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# LOCALLY TOPOLOGICAL ERGODICITY AND WEAKLY MIXING FOR BOUNDED LINEAR OPERATORS[1]

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In this article, we discuss $J^{erg}$-class operators and $J^{wmix}$-class operators which can be viewed as a ”localization” of the notions of topological ergodicity and weakly mixing respectively.

**Key words** : Topological ergodicity, weakly mixing, shift, $J^{erg}$-class, $J^{wmix}$-class.

## 1. INTRODUCTION AND PRELIMINARIES

A discrete dynamical system is simply a continuous mapping $T : X \to X$ where $X$ is a complete separable metric space. For $x \in X$, the orbit of $x$ under $T$ is

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$Orb(T,x) = \{x, T(x), T^2(x), \ldots\}$ where $T^n = T \circ T \circ \cdots \circ T$ is the $n^{\text{th}}$ iteration of $T$.

In this paper, we are interested in the case of that $T$ is a bounded linear operators on a Banach space $X$. There have been many dynamical properties discussed such as hypercyclic, frequently hypercyclic, topological mixing, Devaney chaos and so on. Specially, ones gave characterizations of these properties for unilateral backward weighted shift operators.

Let us recall some basic notions and conclusions in dynamical systems (see [1], [2] and [6]).

Let $A = \{n_k\}_{k=1}^{\infty}$ be a subset of $\mathbb{N}$. Denote $[m,n] = \{m, m+1, \ldots, n-1, n\}$ for any two integers $m, n \in \mathbb{N}$ with $m \leq n$. $A$ is said to be thick, if for each $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $[N+1, N+n] \subseteq A$. Denote $\mathcal{F}_t$ the family of all thick sets. $A$ is said to be syndetic, if $\sup\limits_k (n_{k+1} - n_k) < \infty$, i.e., $A$ has bounded gaps. Denote $\mathcal{F}_s$ the family of all syndetic sets. $A$ is said to be thickly syndetic, if for any $n \in \mathbb{N}$ there exists a syndetic set $s_1^n < s_2^n < \ldots$ such that $\bigcup\limits_{j=1}^{\infty} [s_j^n, s_j^n + n] \subseteq A$. Denote $\mathcal{F}_{ts}$ the family of all thickly syndetic sets.

The lower density of $A$ is defined by

$$\liminf_{n\to\infty} \frac{A \bigcap \{0, 1, \ldots, n-1\})}{n} > 0.$$

Similarly, the upper density of $A$ is defined by

$$\limsup_{n\to\infty} \frac{A \bigcap \{0, 1, \ldots, n-1\})}{n} > 0.$$

Let $T : X \to X$ be an operator acting on a separable Banach space $X$. For any $U, V \subseteq X$ and $x \in X$, define

$$N_T(x, V) = \{n \in \mathbb{N}; T^n(x) \in V\},$$

and

$$N_T(U, V) = \{n \in \mathbb{N}; T^n(U) \bigcap V \neq \phi\}.$$

Then we can use these sets to describe some notations of dynamical properties. For any nonempty open subsets $U, V \subseteq X$, if $N_T(U,V)$ is nonempty (or infinite), then $T$ is called hypercyclic; if $N_T(U,V)$ is a thick set, then $T$ is called topologically weakly mixing; if $N_T(U,V)$ is cofinite (i.e., it contains $[N,+\infty)$ for some $N \in \mathbb{N}$), then $T$ is called topologically mixing; if $N_T(U,V)$ is a syndetic set, then $T$ is called topologically ergodic. If there exists a vector $x \in X$ such that for any open subsets $V \subseteq X$, $N_T(x,V)$ has positive lower density, then $T$ is called frequently hypercyclic.

Let $l^p$ be the classical Banach space of absolutely $p^{\text{th}}$ power summable sequences $x = (x_1, x_2, \ldots)$ and we use $\| \cdot \|_p$ to represent its norm. Let $\Omega = \{\omega_n\}_{n=1}^{\infty}$ be a bounded sequence of nonzero complex numbers and let $\pi_n$ be the projection from $l^p$ to the $n^{\text{th}}$ coordinate, i.e., $\pi_n(x) = x_n$ for $n \geq 1$. The unilateral backward weighted shift operator $T$ with weight sequence $\Omega$ is defined on $l^p$ by

$$\pi_n \circ T(x) = \omega_n x_{n+1}, \quad for\ all\ n \geq 1,$$

where $x = (x_1, x_2, \ldots)$.

