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RESOLUTION OF VERONESE EMBEDDING OF PLANE CURVES

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Let C be a smooth (irreducible) curve of degree d in $\mathbb{P}^2$. Let $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ be the Veronese embedding and let $\mathcal{I}_C$ denote the homogeneous ideal of C on $\mathbb{P}^5$. In this note we explicitly write down the minimal free resolution of $\mathcal{I}_C$ for $d \geq 2$.

Key words : Minimal resolutions, Veronese embedding, plane curves.

1. INTRODUCTION

In [3], the author has remarked, " It is very exceptional to be able to construct the whole resolution explicitly, let alone to be able to do so by hand!." This remark of Lazarfeld motivated us to try to explicitly calculate whole resolutions of projective varieties.

In this paper we have explicitly calculated the whole resolutions of the Veronese embedding of plane curves. We look at the even and odd degree curves separately and get the explicit resolution for both.

Let C be a smooth and irreducible projective curve and L be an ample line bundle on C, generated by its global sections. Then L determines a morphism

$$\Phi_L : C \longrightarrow \mathbb{P}(H^0(L)) = \mathbb{P}^r$$

where $r = h^0(L) - 1$. Also we have that if L is very ample, then Φ_L is an embedding.

Let $\mathcal{I}_C$ be the homogeneous ideal of C in $\mathbb{P}^r$ and S, homogeneous coordinate ring of the projective space, $\mathbb{P}^r$

Let $R = S/\mathcal{I}_C$, then the minimal graded free resolution of R is the following exact sequence of free modules:

$$0 \rightarrow E_n \stackrel{\alpha_n}{\rightarrow} \ldots \stackrel{\alpha_3}{\rightarrow} E_2 \stackrel{\alpha_2}{\rightarrow} E_1 \stackrel{\alpha_1}{\rightarrow} E_0 \rightarrow R \rightarrow 0 \ldots (A)$$

where each E_i is a direct sum of twists of S, i.e.

$$E_i = \oplus_j S(-a_{ij}),$$

And the maps, α_i's in the above exact sequence are given by matrices of homogeneous forms and none of the entries in the above matrices are non-zero constants. Note that $E_0 = S$ and the image of α_1 is the ideal of S, $\mathcal{I}_C$.

In this note we look at C, a smooth(irreducible) curve of degree d such that $C \hookrightarrow \mathbb{P}^2$(here L is $\mathcal{O}_C(1)$). Now because of the Veronese embedding, we get an embedding of C in $\mathbb{P}^5$ which is nothing but the embedding of C in $\mathbb{P}^5$ due to the very ample line bundle, $\mathcal{O}_C(2)$. We explicitly calculate minimal free resolution of $\mathcal{I}_C$ and in particular get the equations defining C in $\mathbb{P}^5$. Most of the definitions in this note are from [1] and [2].

NOTATIONS

The *first syzygy module* is defined as the image of α_2 in E_1 in the exact sequence (A) and is denoted by $\mathrm{Syz}^1(\mathcal{I}_C)$.

The k^{th} *syzygy module* is defined inductively to be the module of syzygies of the $(k-1)^{st}$ *syzygy module*. Hence we have the following inductive relation:

$$\mathrm{Syz}^k(\mathcal{I}_C) = \mathrm{Syz}^1(\mathrm{Syz}^{k-1}(\mathcal{I}_C))$$

2. RESOLUTIONS OF VERONESE EMBEDDING

Consider $\sigma : \mathbb{P}^2 \to \mathbb{P}^5$ such that for $p = (a_0, a_1, a_2) \in \mathbb{P}^2$,

$$\sigma(p) = (a_0^2, a_0a_1, a_0a_2, a_1^2, a_1a_2, a_2^2)$$

This is called the *Veronese embedding* of $\mathbb{P}^2$ in $\mathbb{P}^5$ [2].

Now if $x_{00}, x_{01}, x_{02}, x_{11}, x_{12}, x_{22}$ denote homogeneous coordinates on $\mathbb{P}^5$, then one has a description of $\sigma(\mathbb{P}^2)$ as the zeros of the six minors of the following 3×3 symmetric matrix.

$$\begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{01} & x_{11} & x_{12} \\ x_{02} & x_{12} & x_{22} \end{pmatrix}$$

Moreover we also get a map,

$$\theta : k[x_{00}, x_{01}, x_{02}, x_{11}, x_{12}, x_{22}] \to k[x_0, x_1, x_2]$$

such that, $\theta(x_{ij}) = x_i x_j \ \forall 0 \le i \le j \le 2$.

Also the defining equations of this embedding are:

$$\begin{aligned}
\Delta_{00} &= x_{11}x_{22} - x_{12}^2 \\
\Delta_{01} &= x_{01}x_{22} - x_{12}x_{02} \\
\Delta_{02} &= x_{01}x_{12} - x_{02}x_{11} \\
\Delta_{11} &= x_{00}x_{22} - x_{02}^2 \\
\Delta_{12} &= x_{00}x_{12} - x_{02}x_{01} \\
\Delta_{22} &= x_{00}x_{11} - x_{01}^2
\end{aligned}$$

Notice that,

$$ker(\theta) =< \quad \Delta_{i,j}, \forall 0 \leq i \leq j \leq 2 \quad >$$

From now we will denote $k[x_{00}, x_{01}, x_{02}, x_{11}, x_{12}, x_{22}]$ as S. And for $d \in \mathbb{Z}$, $S(d)$ is the graded S module such that $S(d)_n = S_{d+n}$

