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THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF AN INITIAL BOUNDARY VALUE PROBLEM FOR THE GENERALIZED BENJAMIN-BONA-MAHONY EQUATION[1]

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The asymptotic behaviors of solutions of an initial-boundary value problem for the generalized BBM equation with non-convex flux are discussed in this paper. It is proved that under the conditions of constant boundary data and small perturbation for the initial data, the global solutions exist and converge time-asymptotically to a stationary wave or the superposition of a stationary wave and a rarefaction wave. The proof is given by a technical L^2-weighted energy method.

Key words : Asymptotic behaviors; generalized BBM equation; initial-boundary value problem; stationary solution; rarefaction wave.

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1. INTRODUCTION

Consider the initial-boundary value problem of the generalized Benjamin-Bona-Mahony equation as follows:

$$\begin{cases} u_t + f(u)_x = u_{xx} + u_{xxt}, \;\; x > 0, \;\; t > 0 \\ u(x,t)|_{x=0} = u_-, \;\; t \geq 0 \\ u(x,t)|_{t=0} = u_0(x) = \begin{cases} u_-, & x = 0 \\ u_+, & x \to \infty, \end{cases} \end{cases} \tag{1.1}$$

where $u_\pm$ are constant satisfying $u_- < u_+$ and f satisfies:

$$\begin{cases} f \in C^2 \\ f(0) = f'(0) = 0, \;\; f''(0) > 0 \\ f(u) > 0, \;\; u \in [u_-, 0). \end{cases} \tag{1.2}$$

We also assume the initial data u_0 satisfies the compatibility condition:

$$u(0) = u_-. \tag{1.3}$$

The equation of type $(1.1)_1$ is related to the well-known BBM equation, which was advocated by Benjamin-Bona-Mahony [1] as a model in the study of unidirectional long waves of small amplitudes in water. It has been used to account adequately for observable phenomena such as the interaction of solitary waves and dissipationless, undular shocks. In recent years, the generalized BBM equation has been the subject of numerous investigations.(A complete literature in this direction is beyond the scope of this paper, however, we want to mention [2-6, 8-12]. For the corresponding results on some related models such as the scalar conservation law, the Korteweg-de Vries-Burgers equation, the Navier-Stokes equations, the psedo-parabolic equation etc., see [7, 13-28] and the references cited therein.) In the case of the convex flux functions, the Cauchy problems, the initial boundary value problems and the large time behaviors of solutions to the initial value problem for various generalized BBM equation have been studied, cf. [3, 6-9]. Under certain assumptions both L^2 and L^∞ rates of decay of the solutions to these problems were established, cf. [4, 5, 10-12].

When the flux function is non-convex, the problem becomes complex and difficult. Recently, Hasimoto-Matsumura [13] investigated the large time behavior of the solution to the initial-boundary value problem in the half-space for scalar Burgers equation without convexity. Motivated by the L^2-weighted energy method in [13], our present paper is devoted to studying the existence and the asymptotic behavior of global solutions of the generalized BBM equation (1.1) with non-convex flux function. Under the conditions that

$$u_- < u_+ \leq 0 \text{ and } f(u_+) < f(u) \text{ for } u \in [u_-, u_+) \tag{1.4}$$

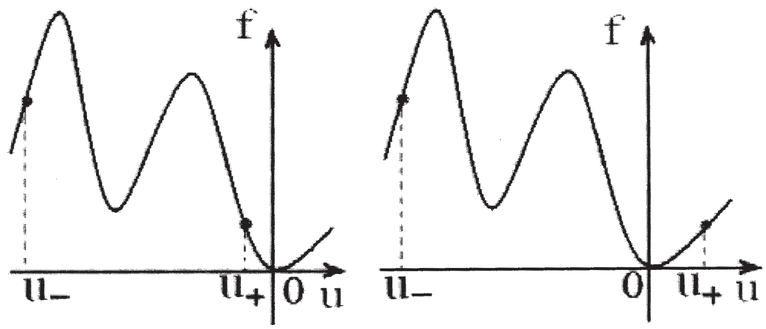
and $u_- < 0 < u_+$, respectively (the function in both cases is as shown in the figure below), using an L^2-weighted energy method as in [13], we prove that solutions for the initial-boundary value problem (1.1) exist and converge time-asymptotically to a stationary wave and the superposition of a stationary wave and a rarefaction wave.