*Proposition* 1.1 — Suppose $T$ is a weighted backward shift operator on $l^p$, $1 \leq p < \infty$, with weight sequence $\{\omega_n\}_{n=1}^{\infty}$. Denote $\beta(n)$ as

$$\beta(n) = \prod_{i=1}^{n} \omega_i, \quad for \quad n = 1, 2, \ldots \quad ,$$

then

$(I)$ ([7]) $T$ is Devaney chaotic if and only if $\sum\limits_{n=1}^{\infty} \frac{1}{|\beta(n)|^p} < \infty$;

$(II)$ ([3]) $T$ is topologically mixing if and only if $\lim\limits_{n\to\infty} |\beta(n)| = \infty$.

$(III)$ ([9]) $T$ is hypercyclic if and only if $T$ is weakly mixing, if and only if $\limsup\limits_{n\to\infty} |\beta(n)| = \infty$.

In the present article, we shall discuss $J^{erg}$-class operators and $J^{wmix}$-class operators which can be viewed as a ”localization” of the notions of topological ergodicity and weakly mixing respectively. This is inspired by the work of

G. Costakis and A. Manoussos. In their article [4], J-class operators are well discussed.

*Definition* 1.2 — Let $T : X \to X$ be an linear operator acting on a Banach space $X$ and $x \in X$. The $J$-set of $x$ is defined by

$$J(x) = \{y \in X : there\ exist\ a\ strictly\ increasing\ sequence\ of\ positive\ integers\ \{k_n\}\ and\ a\ sequence\ \{x_n\} \subseteq X\ such\ that\ x_n \to x\ and\ T^{k_n}(x_n) \to y\}$$

or equivalently

$$J(x) = \{y \in X : for\ every\ pair\ of\ neighborhoods\ U, V\ of\ x, y\ respectively,\ N_T(U, V)\ is\ a\ infinite\ set\}.$$

Furthermore, $x$ is called a J-vector if $J(x) = X$ and $T$ is called a J-class operator if there exists a nonzero J-vector.

A characterization of hypercyclic operators through J-sets had been given.

*Proposition* 1.3 — Let $T : X \to X$ be an operator acting on a separable Banach space $X$. The following are equivalent.

(i) $T$ is hypercyclic;

(ii) For every $x \in X$ it holds that $J(x) = X$;

(iii) The set $A = \{x \in X : J(x) = X\}$ is dense in $X$.

## 2. $J^{erg}$-Class and $J^{wmix}$-Class Operators

*Definition* 2.1 — Let $T : X \to X$ be an linear operator acting on a Banach space

$X$ and $x \in X$. The $J^{erg}$-set of $x$ is defined by

$$J^{erg}(x) = \{y \in X : for\ every\ pair\ of\ neighborhoods\ U, V\ of\ x, y\ respectively,\ N_T(U,V)\ is\ a\ syndetic\ set.\}$$

Furthermore, $x$ is called a $J^{erg}$-vector if $J^{erg}(x) = X$ and $T$ is called a $J^{erg}$-class operator if there exists a nonzero $J^{erg}$-vector.

*Proposition* 2.2 — Let $T : X \to X$ be a linear operator acting on a separable Banach space $X$. Then

(i) The set of all $J^{erg}$-vectors is closed;

(ii) For any $J^{erg}$-vector $x$ and any $\lambda \in \mathbb{C}$, $\lambda x$ is a $J^{erg}$-vector.

PROOF : (i) Let $\{x_n\}$ be a sequence of $J^{erg}$-vectors and $x_n \to x$ as $n \to \infty$. Given any nonempty open set $V$ in $X$ and any neighborhood $W$ of $x$. Then there exists $n \in \mathbb{N}$ such that $x_n \in W$. Consequently, there is a neighborhood $U$ of $x_n$ such that $U \subseteq W$. Since $N_T(U,V) \subseteq N_T(W,V)$, then $N_T(W,V)$ is a syndetic set and hence $x$ is a $J^{erg}$-vector.