***Theorem* 1** [4] — *The ideal $\mathcal{I}_{\mathbb{P}^2}$ of $\sigma(\mathbb{P}^2)$ in $\mathbb{P}^5$ has the following resolution.*

$$0 \to S(-4)^{\oplus 3} \stackrel{M_3}{\to} S(-3)^{\oplus 8} \stackrel{M_2}{\to} S(-2)^{\oplus 6} \stackrel{M_1}{\to} \mathcal{I}_{\mathbb{P}^2} \to 0 \tag{1}$$

where,

$$M_1 = \begin{bmatrix} \Delta_{00}, & \Delta_{01}, & \Delta_{02},, & \Delta_{11}, & \Delta_{12}, & \Delta_{22} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} x_{02} & 0 & x_{01} & -0 & 0 & x_{00} & 0 & 0 \\ -x_{12} & x_{02} & -x_{11} & x_{01} & 0 & 0 & x_{00} & 0 \\ x_{22} & 0 & x_{12} & x_{02} & x_{01} & x_{02} & 0 & x_{00} \\ 0 & -x_{12} & 0 & -x_{11} & 0 & -x_{11} & -x_{01} & 0 \\ 0 & x_{22} & 0 & 0 & -x_{11} & x_{12} & x_{02} & -x_{01} \\ 0 & 0 & 0 & x_{22} & x_{12} & 0 & 0 & x_{02} \end{bmatrix} \tag{2}$$

$$M_3 = \begin{bmatrix} x_{01} & x_{00} & 0 \\ -x_{11} & -x_{01} & 0 \\ -x_{02} & 0 & x_{00} \\ x_{12} & x_{02} & 0 \\ -x_{22} & 0 & x_{02} \\ 0 & -x_{02} & -x_{01} \\ 0 & x_{12} & x_{11} \\ 0 & -x_{22} & -x_{12} \end{bmatrix} \tag{3}$$

3. Resolutions of Plane Curves in the Veronese Embedding

Let C be a smooth(or irreducible) curve such that, $C \hookrightarrow \mathbb{P}^2$. Hence C is given by a irreducible polynomial in three variables. Now recall that $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. Hence we have $C \hookrightarrow \mathbb{P}^2 \xrightarrow{\sigma} \mathbb{P}^5$. We will compute the syzygies of the homogeneous ideal $\mathcal{I}_{\sigma(C)}$ using this embedding and the resolution of the Veronese embedding above. Let C be defined by the polynomial f of degree d in three variables. Hence,

$$C = \mathcal{Z}(f(x_0, x_1, x_2))$$

Let,

$$f = \sum_{i+j+k=d} a_{i,j,k} x_0^i x_1^j x_2^k$$

3.1: Degree of f is even

We have d is even (say $2m$), and

$$f = \sum_{i+j+k=2m} a_{i,j,k} x_0^i x_1^j x_2^k$$

Lemma 2 — $Im(\theta)$ is a sub-algebra of $K[x_0, x_1, x_2]$ and is generated by even polynomials.

Proof : To prove that $f \in Im(\theta)$. We split f in four parts, depending on the parities of i,j,k, i.e., $f = f^I + f^{II} + f^{III} + f^{IV}$ with;

$$f^I = \sum_{\substack{i+j+k=d,\\ i,j,k \text{ even}}} a_{i,j,k} x_0^i x_1^j x_2^k$$

and so on.

Case I : When i, j, k are all even, consider

$$F^I = \sum_{\substack{i+j+k=d\\ i,j,k \text{ even}}} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}}$$

Notice that $\theta(\mathbf{F^I}) = \mathbf{f^I}$

Case II : When i is even, j and k odd, consider

$$F^{II} = \sum_{\substack{i+j+k=d \\ i \text{ even} \\ j,k \text{ odd}}} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k-1}{2}} x_{12}$$

Similarly as Case I, $\theta(\mathbf{F^{II}}) = \mathbf{f^{II}}$

Case III : With i is odd, j is even, k is odd consider

$$F^{III} = \sum_{\substack{i+j+k=d \\ j \text{ even} \\ i,k \text{ odd}}} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k-1}{2}} x_{02}$$

$\theta(\mathbf{F^{III}}) = \mathbf{f^{III}}$

Case IV : i is odd, j is odd, k is even consider,

$$F^{IV} = \sum_{\substack{i+j+k=d \\ k \text{ even} \\ i,j \text{ odd}}} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k}{2}} x_{01}$$

$\theta(\mathbf{F^{IV}}) = \mathbf{f^{IV}}$

Now let,

$$\mathbf{F} = \mathbf{F^I} + \mathbf{F^{II}} + \mathbf{F^{III}} + \mathbf{F^{IV}}$$

Then

$$\theta(\mathbf{F}) = \mathbf{f}$$

Hence $\mathbf{f} \in \mathbf{Im}(\theta)$.

Lemma 3 — Let $G \in S$ such that, G homogeneous and $\mathcal{Z}(\theta(F)) \subset \mathcal{Z}(\theta(G)) \subset \mathbb{P}^2$, where F is a irreducible polynomial of even degree. Then $G \in < F, \Delta_{i,j} : 0 \leq i \leq j \leq 2 >$.

