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ABOUT A SPECIAL CLASS OF TWO-DIMENSIONAL COMPLEX FINSLER SPACES

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In this paper we investigate a class of two-dimensional complex Finsler spaces, called η-Einstein, in [3]. The holomorphic sectional curvatures of such spaces, in directions of the local complex Berwald frames $\{l, m, \bar{l}, \bar{m}\}$ and $\{\lambda, \mu, \bar{\lambda}, \bar{\mu}\}$ are studied. We classify some subclasses of η-Einstein spaces, with respect to the horizontal holomorphic sectional curvature in direction λ. Finally, a special approach is devoted to the holomorphic bisectional curvatures of the two - dimensional η-Einstein spaces.

Key words : Generalized Einstein space, η-Einstein space, complex Berwald frame, holomorphic sectional and bisectional curvatures.

1. INTRODUCTION

In complex Finsler geometry there are not so many known classes of complex Finsler metrics. Besides the significant Kobayashi, Caratheodory and Wu metrics (see [1, 12]), which quickened the study of such Finsler geometry, we know two rather trivial classes of complex Finsler metrics: the complex Finsler metrics which come from Hermitian metrics on the base manifold (the purely Hermitian metrics in [15]), and the locally Minkowski complex metrics. Recently, we initiated the study of the complex Randers metrics, ([6]), or, more generally the complex Finsler spaces with $(\alpha, \beta)-$ metrics. Another class of complex Finsler metrics, introduced by us in [3], is that of the η-Einstein metrics, with its subclass of generalized Einstein metrics, ([4]).

The study of the holomorphic curvatures of the complex Finsler metrics is a challenging problem in complex Finsler geometry. Therefore, the main purpose of the present paper is to characterize the holomorphic sectional and bisectional curvatures of the 2-dimensional η-Einstein spaces.

The tools of our study are succinct described in § 2. These are the Chern-Finsler complex linear connection and the local complex Berwald frames $\{l, m, \bar{l}, \bar{m}\}$ and $\{\lambda, \mu, \bar{\lambda}, \bar{\mu}\}$ in which the orthogonality is with respect to a Hermitian structure, defined by the fundamental metric tensor of a 2-dimensional complex Finsler space on the holomorphic tangent manifold $T'M$, ([5]). The local Berwald frames are not only a local geometrical machinery, but these also satisfy important properties which contain three main real invariants which live on $T'M$: one vertical curvature invariant **I** and two horizontal curvature invariants $\mathbf{K}$ and $\mathbf{W}$.

By means of these invariants we are able to characterize the horizontal and vertical holomorphic sectional curvatures of such spaces, (in §3). Some characterizations of the 2-dimensional η-Einstein spaces come from the exploration of the $h\bar{h}-$ Riemann type tensors, (Theorems 3.1, 3.2). Further on, we find the necessary and sufficient conditions that $\mathbf{K}$ should be a constant, (Theorems 3.3, 3.4). The Proposition 3.1. is the key of the next results. We find when the non-purely Hermitian η-Einstein spaces, with additional condition of weakly Kähler, have $\mathbf{K} = 0$ and $\mathbf{W} \leq 0$, (Theorem 3.5). Moreover, it is shown that there are only two types of weakly Kähler generalized Einstein metrics, in two dimensional case. Namely, these are either purely Hermitian with $\mathbf{K} = \mathbf{W} =$ constant or non-purely Hermitian with $\mathbf{K} = 0$, (Theorem 3.6). Another result is that any 2-dimensional Kähler

η-Einstein space is either purely Hermitian with $\mathbf{K} = \mathbf{W} =$ constant and $\mathbf{I} = 0$ or non-purely Hermitian with $\mathbf{K} = \mathbf{W} = \mathbf{I}_{|j} = 0$, (Theorem 3.7). We obtain an interesting non-purely Hermitian subclass of η-Einstein spaces which has $\mathbf{K}$ equal to a constant $c \neq 0$ and $\mathbf{W} = 2c - L\left|\Omega\right|^2$, (Theorem 3.8).

In §4, we investigate the holomorphic bisectional curvatures, in directions of the local complex Berwald frames for a 2-dimensional complex Finsler space. As applications, we discuss about the holomorphic bisectional curvatures of the 2-dimensional η-Einstein spaces, (Theorems 4.2, 4.3).

The global validity of the all obtained results is incessantly studied.

2. Preliminaries

For the beginning we shall make a survey of 2 - dimensional complex Finsler geometry with Chern-Finsler complex linear connection and the local complex Berwald frames. Here are set the basic notions and terminology. For more details, see [1, 15, 5].

2.1 Complex Finsler spaces

Let M be a 2 - dimensional complex manifold, $z = (z^k)_{k=\overline{1,2}}$ are the complex coordinates in a local chart. The holomorphic tangent bundle $T'M$ is itself a complex manifold, and the coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=\overline{1,2}}$.

A *complex Finsler space* is a pair (M, F), where $F : T'M \to \mathbb{R}^+$ is a smooth homogeneous function, i.e. $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, $\forall \lambda \in \mathbb{C}$, and the Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$ is positive defined, with $g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ and $L := F^2$. Then, $g_{i\bar{j}}$ is called the fundamental metric tensor of the complex Finsler space.

Consider the sections of the complexified tangent bundle of $T'M$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and $VT''M$ its conjugate. The idea of complex nonlinear connection, briefly $(c.n.c.)$, is an instrument in 'linearization' of the geometry of $T'M$ manifold. A $(c.n.c.)$ is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_uT'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the $(c.n.c.)$. The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the $(c.n.c.)$ which obey to the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation, everywhere is obtained an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Next, let us consider the *Sasaki* type lift of the metric tensor $g_{i\bar{j}}$,

$$\mathcal{G} = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j. \tag{2.1}$$

There exists an unique Hermitian connection D, of $(1,0)-$ type, which satisfies in addition $D_{JX}Y = JD_XY$, for all X horizontal vectors and J the natural complex structure of the manifold, called the Chern-Finsler connection (cf. [1]), in brief $C-F$, which have a special meaning in complex Finsler geometry. The $C-F$ connection is locally given by the following coefficients (cf. [15]):

$$N_j^k = g^{\overline{m}k}\frac{\partial g_{l\overline{m}}}{\partial z^j}\eta^l = L^k_{lj}\eta^l \ ; \ L^i_{jk} = g^{\bar{l}i}\delta_k g_{j\bar{l}} \ \ ; \ C^i_{jk} = g^{\bar{l}i}\dot{\partial}_k g_{j\bar{l}}, \tag{2.2}$$

where $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$, $D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$, etc. Further on, the $C-F$ connection is the main tool in this study.

The nonzero curvatures of the $C-F$ connection are denoted by

$$\begin{aligned} R(\delta_h,\delta_{\bar{k}})\delta_j &= R^i_{j\bar{k}h}\delta_i \ ; \ R(\dot{\partial}_h,\delta_{\bar{k}})\delta_j = \Xi^i_{j\bar{k}h}\delta_i \ ; \ R(\delta_h,\dot{\partial}_{\bar{k}})\delta_j \\ &= P^i_{j\bar{k}h}\delta_i; \\ R(\delta_h,\delta_{\bar{k}})\dot{\partial}_j &= R^i_{j\bar{k}h}\dot{\partial}_i \ ; \ R(\dot{\partial}_h,\delta_{\bar{k}})\dot{\partial}_j = \Xi^i_{j\bar{k}h}\dot{\partial}_i \ ; \ R(\delta_h,\dot{\partial}_{\bar{k}})\dot{\partial}_j \\ &= P^i_{j\bar{k}h}\dot{\partial}_i; \\ R(\dot{\partial}_h,\dot{\partial}_{\bar{k}})\delta_j &= S^i_{j\bar{k}h}\delta_i \ ; \ \ R(\dot{\partial}_h,\dot{\partial}_{\bar{k}})\dot{\partial}_j = S^i_{j\bar{k}h}\dot{\partial}_i, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} R^i_{j\overline{h}k} &= -\delta_{\overline{h}}L^i_{jk} - \delta_{\overline{h}}(N^l_k)C^i_{jl} \ \ ; \ \Xi^i_{j\overline{h}k} = -\delta_{\overline{h}}C^i_{jk} = \Xi^i_{k\overline{h}j}; \\ P^i_{j\overline{h}k} &= -\dot{\partial}_{\overline{h}}L^i_{jk} - \dot{\partial}_{\overline{h}}(N^l_k)C^i_{jl} \ \ ; \ S^i_{j\overline{h}k} = -\dot{\partial}_{\overline{h}}C^i_{jk} = S^i_{k\overline{h}j}. \end{aligned} \tag{2.4}$$