(ii) Let $x$ be a $J^{erg}$-vector and $\lambda \in \mathbb{C}$. Suppose $\lambda \neq 0$. Given any nonempty open set $V$ in $X$ and any neighborhood $U$ of $\lambda x$. Denote $U' = \frac{1}{\lambda}U = \{\lambda y; y \in U\}$ and $V' = \frac{1}{\lambda}V = \{\lambda y; y \in V\}$. Then $U'$ and $V'$ are two nonempty open sets and $N_T(U', V')$ is syndetic. Notice that $N_T(U,V) = N_T(U', V')$. Therefore, $\lambda x$ is a $J^{erg}$-vector. Since $\frac{1}{n}x \to 0$ as $n \to \infty$, then $0$ is a $J^{erg}$-vector by (i).

Now it is not difficult to give a characterization of topological ergodicity by $J^{erg}$ sets.

*Proposition* 2.3 — Let $T : X \to X$ be a linear operator acting on a separable Banach space $X$. The following are equivalent.

(I) $T$ is topologically ergodic;

(II) For every $x \in X$ it holds that $J^{erg}(x) = X$;

(III) The set $A = \{x \in X : J^{erg}(x) = X\}$ is dense in $X$.

We shall show another version of a useful lemma in [2].

*Lemma* 2.4 — Let $T : X \to X$ be a linear operator acting on a separable Banach space $X$ and let $x$ be a J-vector.

(i) For any neighborhood $W$ of $0$, any neighborhood $U$ of $x$ and any nonempty open subset $V$ in $X$, the sets $N_T(U, W)$ and $N_T(W, V)$ are thick.

(ii) Suppose that all the sets $N_T(U, V) \bigcap N_T(W, V)$ are nonempty, for $U, V, W$ as above. Then all these sets are thick.

PROOF : Notice that $0$ is a J-vector if $T$ is a J-class operator. If we replace $x$ being J-vector to the condition of $T$ being hypercyclic, the proof of [2] also holds.□

*Proposition* 2.5 — Let $T : X \to X$ be a linear operator acting on a separable Banach space $X$. If $x, y$ are any two $J^{erg}$-vectors, then $x + y$ is a $J$-vector.

PROOF : Suppose that $x$ and $y$ are two $J^{erg}$-vectors. Given any nonempty open set $V$ in $X$ and any neighborhood $U$ of $x + y$. There exist a neighborhood $U_1$ of $x$, a neighborhood $U_2$ of $y$, a neighborhood $W$ of $0$ and a nonempty open set $V' \subseteq V$ such that $U_1 + U_2 = \{a + b; a \in U_1 \; and \; b \in U_2\} \subseteq U$ and $V' + W = \{a+b; a \in V' \; and \; b \in W\} \subseteq V$. By Lemma 2.4, $N_T(U_1, W)$ is thick. In addition, $N_T(U_2, V')$ is syndetic. Thus, $N_T(U, W) \supseteq N_T(U_1, W) \bigcap N_T(U_2, V') \neq \phi$ and consequently $x + y$ is a $J$-vector.

*Lemma* 2.6 [8] — Suppose $T$ is a weighted backward shift operator on $l^p$, $1 \leq p < \infty$, with weight sequence $\{\omega_n\}_{n=1}^{\infty}$. Then the following conditions are equivalent:

$(I)$ $T$ is topologically ergodic;

$(II)$ For any $K > 0$, the set $\{n \in \mathbb{N}; |\beta(n)| > K\}$ is syndetic;

$(III)$ For any $K > 0$, the set $\{n \in \mathbb{N}; |\beta(n)| > K\}$ is thickly syndetic.

***Theorem* 2.7** — *Suppose that $T$ is a weighted backward shift operator on $l^p$, $1 \leq p < \infty$, with weight sequence $\{\omega_n\}_{n=1}^{\infty}$. Then the following conditions are equivalent:*

$(i)$ $T$ *is a $J^{erg}$-class operator;*

$(ii)$ $0$ *is a $J^{erg}$-vector;*

$(iii)$ $T$ *is topologically ergodic.*

PROOF : $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are obvious. It suffices to prove $(ii) \Rightarrow (iii)$. Suppose that $0$ is a $J^{erg}$-vector. Then the proof of $(I) \Rightarrow (II)$ in Lemma 2.6 also holds, consequently the set $\{n \in \mathbb{N}; |\beta(n)| > K\}$ is syndetic for any $K > 0$ and hence $T$ is topologically ergodic.