PROOF : Let $\theta(G) = g$, then g is a homogeneous polynomial and,

$$\mathcal{Z}(f) \subset \mathcal{Z}(g)$$

$\Rightarrow g \in (f)$ as C is a irreducible curve and hence f is irreducible hence,

$$g = f.h \text{ for some } h \text{ homogeneous in } K[x_0, x_1, x_2]$$

Now f and g are even degree implies that h is of even degree hence, $\exists H \in S$, homogeneous such that $\theta(H) = h$.
Thus $\theta(G) = \theta(F).\theta(H) = \theta(F.H)$,

$$\Rightarrow \theta(G - F.H) = 0$$

$$\Rightarrow G - F.H \in ker(\theta)$$

$$\Rightarrow G - F.H = \sum_{0 \leq i \leq j \leq 2} \Delta_{ij} S_{ij} \text{ for some } S_{ij} \in S, S_{ij} \text{ homogeneous}$$

$$\Rightarrow G \in < F, \Delta_{ij} : 0 \leq i \leq j \leq 2 >$$

This completes the proof of the lemma.

Now recall M_2 and M_3 from equations (2) and (3), from now we will denote them as below: Let us denote the i^{th} row of M_2 as W_i and j^{th} row of M_3 as G_j, for $1 \leq i \leq 8$ and $j = 1, 2, 3$. Hence we get,

$$M_2 = \left[\begin{array}{cccccccc} W_1, & W_2, & W_3, & W_4, & W_5, & W_6, & W_7, & W_8 \end{array} \right] \qquad (*)$$

$$M_3 = \left[\begin{array}{ccc} G_1, & G_2, & G_3 \end{array} \right] \qquad (**)$$

***Theorem* 4** — *Let C be an irreducible curve of even degree say $d = 2m$, $m \geq 1$. The homogeneous ideal $\mathcal{I}_C$ of $\sigma(C)$ in $\mathbb{P}^5$ has the following resolution.*

$$\begin{aligned} 0 \to S(-m-4)^{\oplus 3} \quad & \stackrel{\alpha_4}{\to} S(-4)^{\oplus 3} \oplus S(-m-3)^{\oplus 8} \\ & \stackrel{\alpha_3}{\to}\stackrel{\alpha_3}{\to} S(-3)^{\oplus 8} \oplus S(-m-2)^{\oplus 6} \stackrel{\alpha_2}{\to} S(-2)^{\oplus 6} \\ & \oplus S(-m) \stackrel{\alpha_1}{\to} S \to S/\mathcal{I}_C \to 0 \end{aligned} \qquad (4)$$

where,

$$\alpha_1 = \begin{bmatrix} [M_1], & F \end{bmatrix} \tag{5}$$

Let

$$\begin{aligned} \alpha_2 \;\; = \;\; & [W_1', \; W_2', \; W_3', \; W_4', \; W_5', \; W_6', \; W_7', \; W_8', \\ & U_{00}, \; U_{01}, \; U_{02}, \; U_{11}, \; U_{12}, \; U_{22}] \end{aligned} \tag{6}$$

where

$$W_i' = \begin{bmatrix} W_i \\ 0 \end{bmatrix} \qquad \forall i = 1, \ldots, 8$$

with W_i as in (*)

Hence,

$$\alpha_2 = \begin{bmatrix} M_2 & -FI_6 \\ 0 & M_1 \end{bmatrix}$$

Let

$$H_i = \begin{bmatrix} [F.I_i^8] \\ [W_i] \end{bmatrix}$$

where

$$I_i^k = \begin{bmatrix} 0, & 0, & \ldots, & \overset{i^{th} \text{ position}}{1}, & 0, & \ldots, & 0 \end{bmatrix}^T \text{ is a } k \times 1 \text{ vector}$$

$$\alpha_3 = \begin{bmatrix} G_1', & G_2', & G_3', & H_1, & \ldots, & H_8 \end{bmatrix} \tag{7}$$

where,

$$G_i' = \begin{bmatrix} G_i \\ [\bar{0}] \end{bmatrix} \qquad \text{for } i = 1, 2, 3$$

where G_i as in (**) and $[\bar{0}]$ is a 0 matrix of appropriate dimension, hence we have,

$$\alpha_3 = \begin{bmatrix} M_3 & -FI_8 \\ 0 & M_2 \end{bmatrix}$$

Now let

$$\alpha_4 = \left[\begin{pmatrix} [-F.I_1^3] \\ [G_1] \end{pmatrix}, \begin{pmatrix} [-F.I_2^3] \\ [G_2] \end{pmatrix}, \begin{pmatrix} [-F.I_3^3] \\ [G_3] \end{pmatrix} \right] \tag{8}$$

Hence we can write that,

$$\alpha_4 = \begin{bmatrix} [-F.I^3] \\ [M_3] \end{bmatrix}$$

PROOF : From Lemma 3, it is clear that

$$\alpha_1 = \begin{bmatrix} \Delta_{00}, & \Delta_{01}, & \Delta_{02}, & \Delta_{11}, & \Delta_{12}, & \Delta_{22}, & F \end{bmatrix}$$

Now consider,

$[A, B] \in$ ker α_1, where

$$A = \begin{bmatrix} a_{00}, & a_{01}, & a_{02}, & a_{11}, & a_{12}, & a_{22} \end{bmatrix}$$

where $a_{ij} \in S$, homogeneous. And $B \in S,$ homogeneous then $\sum_{i,j} a_{ij}.\Delta_{ij} + B.F = 0$

$$\Rightarrow \theta(B.F) = 0$$

$$\Rightarrow \theta(B).f = 0$$

$$\Rightarrow B \in < \Delta_{ij} : 0 \leq i \leq j \leq 2 >$$

Hence, $B = \sum(b_{ij}\Delta_{ij})$ for some homogeneous polynomials $b_{ij} \in S$.

$$\Rightarrow \sum(a_{ij} + b_{ij}.F).\Delta_{ij} = 0$$

Now if $a_{ij} + b_{ij}.F = 0$ for all (a_{ij}, b_{ij}) then such a $[A, B]$ is generated by U_{ij}. If not then, $\Rightarrow \sum(a_{ij} + b_{ij}F) \in$ Syz$^1(< \Delta_{ij} : 0 \leq i \leq j \leq 2 >)$

Hence, the relations between Δ_{ij} and F are generated by $U_{ij} : 0 \leq i \leq j \leq 2$ and $W'_k : k = 1, \ldots, 8$.