The Riemann tensor

$$\begin{aligned} \mathbf{R}(W,\overline{Z},X,\overline{Y}) \ &: \ = G(R(X,\overline{Y})W,\overline{Z}), \\ \mathbf{R}(W,\overline{Z},X,\overline{Y}) &= \overline{\mathbf{R}(Z,\overline{W},Y,\overline{X})} \end{aligned} \tag{2.5}$$

for $W, X, \overline{Z}, \overline{Y}$ horizontal or vertical vectors, has the following $h\bar{h}-$, $h\bar{v}-$, $v\bar{h}-$, $v\bar{v}-$ components: $R_{\bar{j}i\bar{h}k} := g_{l\bar{j}} R^l_{i\bar{h}k}$; $P_{\bar{j}i\bar{h}k} := g_{l\bar{j}} P^l_{i\bar{h}k}$; $\Xi_{\bar{j}i\bar{h}k} := g_{l\bar{j}} \Xi^l_{i\bar{h}k}$; $S_{\bar{j}i\bar{h}k} =: g_{l\bar{j}} S^l_{i\bar{h}k}$, which have properties $R_{i\bar{j}k\bar{h}} = \overline{R_{\bar{j}i\bar{h}k}}$; $\Xi_{i\bar{j}k\bar{h}} = P_{\bar{j}i\bar{h}k}$; $P_{i\bar{j}k\bar{h}} = \Xi_{\bar{j}i\bar{h}k}$; $S_{i\bar{j}k\bar{h}} = S_{\bar{j}i\bar{h}k} = S_{\bar{h}i\bar{j}k}$, where $R_{i\bar{j}k\bar{h}} := \overline{R_{\bar{i}j\bar{k}h}}$, etc., (see [15], p. 77).

Let us recall that in [1]'s terminology, the complex Finsler space (M, F) is *strongly Kähler* iff $T^i_{jk} = 0$, *Kähler* iff $T^i_{jk}\eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}} T^i_{jk}\eta^j \overline{\eta}^l = 0$, where $T^i_{jk} := L^i_{jk} - L^i_{kj}$. In [11] it is proved that strongly Kähler and Kähler notions actually coincide. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on M, so-called *purely Hermitian metrics* in [15], (i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$), all those nuances of Kähler are the same.

In [3] we introduced and studied a special class of complex Finsler spaces, called by us $\eta-$ *Einstein*, briefly $(\eta - E)$. Similarly, a 2-dimensional complex Finsler space is called $(\eta - E)$, if there exists two smooth functions $K_i(z, \eta)$: $T'M \to \mathbf{R}$, $i = 1, 2$, such that

$$R_{\bar{j}k} = K_1(z,\eta) L g_{k\bar{j}} + K_2(z,\eta)\eta_k \bar{\eta}_j, \tag{2.6}$$

where $R_{\bar{j}k} := R_{\bar{j}i\bar{h}k}\eta^i\overline{\eta}^h = -g_{l\bar{j}}(\delta_{\bar{h}} N^l_k)\overline{\eta}^h$ and $\eta_k := \frac{\partial L}{\partial \eta^k}$.

If we take $K_1(z, \eta) = K_2(z, \eta)$ then such $(\eta - E)$ spaces are referred to as *generalized Einstein*, briefly $(g.E)$, in [4].

2.2. The local complex Berwald frames

In [5] we introduced the local complex Berwald frames. By means of these, an exhaustive study of 2 - dimensional complex Finsler spaces is made in [5]. Here, we shall summarize some basic results. We set $l := l^i \dot{\partial}_i$ and its dual form is $\omega = l_i \delta\eta^i$, where

$$l^i = \frac{1}{F}\eta^i \ \text{ and } \ l_i = \frac{1}{F} g_{i\bar{j}} \overline{\eta}^j = g_{i\bar{j}} l^{\bar{j}}. \tag{2.7}$$

An orthonormal frame in the vertical bundle $VT'M$, which is 2 - dimensional in any point, is given by $l = l^i \dot{\partial}_i$ and $m = m^i \dot{\partial}_i$, where

$$m = \frac{1}{\sqrt{g}}(-l_2\dot{\partial}_1 + l_1\dot{\partial}_2) \tag{2.8}$$

in a fixed chart. Then $\{l, m, \bar{l}, \bar{m}\}$, with m given by (2.8) is called the local complex Berwald frame of the space. Indeed, since the local frame is orthonormal we have: $l^i l_i = m^i m_i = 1$ and $l^i m_i = l_i m^i = 0$, where $l_i = g_{i\bar{j}} l^{\bar{j}}$ and $m_i = g_{i\bar{j}} m^{\bar{j}}$.

We specify that (2.8) provides only a local frame, because the set of natural local basis in every chart does not have tensorial character. More precisely, we can check that $m' = \frac{\mathcal{T}}{|\mathcal{T}|} m$, at the local change of charts, where $\mathcal{T}$ is the Jacobi determinant of changes. This show that m is not a vector, it depends on the local change. Therefore, it will say that m from (2.8) is a pseudo-vector. Although m from (2.2) depends on the local changes of the coordinates, it is very important in our study, in a fixed chart. Certainly, further on we will be very careful with the global validity of our assertions.

With respect to the local complex Berwald frame, $\dot{\partial}_k$ and $g_{i\bar{j}}$ are decomposed as follows $\dot{\partial}_i = l_i l + m_i m$ and hence, $g_{i\bar{j}} = l_i l_{\bar{j}} + m_i m_{\bar{j}}$. From here it is deduced

$$C^i_{jk} = g^{\bar{m}i} \dot{\partial}_k g_{j\bar{m}} = A l^i m_k m_j + B m^i m_k m_j, \tag{2.9}$$

where $A := m^j m^k l_h C^h_{kj}$ and $B := m_h m^k m^j C^h_{jk}$.

Now, via the natural isomorphism between the bundles $VT'M$ and $T'M$, composed with the horizontal lift of $HT'M$, we obtain the following orthonormal local frame on $H_C T'M$, $\{\lambda := l^i \delta_i,\ \ \mu = m^i \delta_i,\ \ \bar{\lambda} := l^{\bar{\imath}} \delta_{\bar{\imath}},\ \ \bar{\mu} = m^{\bar{\imath}} \delta_{\bar{\imath}}\}$.

Using $L^i_{jk} = g^{\bar{m}i} \delta_k g_{j\bar{m}}$ it results

$$\begin{aligned} L^i_{jk} &= J l^i l_j l_k + U l^i m_j l_k + V l^i l_j m_k + X l^i m_j m_k \\ &\quad + O m^i l_j l_k + Y m^i m_j l_k + E m^i l_j m_k + H m^i m_j m_k, \end{aligned} \tag{2.10}$$

where $J := l^j l^k l_i L^i_{jk}$, $U := m^j l^k l_i L^i_{jk}$, $V := l^j m^k l_i L^i_{jk}$, $X := m^j m^k l_i L^i_{jk}$, $O := l^j l^k m_i L^i_{jk}$, $Y := m^j l^k m_i L^i_{jk}$, $E := l^j m^k m_i L^i_{jk}$, $H := m^j m^k m_i L^i_{jk}$.

