condition (1) and Theorem 3.1 imply that there exists a unique mild solution \( u \) to (1.1) on \( I \) which satisfies
\[
u(t) = T(t)u_0 - T(t)H(t_1, \cdots, t_p, u) + \int_0^t T(t-s)F(s, u(s))ds,
\]
for \( t \in I \) and \( u_0 \in \Sigma \).

On the other hand, we know, by hypotheses (1)-(3), that for \( t \in (0, T], h \in (0, T - t] \)
\[
\| u(t + h) - u(t) \| \leq \| T(t + h)u_0 - T(t)u_0 \|
\]
\[
+ \| T(t + h)H(t_1, \cdots, t_p, u) - T(t)H(t_1, \cdots, t_p, u) \|
\]
\[
+ \| \int_0^{t+h} T(t + h - s)f(s, u(s))ds - \int_0^t T(t - s)f(s, u(s))ds \| \leq \int_t^{t+h} \| T(s)Au_0 \| ds + \int_t^{t+h} \| T(s)AH(t_1, \cdots, t_p, u) \| ds
\]
\[
+ \int_t^{t+h} \| T(t + h - s)f(s, u(s)) \| ds
\]
\[ \int_0^t \| (T(t + h - s) - T(t - s)) f(s, u(s)) \| ds \]
\[ \leq M_0 h + \sup_{t \in I} \| T(t) A H(t_1, \cdots, t_p, u) \| h + M_2 t^{-\gamma} + M_3 h^{-\gamma} \]
\[ \leq M_4 h^{-\gamma}, \]

where \( M_i (i = 2, 3, 4) \) are positive constants. Thus \( u \in C^{-\gamma}((0, T]; X) \). Moreover, by hypothesis (1), one can show easily that \( f(\cdot, u(\cdot)) \) is Hölder continuous in \((0, T]\) with exponent \(-\gamma\). Then, by [22, Theorem 4.3], we know that \( u \) is a unique classical solution of (1.1). This completes the proof.

As an application of Theorem 3.2, we consider the existence and uniqueness of classical solutions to the nonlocal problem (1.2). The main result is the following theorem:

**Theorem 3.3** — Let hypothesis \((H_4)\) and hypotheses in Theorem 3.2 hold. Assume in addition that the following semilinear Cauchy problem

\[ \begin{cases} v'(t) = Av(t) + G(v)F(v(t)), & t \in (0, T], \\ v(0) + H(t_1, \cdots, t_p, v) = u_0. \end{cases} \tag{3.4} \]

has a unique classical solution \( v \) on \( I \). Then there exists a unique classical solution \( u \) to (1.2) defined on \([0, \tilde{T}]\), where \( \tilde{T} \) is given by

\[ \tilde{T} = \int_0^T G(v(s)) ds. \]

**Proof**: In fact, we infer by \((H_4)\) and the hypotheses of Theorem 3.2 that the auxiliary problem (3.4) has a unique classical solution on \( I \). If we set

\[ G(\zeta) = \int_0^\zeta G(v(s)) ds, \]

one can show easily that \( G \) is a \( C^1 \) diffeomorphism from \( I \) to \([0, \tilde{T}]\). Thus the ordinary differential equation

\[ \begin{cases} \dot{\theta}(t) = \frac{1}{G(v(\theta(t)))}, & t > 0, \\ \theta(0) = 0 \end{cases} \]

has a unique solution given by \( \theta(t) = G^{-1}(t) \) on \([0, \tilde{T}]\).

Now let

\[ u(t) = v(\theta(t)). \tag{3.5} \]
Then it follows that $u(0) = v(\theta(0)) = u_0 - H(t_1, \cdots, t_p, u(\cdot))$, $u(t)$ is continuous on $I$, and $u$ is continuously differentiable with $u(t) \in D(A)$ for $0 < t \leq \tilde{T}$. Moreover,

$$
\dot{u}(t) = \dot{v}(\theta(t))\dot{\theta}(t)
$$

$$
= (Av(\theta(t)) + G(v(\theta(t)))F(v(\theta(t)))) \frac{1}{G(v(\theta(t)))}
$$

$$
= \frac{1}{G(u)}Au(t) + F(u(t)), \quad 0 < t \leq \tilde{T}.
$$

Hence, $u(t)$ is a classical solution of the nonlocal evolution equation (1.2).

Next we show the uniqueness. Let $u(t)$ be a local solution of (1.1) and let $\sigma(t)$ be the solution of the ordinary differential equation

$$
\begin{cases}
\dot{\sigma}(s) = G(u(\sigma(s))), & s > 0, \\
\sigma(0) = 0.
\end{cases}
$$

(3.6)

Set

$$
v(t) = u(\sigma(t)).
$$

Then we have that $v(0) = u_0 - H(t_1, \cdots, t_p, v(\cdot))$, $v(t)$ is continuous for $t \geq 0$, and $v$ is continuously differentiable with $v(t) \in D(A)$ for $0 < t \leq T$. Moreover,

$$
\dot{v}(t) = \dot{u}(\sigma(t))\dot{\sigma}(t)
$$

$$
= (Av(t) + G(v(t)))f(v(t)), \quad 0 < t \leq T.
$$

Going back to (3.6) one can see that $\sigma$ is given by

$$
\sigma(t) = \int_0^t G(v(s))ds
$$

and thus $\sigma(\theta(t)) = t$. This shows that $u$ is necessarily given by (3.5) and the proof is complete.

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REFERENCES