Hence we get,

$$ker(\alpha_1) \subset Im(\alpha_2)$$

Now consider

$A = \begin{bmatrix} a_{00}, & a_{01}, & a_{02}, & a_{11}, & a_{12}, & a_{22} \end{bmatrix}^T, a_{ij} \in S,$ a_{ij} homogeneous $\forall 0 \leq i \leq j \leq 2$ and,

$$B = \begin{bmatrix} (b_k) \end{bmatrix}, b_k \in S, \text{ homogeneous}$$

such that $[A, B] \in$ ker(α_2) then,

$$\sum_{0 \leq i \leq j \leq 2} a_{ij}.U_{ij} + \sum_{1 \leq k \leq 8} b_k.W'_k = 0$$

$$\Rightarrow \sum_{i,j} a_{ij}\Delta_{ij} = 0$$

As the last column of each W'_k, $k = 1, \ldots, 8$ is zero and the last column of U_{ij} is Δ_{ij} for $0 \leq i \leq j \leq 2$

$$\Rightarrow A \in < W_k : k = 1, \ldots, 8 >$$

Let $A = \sum_k (c_k W_k)$, for some homogeneous

polynomial, $c_k \in S$

$$\Rightarrow -\sum_k c_k W_k F.Id_6 + \sum_k b_k W_k = 0$$

where Id_n is a $n \times n$ identity matrix.

$$\Rightarrow \sum_{i,k} W_k(-c_k F + b_k) = 0$$

Hence if $-c_k.F + b_k = 0$ for all k, this implies $b_k = c_k.F$ for all k then such (b_k, a_{ij}) are generated by $< \left[\left[F.[I_i^8] \right], [W_i] \right] >$ for $i = 1, \ldots, 8$. And if not then, $\Rightarrow [(-c_k F + b_k)I_k]_{k=1,\ldots,6} \in$ Syz^1($< W_j : j = 1, \ldots, 8 >$).

Hence the relations between W'_k and U_{ij} are generated by G'_i and H_k. So, [A B] $\in$ Im(α_3). Hence we get,

$$ker(\alpha_2) \subset Im(\alpha_3)$$

Now consider

Let, $\begin{bmatrix} A & B \end{bmatrix} \in \ker \alpha_3$ then, $A = \begin{bmatrix} a_1, & a_2, & a_3, a_4, & a_5, & a_6, & a_7, & a_8 \end{bmatrix}^T$, $a_i \in S$, homogeneous for $i = 1, \ldots, 8$ $B = \begin{bmatrix} (b_k) \end{bmatrix}$ $b_k \in S$, homogeneous for $k = 1, 2, 3$ and

$$\sum_i a_i.H_i + \sum_k b_k.G'_k = 0$$

$$\Rightarrow \sum_i a_i W_i = 0$$

As the last six columns of each G'_k, $k = 1, 2, 3$ are zero.

$$\Rightarrow A \in < G_p : p = 1, 2, 3 >$$

Let $A = \sum_k (c_p G_p)$, for some homogeneous polynomial, $c_k \in S$.

Then we have, $\sum_p (c_p G_p).(F.Id_8) + \sum_k b_k.G_k = 0$

$$\Rightarrow \sum_p (c_p.F.Id_8 + b_p)\, G_p = 0$$

Now if $c_p.F + b_p = 0$ for every p, then $b_p = -c_p.F$ for all p, then we can say that $([b_p], [c_p])$ is generated by $< \left(\left[-F.I_i^3\right], \left[I_i^3\right]\right) : i = 1, 2, 3 >$, hence $([b_p], [a_k])$ is generated by $< \left(\left[-F.I_i^3\right], [G_i]\right) i = 1, 2, 3 >$

Also from Theorem 1 we have that $G'_k : k = 0, 1, 2$ are independent. Hence $\text{Syz}^1(< G'_i, H_j : i = 1, 2, 3 \text{ and } j = 1, \ldots 8 >) =< \left(\left[-F.I_i^3\right], [G_i]\right) : i = 1, 2, 3 >$

Hence, [A,B] $\in$ Im(α_4).

And so, ker $(\alpha_3) \subset$ Im(α_4)

Also, from the representations of α_i where $1 \leq i \leq 4$, we get that, $\alpha_j.\alpha_{j+1} = 0$ where $1 \leq j \leq 3$

Hence, we know that Im $(\alpha_{j+1}) \subset$ ker (α_j) where $1 \leq j \leq 3$

So the resolution in (**4**) is exact.

3.2. Degree of f is odd
Recall

$$f = \sum_{i+j+k=d} a_{i,j,k} x_0^i x_1^j x_2^k$$

Now let $f_0 = x_0.f$, $f_1 = x_1.f$, $f_2 = x_2.f$

Then f_n is of even degree and hence according to Case A, $f_n \in Im(\theta)$ for $n = 0, 1, 2$

Lemma 5 — Let $G \in k[x_{00}, x_{01}, x_{02}, x_{11}, x_{12}, x_{22}]$ such that, G homogeneous and
$\mathcal{Z}(\theta(F_0)) \cap \mathcal{Z}(\theta(F_1)) \cap \mathcal{Z}(\theta(F_2)) \subset \mathcal{Z}(\theta(G)) \subset \mathbb{P}^2$. Then $G \in < F_k, \Delta_{i,j} : 0 \leq k \leq 2, 0 \leq i \leq j \leq 2 >$.