Note that $U = V$ and $Y = E$ characterize the Kähler spaces and for any weakly Kähler space we have $U = V$.

Denoting by " $_{|}$ " , " $|$ " , " $\bar{|}$" and " $\bar{|}$", the $h-$, $v-$, $\overline{h}-$, $\overline{v}-$ covariant derivatives with respect to $C-F$ connection, respectively, in [5] it is proved:

$$\begin{aligned} l_i|_j &= \frac{-1}{2F}l_il_j \ ; \ l_i|_{\bar{j}} = \frac{1}{2F}l_il_{\bar{j}} + \frac{1}{F}m_im_{\bar{j}} \ ; \ \ F|_j = \frac{1}{2}l_j; \\ l^i|_j &= \frac{1}{F}\delta^i_j - \frac{1}{2F}l_jl^i \ ; \ l^i|_{\bar{j}} = \frac{-1}{2F}l_{\bar{j}}l^i \ ; \ l_{i|j} = l_{i|\bar{j}} = l^i_{|j} = l^i_{|\bar{j}} = 0; \\ m^i|_j &= \frac{-1}{2F}l_jm^i + \frac{B}{2}m_jm^i \ ; \ m^i|_{\bar{j}} = \frac{1}{2F}l_{\bar{j}}m^i - \frac{1}{F}m_{\bar{j}}l^i - \frac{\bar{B}}{2}m_{\bar{j}}m^i; \\ m_i|_j &= \frac{1}{2F}m_il_j - \frac{1}{F}l_im_j - \frac{B}{2}m_im_j \ ; \ m_i|_{\bar{j}} = \frac{-1}{2F}m_il_{\bar{j}} + \frac{\bar{B}}{2}m_im_{\bar{j}}; \\ m_{i|j} &= -\frac{1}{2}[(J+Y)l_j + (V+H)m_j]m_i; \\ m_{i|\bar{j}} &= \frac{1}{2}[(\bar{J}+\bar{Y})l_{\bar{j}} + (\bar{V}+\bar{H})m_{\bar{j}}]m_i \ ; \\ m^i_{|j} &= \frac{1}{2}[(J+Y)l_j + (V+H)m_j]m^i; \\ m^i_{|\bar{j}} &= -\frac{1}{2}[(\bar{J}+\bar{Y})l_{\bar{j}} + (\bar{V}+\bar{H})m_{\bar{j}}]m^i \end{aligned} \tag{2.11}$$

and theirs conjugates.

Also, the formula (4.1) from [5] say that for any 2 - dimensional complex Finsler space it is true

$$A|_{\bar{s}}m^{\bar{s}} = -A\bar{B} + \frac{B}{F}. \tag{2.12}$$

Moreover, the class of purely Hermitian spaces is characterized by $A = 0$ (and owing to (2.12), $B = 0$). So that, any such space with $|A| \neq 0$ is not purely Hermitian.

***Theorem* 2.1** [5] — *Let (M, F) be a 2 - dimensional complex Finsler space. If it is a non-purely Hermitian Kähler space, then $\bar{A}_{|0} + F\bar{A}(J+Y) = 0$.*

With respect to the local complex Berwald frames, the $v\bar{v}-$, $h\bar{v}-$, $v\bar{h}-$, $h\bar{h}-$ Riemann type tensors are

$$S_{\bar{r}j\bar{h}k} = \mathbf{I}m_{\bar{h}}m_{\bar{r}}m_jm_k \ ; \tag{2.13}$$

$$\Xi_{\bar{r}j\bar{h}k} = -[A_{|\bar{h}}l_{\bar{r}} + A(\bar{J}+\bar{Y})l_{\bar{r}}l_{\bar{h}} + A(\bar{V}+\bar{H})l_{\bar{r}}m_{\bar{h}} \tag{2.14}$$
$$+B_{|\bar{h}}m_{\bar{r}} + \frac{B}{2}(\bar{J}+\bar{Y})m_{\bar{r}}l_{\bar{h}} + \frac{B}{2}(\bar{V}+\bar{H})m_{\bar{r}}m_{\bar{h}}]m_j m_k = \overline{P_{\bar{j}r\bar{k}h}}\ ;$$

$$R_{\bar{r}j\bar{h}k} = \mathbf{K}l_{\bar{r}}l_j l_{\bar{h}}l_k + \mathbf{W}m_{\bar{r}}m_j m_{\bar{h}}m_k \tag{2.15}$$
$$-[\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y)]l_{\bar{r}}m_j l_{\bar{h}}l_k - [\frac{1}{F}O_{|\bar{0}} - \frac{1}{2}O(\bar{J}+\bar{Y})]m_{\bar{r}}l_j l_{\bar{h}}l_k$$
$$-\bar{J}_{|s}m^s l_{\bar{r}}l_j l_{\bar{h}}m_k - [O_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}O(\bar{V}+\bar{H})]m_{\bar{r}}l_j m_{\bar{h}}l_k$$
$$-[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})]l_{\bar{r}}l_j m_{\bar{h}}m_k - \frac{1}{F}\bar{E}_{|0}m_{\bar{r}}l_j l_{\bar{h}}m_k$$
$$-\frac{1}{F}E_{|\bar{0}}l_{\bar{r}}m_j m_{\bar{h}}l_k - [\frac{1}{F}Y_{|\bar{0}} + BO_{|\bar{0}} - \frac{1}{2}FBO(\bar{J}+\bar{Y})]m_{\bar{r}}m_j l_{\bar{h}}l_k$$
$$-E_{|\bar{s}}m^{\bar{s}}m_{\bar{r}}l_j m_{\bar{h}}m_k - [\frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0}]m_{\bar{r}}m_j m_{\bar{h}}l_k$$
$$-\bar{E}_{|s}m^s l_{\bar{r}}m_j m_{\bar{h}}m_k - [\frac{1}{F}H_{|\bar{0}} + \frac{1}{2}H(\bar{J}+\bar{Y}) + BE_{|\bar{0}}]m_{\bar{r}}m_j l_{\bar{h}}m_k$$
$$-J_{|\bar{s}}m^{\bar{s}}l_{\bar{r}}l_j m_{\bar{h}}l_k - [\bar{O}_{|s}m^s - \frac{1}{2}\bar{O}(V+H)]l_{\bar{r}}m_j l_{\bar{h}}m_k,$$

where $\mathbf{I} := -B|_{\bar{s}}m^{\bar{s}} - \frac{B\bar{B}}{2}$ is the *vertical curvature invariant*, $\mathbf{K} := -\frac{1}{F}J_{|\bar{0}}$ and $\mathbf{W} := -H_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}H(\bar{V}+\bar{H}) - BFE_{|\bar{s}}m^{\bar{s}}$ are the *horizontal curvature invariants.*

In [5] we speak about two horizontal holomorphic sectional curvatures, one in direction λ and other in direction μ, defined as

$$K^h_{F,\lambda}(z,\eta) := 2\mathbf{R}(\lambda,\bar{\lambda},\lambda,\bar{\lambda}) = 2\mathbf{K}\ ;\ K^h_{F,\mu}(z,\eta) = 2\mathbf{R}(\mu,\bar{\mu},\mu,\bar{\mu}) = 2\mathbf{W}. \tag{2.16}$$

Therefore, there exists two vertical holomorphic sectional curvatures, one in the direction l and other in the direction m, defined as

$$K^v_{F,l}(z,\eta) := 2\mathbf{R}(l,\bar{l},l,\bar{l}) = 0\ ;\ K^v_{F,m}(z,\eta) = 2\mathbf{R}(m,\bar{m},m,\bar{m}) = 2\mathbf{I}. \tag{2.17}$$

Immediately consequences of the weakly Kähler property are the equations (Lemma 4.2, from [5]):

$$\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y) - AO_{|\bar{0}} + \frac{1}{2}FAO(\bar{J}+\bar{Y}) = \bar{J}_{|s}m^s; \tag{2.18}$$
$$\frac{1}{F}\bar{E}_{|0} - FAO_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}FAO(\bar{V}+\bar{H}) = V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H}).$$

3. SPECIAL PROPERTIES OF THE 2-DIMENSIONAL $(\eta - E)$ COMPLEX FINSLER SPACES

In this section, our goal is to characterize 2 - dimensional $(\eta - E)$ complex Finsler spaces by means of the horizontal curvature invariants $\mathbf{K}$ and $\mathbf{W}$. Further on, everywhere the index 0 means the contraction by η, for example $P^i_{0\bar{h}k} := P^i_{j\bar{h}k}\eta^j$.