Notations : Hereafter, we denote several generic positive constants depending on $a, b, ...$ by $C_{a,b,...}$ or simply by C. $L^p = L^p(R^+)(1 \leq p \leq \infty)$ denotes the usual Lebesgue space on $R^+ = (0, \infty)$ with its norm

$$\| f \|_{L^p} = (\int_{R^+} |f(x)|^p dx)^{\frac{1}{p}}, 1 \leq p < \infty, \quad \|f\|_{L^\infty} = \sup_{x \in R^+} |f(x)|.$$

$H^{m,p}(m \geq 0, 1 \leq p < +\infty)$ denotes the usual Sobolev space with the norm

$$\| f \|_{H^{m,p}} = (\sum_{j=0}^{m} \| \partial_x^j f \|_{L^p}^p)^{\frac{1}{p}}.$$



Set $H^m = H^{m,2}$. For simplicity, we write $\|\cdot\|_{H^m} = \|\cdot\|_m$ and $\|\cdot\|_{L^2} = \|\cdot\|$. For brevity, $\|f(\cdot,t)\|$ and $\|f(\cdot,t)\|_m$ are denoted by $\|f(t)\|$ and $\|f(t)\|_m$, respectively. Let T be a positive constant and let B be a Banach space, $C^k([0,T];B)$ denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$ and $L^2([0,T];B)$ denotes the space of B-valued L^2 functions on $[0,T]$. We also denote by $H_0^1 = H_0^1(R^+)$ the space of functions $f \in H^1$ with $f(0) = 0$, as a subspace of $H^1(R^+)$.

2. Main Results

In this section, we present our main results. Corresponding to the cases $u_- < u_+ \le 0$ and $u_- < 0 < u_+$, we divide this section into two cases. Throughout this section, we always assume the condition (1.2) and (1.3) hold. By the condition (1.2) and the properties of continuous function, there exist positive constants r, ν such that

$$f''(u) \ge \nu > 0 \ \text{ for } \ |u| \le r, \ \text{ and } \ f(u) \ge \nu \ \text{ for } \ u \in [u_-, -r]. \tag{2.1}$$

Case 1 : Converge Asymptotically to a Stationary Wave

When $u_\pm$ satisfy the condition (1.4), we will prove that the solutions converge time-asymptotically to a stationary wave $\phi(x)$ for the problem (1.1). Here $\phi(x)$ is the following boundary value problem of the ordinary differential equation:

$$\begin{cases} f(\phi)_x = \phi_{xx}, \ \ x > 0 \\ \phi(0) = u_-, \ \ \phi(+\infty) = u_+. \end{cases} \tag{2.2}$$

For the properties of the solution $\phi(x)$ to the boundary value problem (2.2), we have the following lemma, which is proved in the same way as in [2,7,14].

Lemma 2.1 — Assume (1.2) and (1.4). Then the boundary value problem (2.2) has a unique solution $\phi \in C^3([0,\infty))$ satisfying

$$\begin{cases} u_- < \phi(x) < 0, \;\; \phi_x(x) > 0, \;\; x > 0 \\ |\phi(x) - u_+| \le Ch(\tilde{u})(1+x)^{-1}, \;\; x > 0 \\ |\partial_x^k \phi(x)| \le Ch(\tilde{u})(1+x)^{-2} \;\; (k = 1, 2), \;\; x \ge 0 \end{cases}$$

for the case $f'(u_+) = 0$ and

$$\begin{cases} u_- < \phi(x) < u_+, \;\; \phi_x(x) > 0, \;\; x > 0 \\ |\phi(x) - u_+| \le Ch(\tilde{u}) \exp\left(-|f'(u_+)|x\right), \;\; x > 0 \\ |\partial_x^k \phi(x)| \le Ch(\tilde{u}) \exp\left(-|f'(u_+)|x\right) \;\; (k = 1, 2), \;\; x \ge 0 \end{cases}$$

for the case $f'(u_+) < 0$, where C is a constant, $\tilde{u} = u_+ - u_-$, h is a function of $\tilde{u}$ satisfying $\lim_{\tilde{u} \to 0} h(\tilde{u}) = 0$.