PROOF : Now let $\theta(G) = g$, then degree(g) is even.

$$\mathcal{Z}(f_0) \cap \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2) \subset \mathcal{Z}(g)$$

$$\Rightarrow \mathcal{Z}(f) \subset \mathcal{Z}(g)$$

$\Rightarrow g \in (f)$as C is an irreducible curve and hence f is irreducible

$\Rightarrow g = f.h$ for some $h \in k[x_0, x_1, x_2]$

$\Rightarrow h \neq 1$ as degree f is odd while degree g is even

$\Rightarrow g = \sum_{i=0,1,2} f_i h_i$ for some homogeneous even degree polynomials $h_i \in k[x_0, x_1, x_2]$

$\Rightarrow G = \sum_{i=0,1,2} F_i H_i$, where $\theta(H_i) = h_i \ \forall i = 0, 1, 2$.

Such a H_i, exists as the degree of h_i is even.

$$\begin{aligned}
&\Rightarrow \quad \theta(G - \sum_{i=0,1,2} F_i H_i) = 0 \\
&\Rightarrow \quad G - \sum_{i=0,1,2} F_i H_i \in ker(\theta) \\
&\Rightarrow \quad G = \sum_{i=0,1,2} F_i H_i + \sum_{i,j=0,1,2} \Delta_{ij} S_{ij} \quad \text{for some} \quad S_{ij} \in k[x_{00}, \ldots, x_{22}] \\
&\Rightarrow \quad G \in < F_k, \Delta_{ij} : i, j, k = 0, 1, 2 >
\end{aligned}$$

Lemma 6 — $Im(\theta)$ is a sub-algebra of $K[x_0, x_1, x_2]$ and is generated by even polynomials.

PROOF : Like in the case of degree of f being even we split f in four parts depending on the parities of i,j,k.

Case I : i, j, k are all odd. Let

$$\begin{aligned}
\text{Let } h_I &= \sum_{i,j,k} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k-1}{2}} \\
{F_0}^I &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i+1}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k-1}{2}} x_{12} \\
{F_1}^I &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j+1}{2}} x_{22}^{\frac{k-1}{2}} x_{02} \\
{F_2}^I &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k+1}{2}} x_{01}
\end{aligned}$$

Then,

$$\begin{aligned}
{F_0}^I &= x_{00} x_{12} h_I \\
{F_1}^I &= x_{11} x_{02} h_I \\
{F_2}^I &= 7 x_{22} x_{01} h_I
\end{aligned}$$

Case II : i odd, j even, k even. Now

$$\begin{aligned}
\text{Let } h_{II} &= \sum_{i,j,k} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}} \\
{F_0}^{II} &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i+1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}} \\
{F_1}^{II} &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}} x_{01} \\
{F_2}^{II} &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}} x_{02}
\end{aligned}$$

Then,

$$\begin{aligned}
{F_0}^{II} &= x_{00} h_{II} \\
{F_1}^{II} &= x_{01} h_{II} \\
{F_2}^{II} &= x_{02} h_{II}
\end{aligned}$$

Case III : i even, j odd, k even. Now

$$\begin{aligned}
\text{Let } h_{III} &= \sum_{i,j,k} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k}{2}} \\
{F_0}^{III} &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k}{2}} x_{01} \\
{F_1}^{III} &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j+1}{2}} x_{22}^{\frac{k}{2}} \\
{F_2}^{III} &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k}{2}} x_{12}
\end{aligned}$$

Then,

$$
\begin{aligned}
{F_0}^{III} &= x_{01}h_{III} \\
{F_1}^{III} &= x_{11}h_{III} \\
{F_2}^{III} &= x_{12}h_{III}
\end{aligned}
$$

Case IV : i even, j even, k odd. Now

$$
\begin{aligned}
\text{Let } h_{IV} &= \sum_{i,j,k} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k-1}{2}} \\
{F_0}^{IV} &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k-1}{2}} x_{02} \\
{F_1}^{IV} &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k-1}{2}} x_{12} \\
{F_2}^{IV} &= \sum_{i+j+k=d} a_{i,j,k} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k+1}{2}}
\end{aligned}
$$

Then,

$$
\begin{aligned}
{F_0}^{IV} &= x_{02}h_{IV} \\
{F_1}^{IV} &= x_{12}h_{IV} \\
{F_2}^{IV} &= x_{22}h_{IV} \\
F_n &= 7{F_n}^{I} + {F_n}^{II} + {F_n}^{III} + {F_n}^{IV} \forall n = 0,1,2
\end{aligned}
$$

Also notice $\theta(F_n) = f_n$ for $n = 0, 1, 2$

***Theorem* 7** — *Let C be an irreducible curve of odd degree say $d = 2m - 1$, for $m \geq 2$. The ideal $\mathcal{I}_C$ of $\sigma(C)$ in $\mathbb{P}^5$ has the following resolution.*

$$
\begin{aligned}
0 \to S(-m-4) \quad & \xrightarrow{\beta_4} S(-4)^{\oplus 3} \oplus S(-m-2)^{\oplus 6} \xrightarrow{\beta_3} \xrightarrow{\beta_3} S(-3)^{\oplus 8} \\
& \oplus S(-m-1)^{\oplus 8} \xrightarrow{\beta_2} S(-2)^{\oplus 6} \oplus \\
& S^{\oplus 3}(-m) \xrightarrow{\beta_1} S \to S/\mathcal{I}_C \to 0 \qquad (9)
\end{aligned}
$$