Lemma 3.1 — For any $X \in \Gamma^0(T'M)$ the following properties hold true:

i) $X|_{\bar{k}|\bar{j}} - X_{|\bar{j}}|_{\bar{k}} = C^{\bar{\imath}}_{\bar{j}\bar{k}} X_{|\bar{\imath}}$;

ii) $X|_{\bar{k}|j} - X_{|j}|_{\bar{k}} = -P^i_{0\bar{k}j} X|_i$.

PROOF : Indeed, we have

$\left[\delta_{\bar{j}}, \dot{\partial}_{\bar{k}}\right] X = L^{\bar{\imath}}_{\bar{k}\bar{j}} \left(\dot{\partial}_{\bar{\imath}} X\right) = L^{\bar{\imath}}_{\bar{k}\bar{j}} X|_{\bar{\imath}}$ and

$\left[\delta_j, \dot{\partial}_{\bar{k}}\right] X = -P^i_{0\bar{k}j} \dot{\partial}_i X = -P^i_{0\bar{k}j} X|_i.$

On the other hand,

$\left[\delta_{\bar{j}}, \dot{\partial}_{\bar{k}}\right] X = \delta_{\bar{j}} \left(\dot{\partial}_{\bar{k}} X\right) - \dot{\partial}_{\bar{k}}(\delta_{\bar{j}} X) = \delta_{\bar{j}} \left(X|_{\bar{k}}\right) - \dot{\partial}_{\bar{k}}(X_{|\bar{j}})$

$= X|_{\bar{k}|\bar{j}} + L^{\bar{\imath}}_{\bar{k}\bar{j}} X|_{\bar{\imath}} - X_{|\bar{j}}|_{\bar{k}} - C^{\bar{\imath}}_{\bar{j}\bar{k}} X_{|\bar{\imath}}$ and

$\left[\delta_j, \dot{\partial}_{\bar{k}}\right] X = \delta_j \left(\dot{\partial}_{\bar{k}} X\right) - \dot{\partial}_{\bar{k}}(\delta_j X) = \delta_j \left(X|_{\bar{k}}\right) - \dot{\partial}_{\bar{k}}(X_{|j})$
$= X|_{\bar{k}|j} - X_{|j}|_{\bar{k}}.$

From the above relations it results i) and ii). □

Firstly, writing the identity i) from Lemma 3.1 for J

$$J|_{\bar{r}|\bar{s}} - J_{|\bar{s}}|_{\bar{r}} = C^{\bar{n}}_{\bar{s}\bar{r}} J_{|\bar{n}}. \tag{3.1}$$

and taking into account $J|_{\bar{k}} = -\frac{1}{2F} J l_{\bar{k}} + \frac{1}{F} O m_{\bar{k}}$ (see [5], Proposition 4.2 i) and (2.11), hence

$$J|_{\bar{r}|\bar{s}} = -\frac{1}{2F} J_{|\bar{s}} l_{\bar{r}} + [\frac{1}{F} O_{|\bar{s}} - \frac{1}{2F} O(\bar{J} + \bar{Y}) l_{\bar{s}} - \frac{1}{2F} O(\bar{V} + \bar{H}) m_{\bar{s}}] m_{\bar{r}}. \tag{3.2}$$

Secondly, using $J_{|\bar{s}} = -\mathbf{K}l_{\bar{s}} + J_{|\bar{h}}m^{\bar{h}}m_{\bar{s}}$ and (2.11), we obtain

$$J_{|\bar{s}}|_{\bar{r}} = -\left(\mathbf{K}|_{\bar{r}} + \frac{1}{2F}\mathbf{K}l_{\bar{r}} + \frac{1}{F}J_{|\bar{r}}\right) l_{\bar{s}} + J_{|\bar{h}}|_{\bar{r}}m^{\bar{h}}m_{\bar{s}}. \tag{3.3}$$

Plugging (3.2) and (3.3) into (3.1), it results

$-\frac{1}{2F}J_{|\bar{s}}l_{\bar{r}} + \frac{1}{F}O_{|\bar{s}}m_{\bar{r}} + \left(\mathbf{K}|_{\bar{r}} + \frac{1}{2F}\mathbf{K}l_{\bar{r}} + \frac{1}{F}J_{|\bar{r}} - \frac{1}{2F}O(\bar{J}+\bar{Y})m_{\bar{r}}\right) l_{\bar{s}}$

$-\left(J_{|\bar{h}}|_{\bar{r}}m^{\bar{h}} + \frac{1}{2F}O(\bar{V}+\bar{H})m_{\bar{r}}\right) m_{\bar{s}} = (-\bar{A}\mathbf{K} + \bar{B}J_{|\bar{h}}m^{\bar{h}})m_{\bar{r}}m_{\bar{s}}$ which contracted by $l^{\bar{s}}$ and $m^{\bar{s}}m^{\bar{r}}$ respectively, leads to

$$\begin{aligned} \mathbf{K}|_{\bar{r}} &= -\frac{1}{F}[J_{|\bar{h}}m^{\bar{h}} + \frac{1}{F}O_{|\bar{0}} - \frac{1}{2}O(\bar{J}+\bar{Y})]m_{\bar{r}}; \\ \bar{A}\mathbf{K} &= J_{|\bar{h}}|_{\bar{r}}m^{\bar{h}}m^{\bar{r}} + \bar{B}J_{|\bar{h}}m^{\bar{h}} - \frac{1}{F}O_{|\bar{s}}m^{\bar{s}} + \frac{1}{2F}O(\bar{V}+\bar{H}). \end{aligned} \tag{3.4}$$

***Theorem* 3.1** — *Let (M,F) be a connected 2 - dimensional complex Finsler space. Then it is $(\eta - E)$ if and only if $\bar{J}_{|s}m^s = 0$ and $\frac{1}{F}O_{|\bar{0}} - \frac{1}{2}O(\bar{J}+\bar{Y}) = 0$. In this case, the functions $K_i(z,\eta)$, $i = 1,2$, are related by*

$$K_1(z,\eta) + K_2(z,\eta) = \mathbf{K}(z)\ ;\ K_1(z,\eta) = -\frac{1}{F}\bar{E}_{|0}$$

and

$$F\bar{A}\mathbf{K} = -O_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}O(\bar{V}+\bar{H}).$$

PROOF : By (2.15)

$R_{\bar{r}k} = R_{\bar{r}j\bar{h}k}\eta^j\bar{\eta}^h = LR_{\bar{r}j\bar{h}k}l^jl^{\bar{h}}$

$= L\left\{\mathbf{K}l_{\bar{r}}l_k - [\frac{1}{F}O_{|\bar{0}} - \frac{1}{2}O(\bar{J}+\bar{Y})]m_{\bar{r}}l_k - \bar{J}_{|s}m^sl_{\bar{r}}m_k - \frac{1}{F}\bar{E}_{|0}m_{\bar{r}}m_k\right\}$

$= L\left[-\frac{1}{F}\bar{E}_{|0}g_{k\bar{r}} + \left(\mathbf{K} + \frac{1}{F}\bar{E}_{|0}\right)l_{\bar{r}}l_k\right]$

$-L\left\{[\frac{1}{F}O_{|\bar{0}} - \frac{1}{2}O(\bar{J}+\bar{Y})]m_{\bar{r}}l_k + \bar{J}_{|s}m^sl_{\bar{r}}m_k\right\}$.