Set $u(x,t) = \phi(x) + v(x,t)$, where $\phi = \phi(x)$ is the stationary solution obtained in Lemma 2.1, then the problem (1.1) can be reformulated as

$$\begin{cases} v_t + (f(\phi + v) - f(\phi))_x = v_{xx} + v_{xxt}, \;\; x > 0, t > 0 \\ v(0,t) = 0, \;\; t \ge 0 \\ v(x,0) = v_0(x) := u_0(x) - \phi(x), \;\; x > 0. \end{cases} \tag{2.3}$$

For the existing of the weak solutions, the interested reader is referred to [28-30] and the references cited therein. We want to seek the solution of (2.3) in Banach space:

$$X(0,T) = \{v | v \in C([0,T]; H_0^2) \cap C^1([0,T]; H^1)\},$$

where $0 < T < \infty$.

Our main result in the case (1.4) can be stated as follows.

***Theorem* 2.2** — *(Asymptotics to the Stationary Wave) Assume (1.2), (1.3), (1.4) and $v_0 \in H^2$. If there exists a positive constant η such that $\|v_0\|_{H^2}^2 + h(\tilde{u}) \le \eta$, then the problem (2.3) has a unique global solution $v \in X(0,\infty)$ satisfying*

$$\lim_{t\to\infty} \sup_{x\in R^+} |\frac{\partial^i v(x,t)}{\partial x^i}| = 0 \ \ (i = 0, 1).$$

Here, $h = h(\tilde{u})$ sketch the strength of the stationary wave. The constant η is used to sketch the smallness of the stationary wave and the initial perturbation to control the growth of the solution u.

Case 2 : Converge Asymptotically to Superposition of a Stationary Wave and a Rarefaction Wave

Under the condition $u_- < 0 < u_+$, we expect that the asymptotic state of the solution of (1.1) is the linear superposition of the corresponding stationary solution ϕ connecting u_- to 0 and the rarefaction wave φ_1^R connecting 0 to u_+. Next, we give the definition of ϕ and φ_1^R. The stationary solution ϕ connecting u_- to 0 for (1.1) is defined by Lemma 2.1 in which $u_+ = 0$. The rarefaction wave φ_1^R connecting 0 to u_+ for (1.1) is defined by the restriction of ψ^R on the half line $\psi^R(\frac{x}{t})|_{x>0}$, where ψ^R is the solution of the following Riemann problem for the Burgers equation:

$$\begin{cases} u_t + f(u)_x = 0, \ \ -\infty < x < \infty \ \ t > 0 \\ u(x,0) = \begin{cases} 0, & x > 0 \\ u_+, & x < 0. \end{cases} \end{cases}$$

Under the condition (2.1), ψ^R is exactly given by

$$\psi^R(\tfrac{x}{t}) = \begin{cases} 0, \ \ x \le 0 \\ (f')^{-1}(\tfrac{x}{t}), \quad 0 \le x \le f'(u_+)t \\ u_+, \quad x \ge f'(u_+)t. \end{cases}$$

Because of the non-smoothness of ψ^R, as in the previous paper [2,13-15], we first make a smooth approximation of the rarefaction wave φ_1^R. Let $w(x,t)$ be the unique smooth solution of the Cauchy problem:

$$\begin{cases} w_t + ww_x = 0, \ \ -\infty < x < \infty, \ \ t > 0 \\ w(x,0) = w_{10}(x) = w_+ \cdot k_q \int_0^x (1+y^2)^{-q} dy = 1, \ \ q > \frac{1}{2}, \end{cases}$$

where $k_q \int_0^\infty (1+y^2)^{-q} dy = 1$. We define the smooth approximation of φ_1^R by

$$\varphi = \varphi(x,t) = (f')^{-1}(w(x,t))|_{x>0}.$$

Then as in [14,15], we have the following lemma for φ.