Similarly, we can introduce a ”localization” of the notion of weakly mixing.

*Definition* 2.8 — Let $T : X \to X$ be an linear operator acting on a Banach space $X$ and $x \in X$. The $J^{wmix}$-set of $x$ is defined by

$$J^{wmix}(x) = \{y \in X : for\ every\ pair\ of\ neighborhoods\ U, V\ of\ x, y\ respectively,\ N_T(U,V)\ is\ a\ thick\ set\}.$$

Furthermore, $x$ is called a $J^{wmix}$-vector if $J^{wmix}(x) = X$ and $T$ is called a $J^{wmix}$-class operator if there exists a nonzero $J^{wmix}$-vector.

A characterization of topological weakly mixing by $J^{wmix}$ sets is easily obtained.

*Proposition* 2.9 — Let $T : X \to X$ be a linear operator acting on a separable Banach space $X$. The following are equivalent.

(I) $T$ is weakly mixing;

(II) For every $x \in X$ it holds that $J^{wmix}(x) = X$;

(III) The set $A = \{x \in X : J^{wmix}(x) = X\}$ is dense in $X$.

***Theorem* 2.10** — *Let $T : X \to X$ be a linear operator acting on a separable Banach space $X$.If $x$ is a $J^{erg}$-vector, then $x$ is a $J^{wmix}$-vector. Furthermore, if $T$ is a $J^{erg}$-class operator, then $T$ is a $J^{wmix}$-class operator.*

PROOF : Suppose that $x$ is a $J^{erg}$-vector. Given any neighborhood $U$ of $x$ and any nonempty open subset $V$ in $X$. There exist a neighborhood $U'$ of $x$, a nonempty open subset $V'$ and a neighborhood $W$ of $0$ such that $U' + W \subseteq U$ and $V' + W \subseteq V$. By Lemma 2.4, $N_T(U', W)$ is thick. In addition, $N_T(W, V')$ is syndetic. Thus, $N_T(U, V) \supseteq N_T(U', W) \bigcap N_T(W, V') \neq \phi$ and consequently $N_T(U, V)$ is thick by Lemma 2.4. □

In [4], $J^{mix}$-class operators have been discussed.

*Definition* 2.11 — Let $T : X \to X$ be an linear operator acting on a Banach space $X$ and $x \in X$. The $J^{mix}$-set of $n$ is defined by

$$J^{mix}(x) = \{y \in X : there\ exist\ a\ sequence\ \{x_n\} \subseteq X\ such\ that\ x_n \to x\ and\ T^n(x_n) \to y\}$$

or equivalently

$$J^{mix}(x) = \{y \in X : for\ every\ pair\ of\ neighborhoods\ U, V\ of\ x, y\ respectively,\ N_T(U, V)\ is\ a\ cofinite\ set\}.$$

Furthermore, $x$ is called a $J^{mix}$-vector if $J^{mix}(x) = X$ and $T$ is called a $J^{mix}$-class operator if there exists a nonzero $J^{mix}$-vector.

*Proposition* 2.12 [4] — Let $T : X \to X$ be a linear operator acting on a separable Banach space $X$. The following are equivalent.

(i) $T$ is mixing;

(ii) For every $x \in X$ it holds that $J^{mix}(x) = X$;

(iii) The set $A = \{x \in X : J^{mix}(x) = X\}$ is dense in $X$.

Now one can see

$$J^{mix} - class \Rightarrow J^{erg} - class \Rightarrow J^{wmix} - class \Rightarrow J - class.$$

Specially, they are equivalent for backward unilateral weighted shifts on $l^{\infty}(\mathbb{N})$. This is a straightforward corollary of the following conclusion from [5].

*Proposition* 2.13 [5] — Let $T : l^{\infty}(\mathbb{N}) \to l^{\infty}(\mathbb{N})$ be a backward unilateral weighted shifts with positive weights $\{\alpha_n\}_{n\in\mathbb{N}}$. Then the following are equivalent.

(i) $T$ is a $J$-class operator;

(ii) $T$ is a $J^{mix}$-class operator;

(iii) $\lim_{n\to\infty}(\inf_{j\geq 0}\prod_{i=1}^{n}\alpha_{i+j}) = +\infty$.

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