So that, (M,F) is $(\eta - E)$, i.e. $R_{\bar{r}k} = K_1(z,\eta)Lg_{k\bar{r}} + K_2(z,\eta)\eta_k\bar{\eta}_r$ if and only if $\bar{J}_{|s}m^s = 0$ and $\frac{1}{F}O_{|\bar{0}} - \frac{1}{2}O(\bar{J}+\bar{Y}) = 0$, which work globally on M. Thus, it results $K_1(z,\eta) = -\frac{1}{F}\bar{E}_{|0}$ and $K_2(z,\eta) = \mathbf{K} + \frac{1}{F}\bar{E}_{|0}$.

Now, by the first identity from (3.4), we have $\mathbf{K}|_{\bar{r}} = 0$ and by conjugation, $\mathbf{K}|_{r} = 0$. These mean that $\mathbf{K}$ depends on z only on M. The second identity from (3.4) complete the proof. □

***Theorem* 3.2** — *Let (M, F) be a connected 2 - dimensional complex Finsler space. Suppose that it is weakly Kähler, $\mathbf{K}$ depends on z only and $|A|^2 \neq \frac{4}{L}$ on M. Then, the space is $(\eta - E)$.*

PROOF : Since $\mathbf{K} = \mathbf{K}(z)$, the first identity from (2.18) is $J_{|\bar{h}} m^{\bar{h}} = -\frac{1}{F} O_{|\bar{0}} + \frac{1}{2} O(\bar{J} + \bar{Y})$, which plugged into (2.18) gives $2\bar{J}_{|s} m^s = AFJ_{|\bar{h}} m^{\bar{h}}$. By conjugation, $2J_{|\bar{h}} m^{\bar{h}} = \bar{A} F \bar{J}_{|s} m^s$. The last two relations lead to $(4 - |A|^2 L) J_{|\bar{h}} m^{\bar{h}} = 0$, and so, it follows $J_{|\bar{h}} m^{\bar{h}} = -\frac{1}{F} O_{|\bar{0}} + \frac{1}{2} O(\bar{J} + \bar{Y}) = 0$. □

Contracting the Bianchi identity

$$\mathcal{A}_{kl} \left\{ R_{\bar{r}j\bar{h}k|l} - P_{\bar{r}j\bar{s}k} R^{\bar{s}}_{\bar{0}l\bar{h}} \right\} + R_{\bar{r}j\bar{h}n} T^n_{kl} = 0, \tag{3.5}$$

(see [15], p. 77), by $\bar{\eta}^r \eta^j \bar{\eta}^h \eta^k$ it implies

$$F\mathbf{K}_{|l} - \mathbf{K}_{|0} l_l - \bar{J}_{|s|0} m^s m_l + F\mathbf{K} l^k l_n T^n_{kl} - F\bar{J}_{|s} m^s l^k m_n T^n_{kl} = 0. \tag{3.6}$$

***Theorem* 3.3** — *Let (M, F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space. If it is weakly Kähler, then $\mathbf{K}$ is a constant on (M, F).*

PROOF : Since (M, F) is $(\eta - E)$, $\mathbf{K}$ does not depend on η. The identity (3.6), together with the weakly Kähler property and $\bar{J}_{|s|0} m^s = 0$, yield $\mathbf{K}_{|l} = \frac{1}{F} \mathbf{K}_{|0} l_l$. But, using ii) from Lemma 3.1, we have $0 = \mathbf{K}|_{\bar{k}|j} = \mathbf{K}_{|j}|_{\bar{k}}$.

On the other hand,

$$\mathbf{K}_{|l}|_{\bar{r}} = -\frac{1}{2L} \mathbf{K}_{|0} l_l l_{\bar{r}} + \frac{1}{F} \mathbf{K}_{|0}|_{\bar{r}} l_l + \frac{1}{F} \mathbf{K}_{|0} \left(\frac{1}{2F} l_l l_{\bar{r}} + \frac{1}{F} m_l m_{\bar{r}} \right).$$

It follows that $\mathbf{K}_{|0} = 0$ and so $\mathbf{K}_{|l} = 0$, which is equivalent to $\frac{\partial \mathbf{K}}{\partial z^l} = 0$, i.e. $\mathbf{K}$ is a constant on (M, F).

***Theorem* 3.4** — *Let (M, F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space with $\mathbf{K} = c$ a nonzero constant on M. Then (M, F) is weakly Kähler.*

PROOF : Substituting $\mathbf{K}_{|l} = \bar{J}_{|s}m^s = 0$ in the relation (3.6) it follows that $L\mathbf{K}l^k l_n T^n_{kl} = 0$. Consequently, $l^k l_n T^n_{kl} = 0$, since $\mathbf{K} = c \neq 0$ □

Observe that the above Theorems give necessary and sufficient conditions that a connected 2 - dimensional $(\eta - E)$ complex Finsler space should be of constant horizontal holomorphic curvature in direction λ, i.e. $K^h_{F,\lambda}(z,\eta) = 2c$. It is interesting for us to see what happen with the horizontal holomorphic curvature in direction μ, in this case.

Let us set $\Phi := A_{|\bar{0}} + AF(\bar{J} + \bar{Y})$, $\Omega := A_{|\bar{k}}m^{\bar{k}} + A(\bar{V} + \bar{H})$. The following is a list of properties concerning the 2 - dimensional weakly Kähler $(\eta - E)$ spaces.

Proposition 31. — Let (M, F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space. If it is weakly Kähler, then

i) $cAL = \Phi_{|0} - F(J + Y)\Phi$;

ii) $E_{|\bar{0}} = -\frac{Fc}{2}(1 + |A|^2 L)$;

iii) $V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H}) = -\frac{c}{2}(1 - |A|^2 L)$;

iv) $Y_{|\bar{0}} = -\frac{Fc}{2}(1 - |A|^2 L) + F|\Phi|^2$;

v) $H_{|\bar{0}} + \frac{F}{2}H(\bar{J} + \bar{Y}) = -\frac{ALFc}{2}\left(\bar{B} - F\bar{A}B\right) + \frac{\bar{A}L}{2}[\Phi_{|k}m^k - (V + H)\Phi] + L\Phi\bar{\Omega}$;

vi) $E_{|\bar{s}}m^{\bar{s}} = -\frac{Fc}{2}\left(\bar{B} - F\bar{A}B\right) - \frac{AF}{2}[\bar{\Phi}_{|\bar{k}}m^{\bar{k}} - (\bar{V} + \bar{H})\bar{\Phi}]$;

vii) $\mathbf{W} = c(1 + |A|^2 L) - F(E_{|\bar{s}}m^{\bar{s}})|_l m^l - \frac{3}{2}BFE_{|\bar{s}}m^{\bar{s}} - L|\Omega|^2$, where c is the constant value of $\mathbf{K}$.

PROOF : Let us consider the Bianchi identity, (see [15], p. 77),

$$R_{\bar{r}j\bar{h}k}|_l - \Xi_{\bar{r}j\bar{h}l|k} - P_{\bar{r}j\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{h}} + S_{\bar{r}j\bar{s}l}R^{\bar{s}}_{\bar{0}k\bar{h}} + R_{\bar{r}j\bar{h}n}C^n_{kl} = 0. \tag{3.7}$$

In order to prove the statements i)-vii), we use Theorem 3.3, the covariant derivatives (2.11) and the expressions of the $v\bar{v}-$, $h\bar{v}-$, $v\bar{h}-$, $h\bar{h}-$ Riemann type tensors: (2.13), (2.14) and (2.15).