Lemma 2.3 — Assume that $f \in C^2$, $f'' > 0$, $f(0) = f'(0) = 0$ and $u_+ > 0$, then φ satisfying

$$\begin{cases} \varphi_t + f(\varphi)_x = 0, \quad x, t > 0 \\ \varphi(0,t) = 0, \quad t \geq 0 \\ \varphi(x,0) = \varphi_0(x) := (f')^{-1}(w_0(x,t)) = \begin{cases} 0, \ \ x = 0 \\ \to u_+, \ \ x \to \infty \end{cases} \end{cases}$$

and

(i) $0 = u_- < \varphi(x,t) < u_+, \varphi_x(x,t) > 0$ for $t \geq 0, x > 0$;

(ii) for any p $(1 \leq p \leq \infty)$, there exists a constant $C_{p,q}$ such that

$$\|\varphi_x(t)\|_{L^p}^p \leq C_{p,q} h(u_+)(1+t)^{-p+1},$$

$$\|\varphi_{xx}(t)\|_{L^p}^p \leq C_{p,q} h(u_+)(1+t)^{-p-\frac{p-1}{2q}},$$

$$\|\varphi_{xxt}(t)\|_{L^p}^p \leq C_{p,q} h(u_+)(1+t)^{-p-\frac{2p-1}{2q}},$$

$$\|\varphi_{xxtt}(t)\|_{L^p}^p \leq C_{p,q} h(u_+)(1+t)^{-p-\frac{3p-1}{2q}};$$

(iii) there exists a constant C_q such that

$$\|\frac{\varphi_{xx}^2}{\varphi_x}(t)\|_{L^1} \leq C h_1(u_+)(1+t)^{-1-\frac{1}{2q}},$$

$$\|\frac{\varphi_{xxt}^2}{\phi_x}(t)\|_{L^1} \leq C h_1(u_+)(1+t)^{-1-\frac{1}{q}},$$

where h_1 is a function of u_+ satisfying $\lim_{u_+\to 0} h_1(u_+) = 0$;

(iv) $\lim_{t\to\infty} \sup_{x\in R^+} |\varphi(x,t) - \varphi_1^R(x,t)| = 0.$

Set

$$\Phi(x,t) = \phi(x) + \varphi(x,t). \tag{2.4}$$

Substituting (2.4) into $(1.1)_1$, we have

$$\Phi_t + f(\Phi)_x - \Phi_{xx} - \Phi_{xxt} = -F(\phi, \varphi), \tag{2.5}$$

where $F(\phi,\varphi) = -(f'(\phi+\varphi) - f'(\phi))\phi_x - (f'(\phi+\varphi) - f'(\varphi))\varphi_x + \varphi_{xx} + \varphi_{xxt}$.

Let $v(x,t) = u(x,t) - \Phi(x,t) = u(x,t) - \phi(x) - \varphi(x,t)$, then the problem (1.1) is reformulated in the form

$$\begin{cases} v_t + (f(\Phi + v) - f(\Phi))_x - v_{xx} - v_{xxt} = F(\phi, \varphi), \ \ x > 0, \ \ t > 0 \\ v(0,t) = 0, \ \ t \geq 0 \\ v(x,0) = v_0(x) := u_0(x) - \phi(x) - \varphi(x,0), \ \ x > 0. \end{cases} \tag{2.6}$$

Our main result in the case $u_- < 0 < u_+$ can be stated as follows.

***Theorem* 2.4** — *(Asymptotic to Superposition of a Stationary Wave and a Rarefaction Wave) Assume (1.2), (1.3), $u_- < 0 < u_+ < r$ and $v_0 \in H^2$. Then, there exists a positive constant η such that, if $\|v_0\|_{H^2}^2 + d_0 \leq \eta$, then the initial-boundary value problem (2.6) has a unique global solution $v \in X(0,\infty)$ satisfying*

$$\lim_{t\to\infty} \sup_{x\in R^+} |\frac{\partial^i v(x,t)}{\partial x^i}| = 0 \ \ (i = 0, 1),$$

where r is defined by (2.1), $d_0 = max\{h(-u_-), h_1(u_+)\}$, h and h_1 are defined by Lemma 2.1 and Lemma 2.3, respectively.

3. The Proof of Main Results

In this section, we will prove our main results. For brevity, we only prove Theorem 2.4 in details. Theorem 2.2 can be proved in the same way as the proof of Theorem 2.4.

The proof of Theorem 2.4 combines the local existence results with the a priori estimates. We now state the local existence result as follows.

Proposition 3.1 — (Local Existence) Under the assumptions of Theorem 2.4, the problem (2.3) has a unique solution $v(x,t) \in X(0,t_0)$, where t_0 depends only on $\|v_0\|_{H^2}$.