PROOF : From Lemma 3 and Lemma 5, it is clear that

$$\beta_1 = \begin{bmatrix} \Delta_{00}, & \Delta_{01}, & \Delta_{02}, & \Delta_{11}, & \Delta_{12}, & \Delta_{22}, & F_0, & F_1, & F_2, \end{bmatrix}$$

Now consider $A = \begin{bmatrix} a_{00}, & a_{01}, & a_{02}, & a_{11}, & a_{12}, & a_{22} \end{bmatrix}$, $a_{ij} \in S$, homogeneous $\forall 0 \leq i \leq \jmath \leq 2$ and $B = \begin{bmatrix} b_0, & b_1, & b_2 \end{bmatrix}$ where $b_l \in S$, homogeneous, for $k = 0, 1, 2$ such that $[A\ B] \in$ ker β_1. Hence

$$\sum_{i,j} a_{ij}.\Delta_{ij} + \sum_k b_k.F_k = 0 \tag{10}$$

$$\Rightarrow \theta(\sum_k (b_k.F_k)) = 0$$

$$\Rightarrow \sum_k (\theta(b_k).f_k) = 0$$

$$\Rightarrow \sum_k (\theta(b_k).f.x_k) = 0$$

$$\Rightarrow \sum_k (\theta(b_k).x_k) = 0$$

Let $\theta(b_k) = B_k$, then degree of B_k is even. Then,

$$B = (B_0, B_1, B_2)^T \in \text{Syz}^1(x_0, x_1, x_2)$$

Now by simple computation we get

$$\text{Syz}^1(x_0, x_1, x_2) = < \begin{pmatrix} x_1 \\ -x_0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \\ -x_0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \\ -x_1 \end{pmatrix} >$$

hence $B \in < Y_0, Y_1, Y_2 >$ where,

$$Y_0 = \begin{pmatrix} x_1, & -x_0, & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} x_2, & 0, & -x_0 \end{pmatrix}$$

$$Y_2 = \begin{pmatrix} 0, & x_2, & -x_1 \end{pmatrix}$$

But degree of B_k is even, hence, $B \in < x_k Y_l : k, l = 0, 1, 2 >$.

Hence, $(b_0, b_1, b_2) \in < Y_{lk} : k, l = 0, 1, 2 >$

where

$$Y_{00} = \begin{pmatrix} x_{01}, & -x_{00}, & 0 \end{pmatrix}$$
$$Y_{01} = \begin{pmatrix} x_{11}, & -x_{01}, & 0 \end{pmatrix}$$
$$Y_{02} = \begin{pmatrix} x_{12}, & -x_{02}, & 0 \end{pmatrix}$$

$$Y_{10} = \begin{pmatrix} x_{02}, & 0, & -x_{00} \end{pmatrix}$$
$$Y_{11} = \begin{pmatrix} x_{12}, & 0, & -x_{01} \end{pmatrix}$$
$$Y_{12} = \begin{pmatrix} x_{22}, & 0, & -x_{02} \end{pmatrix}$$

$$Y_{20} = \begin{pmatrix} 0, & x_{02}, & -x_{01} \end{pmatrix}$$
$$Y_{21} = \begin{pmatrix} 0, & x_{12}, & -x_{11} \end{pmatrix}$$
$$Y_{22} = \begin{pmatrix} 0, & x_{22}, & -x_{12} \end{pmatrix}$$

Also note that,

$$Y_{02} = Y_{11} - Y_{20}$$

Claim : The set $\{Y_{00}, Y_{01}, Y_{10}, Y_{11}, Y_{12}, Y_{20}, Y_{21}, Y_{22}\}$ is linearly independent
If not, then there exists a_{ij}, $i, j = 0, 1, 2$ and $(i, j) \neq (0, 2)$, such that,

$$\sum_{i,j} a_{ij} Y_{ij_k} = 0 \text{ for } k = 0,1,2$$

This implies that,

$$a_{00}x_{01} + a_{01}x_{11} + a_{10}x_{02} + a_{11}x_{12} + a_{12}x_{22} = 0$$

Now as, x_{ij} are independent variables for $i, j = 0, 1, 2$, we get that

$$a_{00} = a_{01} = a_{10} = a_{11} = a_{12} = 0$$

Similarly we also have that,

$$a_{02}x_{02} + a_{21}x_{12} + a_{22}x_{22} = 0$$

So we get that,

$$a_{00} = a_{01} = a_{10} = a_{11} = a_{12} = a_{20} = a_{21} = a_{22} = 0$$

Hence the set $\{Y_{00}, Y_{01}, Y_{10}, Y_{11}, Y_{12}, Y_{20}, Y_{21}, Y_{22}\}$ is linearly independent.

Now substituting all Y_{ij} for $i, j = 0, 1, 2$ except for Y_{02} for b in (10) we get, the following 8 vectors,

$$V_{00} = \begin{bmatrix} 0, & 0, & -x_{00}h_I, & 0, & h_{IV}, & h_{III}, & [Y_{00}] \end{bmatrix}^T$$

$$V_{01} = \begin{bmatrix} 0, & 0, & h_{IV}, & 0, & -x_{11}h_I, & -h_{II}, & [Y_{01}] \end{bmatrix}^T$$

$$V_{10} = \begin{bmatrix} 0, & x_{00}h_I, & 0, & h_{IV}, & h_{III}, & 0, & [Y_{10}] \end{bmatrix}^T$$

$$V_{11} = \begin{bmatrix} x_{00}h_I, & h_{IV}, & 0, & 0, & -h_{II}, & -x_{22}h_I, & [Y_{11}] \end{bmatrix}^T$$

$$V_{12} = \begin{bmatrix} 0, & -h_{III}, & 0, & -h_{II}, & -x_{22}h_I, & 0, & [Y_{12}] \end{bmatrix}^T$$

$$V_{20} = \begin{bmatrix} 0, & h_{IV}, & h_{III}, & x_{11}h_I, & 0, & -x_{22}h_I, & [Y_{20}] \end{bmatrix}^T$$

$$V_{21} = \begin{bmatrix} h_{IV}, & x_{11}h_I, & -h_{II}, & 0, & 0, & 0, & [Y_{21}] \end{bmatrix}^T$$

$$V_{22} = \begin{bmatrix} -h_{III}, & -h_{II}, & x_{22}h_I, & 0, & 0, & 0, & [Y_{22}] \end{bmatrix}^T$$