Contracting into (3.7) by $\bar{\eta}^r m^j \bar{\eta}^h \eta^k$, using

$R_{\bar{r}j\bar{h}k}|_l \bar{\eta}^r m^j \bar{\eta}^h \eta^k = -R_{\bar{0}j\bar{0}l} m^j = F^2[\bar{O}_{|s} m^s - \frac{1}{2}\bar{O}(V+H)]m_l = -F^3 Acm_l$;

$P_{\bar{r}j\bar{s}k} \bar{\eta}^r = S_{\bar{r}j\bar{s}l} \bar{\eta}^r = C^n_{kl} \eta^k = 0$ and

$\Xi_{\bar{r}j\bar{h}l|k} \bar{\eta}^r m^j \bar{\eta}^h \eta^k = -F[\Phi_{|0} - F(J+Y)\Phi]m_l$, the assertion i) follows.

The contraction with $\bar{\eta}^r \eta^j \bar{m}^h \eta^k$ of (3.7),

$R_{\bar{r}j\bar{h}k}|_l \bar{\eta}^r \eta^j \bar{m}^h \eta^k = -R_{\bar{0}l\bar{h}0} \bar{m}^h - R_{\bar{0}0\bar{h}l} \bar{m}^h + \frac{1}{L} R_{\bar{0}0\bar{0}0} m_l$

$= F^2[\frac{1}{F}\bar{E}_{|0} + V_{|\bar{s}} m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H}) + c]m_l$ and $\Xi_{\bar{r}j\bar{h}k}\eta^j = 0$ lead to

$$\frac{1}{F}\bar{E}_{|0} + V_{|\bar{s}} m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H}) = -c.$$

On the other hand, by the second identity from (2.18), we have

$$\frac{1}{F}\bar{E}_{|0} - V_{|\bar{s}} m^{\bar{s}} - \frac{1}{2}V(\bar{V}+\bar{H}) = -LA\bar{A}c.$$

The last two relations give ii) and iii).

Now, contracting again (3.7) by $\bar{m}^r \eta^j \bar{\eta}^h \eta^k$, hence

$R_{\bar{r}j\bar{h}k}|_l \bar{m}^r \eta^j \bar{\eta}^h \eta^k = F^2(c + \frac{1}{F}Y_{|\bar{0}} + \frac{1}{F}E_{|\bar{0}})m_l$ and

$P_{\bar{r}j\bar{s}k} P^{\bar{s}}_{\bar{0}l\bar{h}} \bar{m}^r \eta^j \bar{\eta}^h \eta^k = F^2 \left|\Phi\right|^2 m_l$.

It results $[c + \frac{1}{F}Y_{|\bar{0}} + \frac{1}{F}E_{|\bar{0}} - |\Phi|^2]m_l = 0$. Hereby, $Y_{|\bar{0}} = -cF - E_{|\bar{0}} + F\left|\Phi\right|^2$, which together with ii) implies iv).

Next we prove v) and vi). First we contract (3.7) with $\bar{\eta}^r m^j \bar{\eta}^h m^k m^l$ and so,

$(R_{\bar{0}j\bar{0}k} m^j m^k)|_l m^l - BR_{\bar{0}j\bar{0}k} m^j m^k - \Xi_{\bar{0}j\bar{0}l|k} m^j m^k m^l + R_{\bar{0}j\bar{0}n} C^n_{kl} m^j m^k m^l = 0$.

This implies that

$$A|_l m^l Lc = -[\Phi_{|k} m^k + (V+H)\Phi]. \qquad (3.8)$$

The contraction of (3.7) by $\bar{m}^r \eta^j \bar{\eta}^h m^k m^l$ implies

$(R_{\bar{r}0\bar{0}k}\bar{m}^r m^k)|_l m^l - (R_{\bar{r}l\bar{0}k}\bar{m}^r m^k + P_{\bar{r}0\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{0}}\bar{m}^r m^k - R_{\bar{r}0\bar{0}n}C^n_{kl}\bar{m}^r m^k)m^l = 0$, which gives

$H_{|\bar{0}} + \frac{F}{2}H(\bar{J} + \bar{Y}) = F\bar{E}_{|0}|_l m^l + L\Phi\bar{\Omega}$. Now, this together with ii), (3.8) and (2.12) invoice v).

The contraction of (3.7) by $\bar{m}^r\eta^j\bar{m}^h\eta^k m^l$ gives

$(R_{\bar{r}0\bar{h}0}\bar{m}^r\bar{m}^h)|_l m^l + BR_{\bar{r}0\bar{h}0}\bar{m}^r\bar{m}^h - R_{\bar{r}l\bar{h}0}\bar{m}^r\bar{m}^h m^l - R_{\bar{r}0\bar{h}l}\bar{m}^r\bar{m}^h m^l$

$-P_{\bar{r}0\bar{s}0}P^{\bar{s}}_{\bar{0}l\bar{h}}\bar{m}^r\bar{m}^h m^l = 0$, which is equivalent to

$Lc\bar{A}|_l m^l + \frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0} + E_{|\bar{s}}m^{\bar{s}} + B\bar{A}Lc = F\bar{\Phi}\Omega.$

Using ii), v) and (2.12), we obtain vi).

For vii) we contract (3.7) with $\bar{m}^r\eta^j\bar{m}^h m^k m^l$ and we deduce

$(R_{\bar{r}0\bar{h}k}\bar{m}^r\bar{m}^h m^k)|_l m^l + \frac{B}{2}R_{\bar{r}0\bar{h}k}\bar{m}^r\bar{m}^h m^k$

$+\frac{1}{L}R_{\bar{0}0\bar{h}k}\bar{m}^h m^k - R_{\bar{r}l\bar{h}k}\bar{m}^r\bar{m}^h m^k m^l + \frac{1}{L}R_{\bar{r}0\bar{0}k}\bar{m}^r m^k$

$-P_{\bar{r}0\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{h}}\bar{m}^r\bar{m}^h m^k m^l + R_{\bar{r}0\bar{h}n}C^n_{kl}\bar{m}^r\bar{m}^h m^k m^l = 0.$

From here it follows

$-F(E_{|\bar{s}}m^{\bar{s}})|_l m^l - \frac{B}{2}FE_{|\bar{s}}m^{\bar{s}} - V_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}V(\bar{V}+\bar{H}) - \mathbf{W} - \frac{1}{F}\bar{E}_{|0} + A\bar{A}Lc - BFE_{|\bar{s}}m^{\bar{s}} = L\,|\Omega|^2$, which leads to vii). The global validity of the above statements results by straightforward computations. □

Note that $\Phi = 0$ implies $\Omega = 0$, but the converse is not true, (see [5]). Next, some consequences of the above Proposition will be established. From vii) and (2.16), it immediately results.

Corollary 3.1 — Let (M, F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space. If it is weakly Kähler, then

$$K^h_{F,\mu}(z,\eta) = 2c(1 + |A|^2\,L) - 2F(E_{|\bar{s}}m^{\bar{s}})|_l m^l - 3BFE_{|\bar{s}}m^{\bar{s}} - 2L\,|\Omega|^2\,.$$

Remark that the conditions which assure that $K^h_{F,\lambda}(z,\eta) = 2c$ is a constant on M are not suffice such that to imply $K^h_{F,\mu}(z,\eta)$ is also a constant on M.

Corollary 3.2 — Let (M,F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space which is weakly Kähler and $|A| \neq 0$.

i) Then, $K^h_{F,\lambda}(z,\eta) = 0$ if and only if $\Phi_{|0} = F(J+Y)\Phi$.

ii) If $\Phi_{|k}m^k = (V+H)\Phi$ then

$$\mathbf{W} = c(1+|A|^2 L) - \frac{Lc}{2}(\mathbf{I} + \frac{1}{2}F\bar{A}B^2 + F\bar{A}B_{|k}m^k) - L\,|\Omega|^2 . \tag{3.9}$$

PROOF : Proposition 3.1 i) gives the equivalence i), which work globally on M. By Proposition 3.1 vi) we have $E_{|\bar{s}}m^{\bar{s}} = -\frac{Fc}{2}\left(\bar{B} - F\bar{A}B\right)$, which substituted into vii) gives (3.9), owing to (2.11). Their global validity complete the proof. □

***Theorem* 3.5** — *Let (M,F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space which is weakly Kähler, $|A| \neq 0$ and*

$$\Phi_{|k} = \Phi[(J+Y)l_k + (V+H)m_k].$$

Then

$$K^h_{F,\lambda}(z,\eta) = 0 \text{ and } K^h_{F,\mu}(z,\eta) = -2L\,|\Omega|^2 \leq 0.$$

Moreover, $K^h_{F,\mu}(z,\eta)$ is a constant on M if and only if $\Omega = \frac{\alpha}{F}$, where $\alpha \in \mathbf{C}$.