Similar to [10], we can prove Proposition 3.1 by a standard iterative method, we omit the proof of Proposition 3.1. To extend the local solution $v(x,t)$ obtained in globally, we need the a priori estimate as follows.

Proposition 3.2 — (*a Priori* Estimate) Suppose that $v(x,t) \in X(0,T)$ is a solution of the problem (2.6) for some positive constant T with

$$N(T) = \sup_{t\in[0,T]} \|v(t)\|_2^2 \leq \delta^2, \;\; 0 < \delta \ll 1. \tag{3.1}$$

If $\|v_0\|_{H^2}^2 + d_0$ is sufficiently small, then it holds

$$\|v(t)\|_2^2 + \int_0^t (\|\sqrt{\Phi_x} v(\tau)\|^2 + \|v_x(\tau)\|_1^2 + \|v_t(\tau)\|_1^2) d\tau \leq C(\|v_0\|_2^2 + d_0^{\frac{1}{6}}), \tag{3.2}$$

where C is a positive constant independent of T.

We shall prove Proposition 3.2 by an L^2-weighted energy method. Because the flux function f is not convex, we can not use the standard L^2-energy method directly to derive the a priori estimate of v and v_x in (3.2). We now give a simple explanation. Since v and u_+ is sufficiently small in the ***a Priori*** estimate (3.2), the linear problem of (2.3) with $u_+ = 0$ (accordingly, $\Phi(x,t) = \phi(x)$) can be taken as:

$$\begin{cases} v_t + (f'(\phi)v)_x = v_{xx} + v_{xxt}, \;\; x > 0, t > 0 \\ v(0,t) = 0, \;\; t \geq 0 \\ v(x,0) = v_0(x), \;\; x > 0, v_0 \in H^2. \end{cases} \tag{3.3}$$

Let $v \in C([0,T]; H_0^2) \cap C^1([0,T]; H^1)$ be a solution of (3.3). Multiply (3.3) by v and integrate the resulting equation with respect to x over $(0,\infty)$. Then from the integration by parts, we have

$$\frac{d}{dt}\int_0^\infty \frac{1}{2}(v^2+v_x^2)dx+\frac{1}{2}\int_0^\infty f''(\phi)\phi_x v^2 dx+\int_0^\infty v_x^2 dx=0. \tag{3.4}$$

If we note $\phi_x > 0$, the estimate of (3.4) works well in the case $f'' > 0$, but not in the case that f is not convex because f'' changes its sign. In order to overcome the above difficulty, we try to apply a weighted energy method as in [13], where in order to show the asymptotic stability of solutions for non-convex Burgers equation, a weight function w is manipulated as a function of the solution ϕ for (2.2). Here we choose the same weight function w as in [13] by $w = w(u) = f(u) + \delta g(u)$, where $g(u) = -u^{2m} + r^{2m}$, $\delta > 0$ and $m \geq 1$ are constant. Now we state the following lemma about the weight function.

Lemma 3.3 — (Weight Function) Under the condition (1.2), if we take δ sufficiently small and m sufficiently large, then it holds

$$\frac{1}{2}(f''(u)w(u)-f(u)w''(u))>0, \quad w(u)>0 \;\; for \;\; u\in[u_-,r].$$

Motivated by the argument in [13], we introduce a new unknown function $\tilde{v}$ by

$$v(x,t)=w(\Phi(x,t))\tilde{v}(x,t), \tag{3.5}$$

where $\Phi(x,t) = \phi(x) + \varphi(x,t)$, $w = f + \delta g$ is the weight function in Lemma 3.3. We note that w is well defined by Lemma 3.3, that is, smooth and satisfied

$$\nu \leq w(\Phi(x,t)) \leq C, \;\; C_1 \leq (\tfrac{w'}{w})(\Phi) \leq C_2, \;\; x\geq 0, \;\; t\geq 0 \tag{3.6}$$

for some positive constants C, C_1 and C_2.

Next, we proceed to prove Proposition 3.2. The proof of Proposition 3.2 can be obtained by a series of lemmas.