Let,

$$V_{00} = \begin{bmatrix} [H_{00}] & [Y_{00}] \end{bmatrix}^T$$

$$V_{01} = \begin{bmatrix} [H_{01}] & [Y_{01}] \end{bmatrix}^T$$

$$V_{10} = \begin{bmatrix} [H_{10}] & [Y_{10}] \end{bmatrix}^T$$

$$V_{11} = \begin{bmatrix} [H_{11}] & [Y_{11}] \end{bmatrix}^T$$

$$V_{12} = \begin{bmatrix} [H_{12}] & [Y_{12}] \end{bmatrix}^T$$

$$V_{20} = \begin{bmatrix} [H_{20}] & [Y_{20}] \end{bmatrix}^T$$

$$V_{21} = \begin{bmatrix} [H_{21}] & [Y_{21}] \end{bmatrix}^T$$

$$V_{22} = \begin{bmatrix} [H_{22}] & [Y_{22}] \end{bmatrix}^T$$

And, $\mathbf{H} = \begin{bmatrix} [H_{00}], & [H_{01}], & [H_{10}], & [H_{11}], & [H_{12}], & [H_{20}], & [H_{21}], & [H_{22}] \end{bmatrix}$

Now all the relations between F_n's and Δ_{ij}'s are generated by V_{kl}'s and W'_m's and all the relations between only Δ_{ij}'s are generated by W_m's. Hence all relations between F_n, Δ_{jk} are generated by V_{kl}, W'_m.

Hence $\text{Syz}^1(< F_n, \Delta_{ij} >) =< W'_m, V_{kl} >$. Hence

$$\beta_2 = \begin{bmatrix} [W'_1] & [W'_2] & \dots & [W'_8] & [\mathbf{V}] \end{bmatrix}$$

where $W'_k = \begin{bmatrix} [W_k] & [\bar{0}] \end{bmatrix}$ with $[\bar{0}]$ a 1×3 zero vector

Hence we can write β'_2 as

,

$$\beta_2 = \begin{bmatrix} [M_2] & \mathbf{H} \\ \bar{0} & Y \end{bmatrix}$$

This implies that ker$(\beta_1) \subset$ Im (β_2).

Also from the representation of the β_1 and β_2 and from the fact that $M_1 M_2 = 0$ we get that, $\beta_1 \beta_2 = 0$.

So we have, ker$(\beta_1) =$ Im(β_2) .

Now consider, $\bar{A} = (a_{ij})$ such that $a_{ij} \in S$ homogeneous and $\bar{B} = (b_k)$, where $b_k \in S$, homogeneous such that $[A\ B] \in$ ker(β_2). Then we have,

$$\sum A.[\ \mathbf{H} \quad Y\ ] + \sum_k b_k W'_k = 0 \tag{11}$$

Now as all the entries in the last 3 columns in each of W'_i are zero we have,

$$\sum_{ij} a_{ij} Y_{ij} = 0$$

.

Now it can be computed that Syz$^1(Y_{ij}) =< L_i : 1 \leq i \leq 6 >$ where,

$$L_1 = \begin{bmatrix} x_{02}, & 0, & -x_{01}, & 0, & 0, & x_{00}, & 0, & 0 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} x_{12}, & x_{02}, & -x_{11}, & -x_{01}, & 0, & x_{01}, & x_{00}, & 0 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} x_{22}, & 0, & -x_{12}, & x_{02}, & -x_{01}, & 0, & 0, & x_{00} \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 0, & x_{12}, & 0, & -x_{11}, & 0, & 0, & x_{01}, & 0 \end{bmatrix}$$

$$L_5 = \begin{bmatrix} 0, & x_{22}, & 0, & 0, & -x_{11}, & -x_{12}, & x_{02}, & x_{01} \end{bmatrix}$$

$$L_6 = \begin{bmatrix} 0, & 0, & 0, & x_{22}, & -x_{12}, & -x_{22}, & 0, & x_{02} \end{bmatrix}$$

So substituting L_i, $i = 0, \dots, 6$ for $\bar{B}$ in (11) we get the following 6 vectors,

$$K_1' = \begin{bmatrix} 0, & 0, & 0, & x_{00}h_I, & 0, & 0, & -h_{IV}, & h_{III}, & [L_1] \end{bmatrix}^T$$

$$K_2' = \begin{bmatrix} 0, & 0, & x_{00}h_I, & 0, & -h_{III}, & -h_{IV}, & x_{11}h_I, & h_{II}, & [L_2] \end{bmatrix}^T$$

$$K_3' = \begin{bmatrix} -x_{00}h_I, & -h_{IV}, & 0, & -h_{III}, & 0, & h_{III}, & h_{II}, & x_{22}h_I, & [L_3] \end{bmatrix}^T$$

$$K_4' = \begin{bmatrix} 0, & 0, & -h_{IV}, & -x_{11}h_I, & h_{II}, & x_{11}h_I, & 0, & 0, & [L_4] \end{bmatrix}^T$$

$$K_5' = \begin{bmatrix} -h_{IV}, & -x_{11}h_I, & h_{III}, & h_{II}, -x_{22}h_I, & 0, & 0, & 0 & [L_5] \end{bmatrix}^T$$

$$K_6' = \begin{bmatrix} h_{III}, & h_{II}, & 0, & 0, & 0, & -x_{22}h_I, & 0, & 0, & [L_6] \end{bmatrix}^T$$