PROOF : It results by Corollary 3.2 and Proposition 3.1 vi) and vii). □

As a consequence of Theorem 3.1 and Proposition 3.1 ii), we obtain

Proposition 3.2 — Let (M,F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space. If it is weakly Kähler, then $K_1(z,\eta) = \frac{c}{2}(1+|A|^2 L)$ and $K_2(z,\eta) = \frac{c}{2}(1-|A|^2 L)$.

***Theorem* 3.6** — *Let (M,F) be a connected 2 - dimensional $(g.E)$ complex Finsler space. If (M,F) is weakly Kähler, then it is either purely Hermitian with $K^h_{F,\lambda}(z,\eta) = K^h_{F,\mu}(z,\eta) = 2c$ or it is non-purely Hermitian with $K^h_{F,\lambda}(z,\eta) = 0$.*

PROOF : By Proposition 3.2, under assumption of $(g.E)$, i.e. $K_1(z,\eta) = K_2(z,\eta)$, it follows $c\,|A|^2\,L = 0$. This implies either $|A| = 0$, i.e. (M,F) is purely Hermitian or $|A| \neq 0$ and $c = 0$. Now, making $|A| = 0$ in Proposition 3.1 it results $\mathbf{K} = \mathbf{W} = \mathbf{c}$. □

***Theorem* 3.7** — *Let (M,F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space. If (M,F) is Kähler, then it is either purely Hermitian or it is non-purely Hermitian with $\mathbf{K} = \mathbf{W} = \mathbf{I}_{|j} = 0$.*

PROOF : The assumption of Kähler gives $\Phi = \Omega = 0$. Now, using Proposition 3.2 i), it results $cAL = 0$, and so, either $|A| = 0$, i.e. (M,F) is purely Hermitian, or $|A| \neq 0$ and $c = 0$. Moreover, $\Omega = 0$ and $|A| \neq 0$ imply $\mathbf{K} = \mathbf{W} = 0$.

Also, in this second case, the identity ii) from Lemma 3.1 is $B|_{\bar{k}|j} - B_{|j}|_{\bar{k}} = 0$. But, the relations (2.11), (2.12) and $\mathbf{I} := -B|_{\bar{s}}m^{\bar{s}} - \frac{B\bar{B}}{2}$ lead to $B|_{\bar{k}|j} - B_{|j}|_{\bar{k}} = -\mathbf{I}_{|j}m_{\bar{k}}$. Hence, $\mathbf{I}_{|j} = 0$. □

Note that the first class of spaces, obtained in the preceding Theorem, has the horizontal holomorphic sectional curvatures in directions λ and μ equal to the same constant value $2c$ and $\mathbf{I} = 0$. But, there are 2 - dimensional complex Finsler spaces with $\mathbf{I} = 0$, which are not purely Hermitian, (see Proposition 4.1 from [5]). Moreover, the second class, obtained above, has property $R_{\bar{r}j\bar{h}k} = 0$. Thus, the horizontal holomorphic sectional curvature in any direction of such spaces is also zero.

***Theorem* 3.8** — *Let (M,F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space with $\mathbf{K} = c$ a nonzero constant on M. If $|A|^2 = \frac{1}{L}$ then,*

$$R_{\bar{r}k} = cLg_{k\bar{r}} \text{ and } \mathbf{W} = 2c - L\,|\Omega|^2\,.$$

Moreover, $K^h_{F,\mu}(z,\eta) = 4c - 2|\alpha|^2$ is a constant on M if and only if $\Omega = \frac{\alpha}{F}$, where $\alpha \in \mathbf{C}$.

PROOF : Taking into account Theorem 3.4 and Proposition 3.2 it results $K_1 = c$ and $K_2 = 0$. Hence, $R_{\bar{r}k} = cLg_{k\bar{r}}$.

Now, the assumption $|A|^2 = \frac{1}{L}$ together with (2.12), (3.8) and Proposition 3.1 vi) gives $E_{|\bar{s}}m^{\bar{s}} = 0$. So that, by Proposition 3.1 vii), we obtain $\mathbf{W} = 2c - L\,|\Omega|^2$, which implies $K^h_{F,\mu}(z,\eta) = 4c - 2L\,|\Omega|^2$. By means of this form of $K^h_{F,\mu}(z,\eta)$, the equivalence from Theorem is true. □

Remark 3.1 : Note that any 2 - dimensional complex Finsler space which satisfies $R_{\bar{r}k} = \Psi(z,\eta)Lg_{k\bar{r}}$ is a $(\eta - E)$ space with $K_1(z,\eta) = \Psi(z,\eta)$ and $K_2(z,\eta) = 0$. Moreover, if it is weakly Kähler, the Proposition 3.1 involves $|A|^2 = \frac{1}{L}$ and $\Psi = c$.

4. Holomorphic Bisectional Curvatures

We define the *horizontal holomorphic bisectional curvature* in directions $\{\lambda,\mu\}$ by

$$B^h_{F,\lambda,\mu}(z,\eta) := 2\mathbf{R}(\lambda,\bar{\lambda},\mu,\bar{\mu}) + \mathbf{2R}(\mu,\bar{\mu},\lambda,\bar{\lambda}) \tag{4.1}$$

and the *vertical holomorphic bisectional curvature* in directions $\{l,m\}$ by

$$B^v_{F,l,m}(z,\eta) := 2\mathbf{R}(l,\bar{l},m,\bar{m}) + 2\mathbf{R}(m,\bar{m},l,\bar{l}). \tag{4.2}$$

***Theorem* 4.1** — *Let (M,F) be a 2 - dimensional complex Finsler space. Then*

i) $B^h_{F,\lambda,\mu}(z,\eta) = -2[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})] - 2[\frac{1}{F}Y_{|\bar{0}} + BO_{|\bar{0}} - \frac{1}{2}FBO(\bar{J}+\bar{Y})]$;

ii) $B^v_{F,l,m}(z,\eta) = 0$.

Proof : By (2.5) and (2.15),

$\mathbf{R}(\lambda,\bar{\lambda},\mu,\bar{\mu}) = l^{\bar{r}}l^j m^{\bar{h}} m^k R_{\bar{r}j\bar{h}k} = -[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})]$ and

$\mathbf{R}(\mu,\bar{\mu},\lambda,\bar{\lambda}) = m^{\bar{r}}m^j l^{\bar{h}} l^k R_{\bar{r}j\bar{h}k} = \frac{1}{F}Y_{|\bar{0}} + BO_{|\bar{0}} - \frac{1}{2}FBO(\bar{J}+\bar{Y})$, which imply i).

Similarly, by (2.5) and (2.13), $\mathbf{R}(l,\bar{l},m,\bar{m}) = l^{\bar{r}}l^j m^{\bar{h}} m^k S_{\bar{r}j\bar{h}k} = 0$ and $\mathbf{R}(m,\bar{m},l,\bar{l}) = m^{\bar{r}}m^j l^{\bar{h}} l^k S_{\bar{r}j\bar{h}k} = 0$, and so ii).

Changing the local coordinates $(z^k,\eta^k)_{k=\overline{1,2}}$ into $(z'^k,\eta'^k)_{k=\overline{1,2}}$, it results

$V'_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V'(\bar{V}' + \bar{H}') = V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H})$ and

$\frac{1}{F}Y'_{|\bar{0}} + B'O'_{|\bar{0}} - \frac{1}{2}FB'O'(\bar{J}' + \bar{Y}') = \frac{1}{F}Y_{|\bar{0}} + BO_{|\bar{0}} - \frac{1}{2}FBO(\bar{J} + \bar{Y})$, which complete the proof. □

Suggested by [10], we consider here two *mixed holomorphic bisectional curvatures* in directions $\{\lambda, m\}$ and $\{l, \mu\}$, respectively. Namely,

$$\begin{aligned} B_{F,\lambda,m}(z,\eta) \;&:= 2\mathbf{R}(\lambda,\bar{\lambda},m,\bar{m}) + 2\mathbf{R}(m,\bar{m},\lambda,\bar{\lambda}); \\ B_{F,l,\mu}(z,\eta) \;&:= 2\mathbf{R}(l,\bar{l},\mu,\bar{\mu}) + 2\mathbf{R}(\mu,\bar{\mu},l,\bar{l}). \end{aligned}$$

But, due to (2.3), (2.4) and (2.5), it results $\mathbf{R}(\lambda,\bar{\lambda},m,\bar{m}) = \mathbf{R}(\mu,\bar{\mu},l,\bar{l}) = 0$. So that,

$B_{F,\lambda,m}(z,\eta) = 2\mathbf{R}(m,\bar{m},\lambda,\bar{\lambda}) = 2R_{\bar{r}j\bar{h}k}m^{\bar{r}}m^{j}l^{\bar{h}}l^{k}$

$= -2[\frac{1}{F}Y_{|\bar{0}} + BO_{|\bar{0}} - \frac{1}{2}FBO(\bar{J} + \bar{Y})]$ and

$B_{F,l,\mu}(z,\eta) = 2\mathbf{R}(l,\bar{l},\mu,\bar{\mu}) = 2R_{\bar{r}j\bar{h}k}l^{\bar{r}}l^{j}m^{\bar{h}}m^{k} = -2[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H})]$.

These, together with Theorem 4.1, involve

Corollary 4.1 — Let (M, F) be a 2 - dimensional complex Finsler space. Then,

$$B^{h}_{F,\lambda,\mu}(z,\eta) = B_{F,l,\mu}(z,\eta) + B_{F,\lambda,m}(z,\eta).$$

***Theorem* 4.2** — *Let (M, F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space. Then*

$$B^{h}_{F,\lambda,\mu}(z,\eta) = -2[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H})] - \frac{2}{F}Y_{|\bar{0}}. \tag{4.3}$$

Moreover,

i) if it is weakly Kähler, then $B^{h}_{F,\lambda,\mu}(z,\eta) = 2c(1 - |A|^2 L) - 2|\Phi|^2$;

ii) if it is Kähler non-purely Hermitian , then $B^{h}_{F,\lambda,\mu}(z,\eta) = 0$.

iii) if it is Kähler purely Hermitian, then $B^{h}_{F,\lambda,\mu}(z,\eta) = 2c$.

PROOF : Since (M, F) is $(\eta - E)$, $O_{|\bar{0}} - \frac{1}{2}FO(\bar{J} + \bar{Y}) = 0$, and so (4.3) is true. i) results by Proposition 3.1 iii) and iv). The assertions ii) and iii) follow by Theorem 3.7. □

Further more by above considerations, immediately results

Corollary 4.2 — Let (M, F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space. Then,

$$B_{F,\lambda,m}(z,\eta) = -\frac{2}{F}Y_{|\bar{0}}\ ;\ B_{F,l,\mu}(z,\eta) = -2[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H})]. \tag{4.4}$$

Moreover,

i) if it is weakly Kähler, then $B_{F,\lambda,m}(z,\eta) = B_{F,l,\mu}(z,\eta) - 2\,|\Phi|^2$ and $B_{F,l,\mu}(z,\eta) = c(1 - |A|^2\,L)$;

ii) if the space is Kähler non-purely Hermitian, then it is with vanishing mixed holomorphic bisectional curvatures;

iii) if it is Kähler purely Hermitian, then $B_{F,l,\mu}(z,\eta) = B_{F,\lambda,m}(z,\eta) = c$.

***Theorem* 4.3** — *Let (M, F) be a connected 2 - dimensional $(\eta - E)$ complex Finsler space. If (M, F) is weakly Kähler and $|A|^2 = \frac{1}{L}$, then*

$$B^h_{F,\lambda,\mu}(z,\eta) = -2\,|\Phi|^2 \leq 0.$$

Moreover, $B^h_{F,\lambda,\mu}(z,\eta)$ is a constant on M if and only if $\Phi = \beta$, where $\beta \in \mathbf{C}$.

PROOF : It results by Theorem 4.2 i), under assumption $|A|^2 = \frac{1}{L}$. □

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References

1. M. Abate and G. Patrizio, *Finsler Metrics - A Global Approach*, Lecture Notes in Math., **1591**, Springer-Verlag, 1994.

2. T. Aikou, *Some remarks on locally conformal complex Berwald spaces*, Finsler geometry (Seattle, WA, 1995), 109–120, Contemp. Math., **196**, AMS Prov. RI, 1996.

3. N. Aldea, On holomorphic curvature of η - Einstein complex Finsler spaces, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, **51**(99), **4** (2008), 265-277.

4. N. Aldea, *Complex Finsler spaces of constant holomorphic curvature*, Diff. Geom. and its Appl., Proc. Conf. Prague 2004, Charles Univ. Prague (Czech Republic) 2005, 179-190.

5. N. Aldea and G. Munteanu, On two-dimensional complex Finsler manifolds, arXiv: 1010.3409.

6. N. Aldea and G. Munteanu, On complex Finsler spaces with Randers metrics, *Journal Korean Math. Soc.*, **46**, 5(2009), 949-966.

7. D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemannian Finsler Geom.*, Graduate Texts in Math., **200**, Springer-Verlag, 2000.

8. A. Bejancu and H. R. Faran, *The geometry of pseudo-Finsler submanifolds*, Kluwer Acad. Publ., 2000.

9. L. Berwald, On Finsler and Cartan geometries, III. Two-dimensional Finsler spaces with rectilinear extremals, *Ann. of Math.*, (20), **42** (1941), 84-112.

10. J.-G. Cao and P.-M. Wong, Finsler geometry of projectivized vector bundle, *J. Math. Kyoto Univ.* 43, 2 (2003), 369–410.

11. B. Chen and Y. Shen, Kähler Finsler metrics are actually strongly Kähler, *Chin. Ann. Math. Ser. B*, **30**(2) (2009), 173-178.

12. C. K. Cheung and K.-T. Kim, Analysis of the Wu metric. I. The case of convex Thullen domains, *Trans. Amer. Math. Soc.* **348**, 4 (1996), 1429–1457.

13. M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa, Otsu, 1986.

14. R. Miron and M. Anastasiei, *The geometry of Lagrange Spaces; Theory and Applications*, Kluwer Acad. Publ., **59**, FTPH, 1994.

15. G. Munteanu, *Complex spaces in Finsler, Lagrange and Hamilton geometries*, Kluwer Acad. Publ., **141**, FTPH, 2004.

16. S. Nishikawa, *Harmonic maps of Finsler manifolds*. *Topics in differential geometry*, Ed. Acad. Române, Bucharest, 2008, 207-247.

17. Z. Shen, Two-dimensional Finsler metrics with constant flag curvature, *Manuscripta Mathematica*, **109**, 3 (2002), 349-366.

18. P.-M. Wong, A survey of complex Finsler geometry, *Advanced Studied in Pure Math.*, Vol. **48**, Math. Soc. of Japan, (2007), 375-433.

19. R.-M. Yan, Connections on complex Finsler manifold, *Acta Math. Appl. Sinica, English Serie*, **19**, 3 (2003), 431-436.