Let

$$\begin{aligned} K_1' &= \begin{bmatrix} [\mathbf{K}_1] & [L_1] \end{bmatrix}^T \\ K_2' &= \begin{bmatrix} [\mathbf{K}_2] & [L_2] \end{bmatrix}^T \\ K_3' &= \begin{bmatrix} [\mathbf{K}_3] & [L_3] \end{bmatrix}^T \\ K_4' &= \begin{bmatrix} [\mathbf{K}_4] & [L_1] \end{bmatrix}^T \\ K_5' &= \begin{bmatrix} [\mathbf{K}_5] & [L_5] \end{bmatrix}^T \\ K_6' &= \begin{bmatrix} [\mathbf{K}_6] & [L_6] \end{bmatrix}^T \end{aligned}$$

And,

$$\begin{aligned} \mathbf{K} &= \begin{bmatrix} [\mathbf{K}_1] & [\mathbf{K}_2] & [\mathbf{K}_3] & [\mathbf{K}_4] & [\mathbf{K}_5] & [\mathbf{K}_6] \end{bmatrix}^T , \\ \mathbf{L} &= \begin{bmatrix} [\mathbf{L}_1] & [\mathbf{L}_2] & [\mathbf{L}_3] & [\mathbf{L}_4] & [\mathbf{L}_5] & [\mathbf{L}_6] \end{bmatrix}^T \end{aligned}$$

Now all the relations between V_{ij}'s and W'_m's are generated by $\{K'_l$'s, G'_k's, $1 \leq l \leq 6, k = 1, 2, 3\}$ and all the relations between only W'_m's (which are actually W_m) are generated by G'_k's. Hence we have that all relations between $\{\{V_{ij}\}, \{W'_m\}\}$ are generated by $\{K_l$'s,G'_k's $\}$. So ,
Syz$^1(< V_i, W'_j >) =< G'_k, K'_l >$. So we get that,

$$\beta_3 = \begin{bmatrix} [G'_1] & [G'_2] & [G'_3] & [K'_1] & \dots & [K'_6] \end{bmatrix}$$

where, $G'_i = \begin{bmatrix} [G_i] & [\bar{0}] \end{bmatrix}$ where $[\bar{0}]$ is an appropriate dimensional zero matrix.

Hence we can write that,

$$\beta_3 = \begin{bmatrix} M_3 & \mathbf{K} \\ 0 & \mathbf{L} \end{bmatrix}$$

This implies that ker$(\beta_2) \subset$ Im(β_3). Also from the representation of the β_2 and β_3 and from the fact that $M_2 M_3 = 0$ we get that, $\beta_2 \beta_3 = 0$.

So we have, ker$(\beta_2) =$ Im(β_3) .

Now consider $\bar{A} = (A_i)$, such that $A_i \in S$, homogeneous and $\bar{B} = (B_k)$, such that $B_k \in S$, homogeneous such that, $[A\ B] \in$ ker(β_3). Then we have,

$$\sum_l A.[\ K \quad L\ ] + \sum_k B_k G'_k = 0 \tag{12}$$

Hence we have,

$$\sum_l A_l L_l^T = 0$$

(as the last eight columns of G'_i's are zero entries)

Now it can be computed that Syz$^1(L_l) =< J' >$ where,

$$J' = \begin{bmatrix} {x_{12}}^2 - x_{11}x_{22} \\ -x_{02}x_{12} + x_{01}x_{22} \\ x_{11}x_{02} - x_{01}x_{12} \\ {x_{02}}^2 - x_{00}x_{22} \\ -x_{01}x_{02} + x_{00}x_{12} \\ {x_{01}}^2 - x_{00}x_{11} \end{bmatrix}$$

Like earlier, substituting J' in (12) we get J.

$$\mathcal{J} = \begin{bmatrix} -x_{00}x_{12}h_I - x_{00}h_{II} - x_{01}h_{III} - x_{02}h_{IV} \\ \\ -x_{11}x_{02}h_I + x_{01}h_{II} + x_{11}h_{III} + x_{12}h_{IV} \\ \\ -x_{01}x_{22}h_I - x_{02}h_{II} - x_{12}h_{III} - x_{22}h_{IV} \\ \\ [J'] \end{bmatrix}$$

$$\text{Let, } \mathcal{J} = \begin{bmatrix} J \\ J' \end{bmatrix}$$

Now all the relations between K_l's and G'_k's are generated by J and there are no relations between only G'_k's as there are no non-trivial relations between G_k's. Hence all relations between K_l, G'_k are generated by J. Hence Syz$^1(< K_l, G'_k >) =< J >$. Hence

$$\beta_4 = \begin{bmatrix} J \\ J' \end{bmatrix}$$

This implies that ker$(\beta_3) \subset$ Im(β_4).

But from the representation of β_3 and β_4 and from the fact that there are no non-trivial relations between the columns of M_3 we get that, $\beta_3\beta_4 = 0$. So we have, ker$(\beta_3) =$ Im(β_4).

This proves that, the resolution in (9) is a complex. And hence this completes the proof of the theorem.

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