Lemma 3.4 — Under the assumption of Proposition 3.2, the following estimate is valid:

$$\|v(t)\|_1^2+\int_0^t(\|\sqrt{\Phi_x}v(\tau)\|^2+\|v_x(\tau)\|^2)d\tau$$
$$\leq C(\|v_0\|_1^2+\int_0^t\|F(\tau)\|_{L^1}^{\frac{4}{3}}d\tau+d_0+\varepsilon\int_0^t\|v_{xx}(\tau)\|d\tau+d_0\int_0^\infty\|v_t(\tau)\|^2)d\tau, \tag{3.7}$$

where ε is a sufficiently small positive constant.

PROOF : Substituting (3.5) into $(2.6)_1$ and multiplying the resulting equation by $\tilde{v}$, we obtain after integrating it over $[0,\infty)$ that

$$\begin{aligned}&\int_0^\infty (w(\Phi)\tilde{v})_t\tilde{v}dx + \int_0^\infty (f(\Phi + w(\Phi)\tilde{v}) - f(\Phi))_x\tilde{v}dx \\ &\quad - \int_0^\infty (w(\Phi)\tilde{v})_{xx}\tilde{v}dx - \int_0^\infty (w(\Phi)\tilde{v})_{xxt}\tilde{v}dx = \int_0^\infty F\tilde{v}dx.\end{aligned} \tag{3.8}$$

We rewrite the second term on the left hand side of (3.8) as

$$\begin{aligned}&\int_0^\infty (f(\Phi + w(\Phi)\tilde{v}) - f(\Phi))_x\tilde{v}dx \\ &\quad = -\int_0^\infty (f(\Phi + w(\Phi)\tilde{v}) - f(\Phi))\tilde{v}_x dx \\ &\quad = \int_0^\infty -\frac{w'}{w^2}\left(\int_\Phi^{\Phi+w\tilde{v}} f(s)ds - f(\Phi + w\tilde{v})w\tilde{v}\right)\Phi_x dx \\ &\qquad + \int_0^\infty \frac{1}{w}(f(\Phi + w\tilde{v}) - f(\Phi) - f'(\Phi)w\tilde{v})\Phi_x dx \\ &\quad =: I_3 + I_4.\end{aligned}$$

We further rewrite I_3 and I_4 by the Taylor's formula as

$$\begin{aligned}I_3 &= \int_0^\infty \frac{1}{2}w'f'(\Phi + w\tilde{v})\tilde{v}^2\Phi_x dx \\ &\quad - \int_0^\infty \frac{w'}{w^2}\left(\int_\Phi^{\Phi+w\tilde{v}} f(s)ds - f(\Phi + w\tilde{v})w\tilde{v} + \frac{1}{2}f'(\Phi + w\tilde{v})w^2\tilde{v}^2\right)\Phi_x dx \\ &= \int_0^\infty \frac{1}{2}w'f'(\Phi)\Phi_x\tilde{v}^2dx + \int_0^\infty O(|\tilde{v}|)\Phi_x\tilde{v}^2dx,\end{aligned}$$

$$\begin{aligned}I_4 &= \int_0^\infty \frac{1}{2}wf''(\Phi)\tilde{v}^2\Phi_x dx \\ &\quad + \int_0^\infty \frac{1}{w}\left(f(\Phi + w\tilde{v}) - f(\Phi) - f'(\Phi)w\tilde{v} - \frac{1}{2}f''(\Phi)w^2\tilde{v}^2\right)\Phi_x dx \\ &= \int_0^\infty \frac{1}{2}wf''(\Phi)\Phi_x\tilde{v}^2dx + \int_0^\infty O(|\tilde{v}|)\Phi_x\tilde{v}^2dx.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\int_0^\infty (f(\Phi + w(\Phi)\tilde{v}) - f(\Phi))_x\tilde{v}dx \\ &\quad = \int_0^\infty \frac{1}{2}(wf'' + w'f')(\Phi)\Phi_x\tilde{v}^2dx + \int_0^\infty O(|\tilde{v}|)\Phi_x\tilde{v}^2dx.\end{aligned} \tag{3.9}$$

We also rewrite the third term on the left hand side of (3.8) as

$$
\begin{aligned}
-\int_0^\infty (w(\Phi)\tilde{v})_{xx}\tilde{v}dx &= \int_0^\infty (w\tilde{v}_x^2 - \tfrac{1}{2}w'\Phi_{xx}\tilde{v}^2 - \tfrac{1}{2}w''\Phi_x^2\tilde{v}^2)dx \\
&=: \int_0^\infty w\tilde{v}_x^2 dx + I_5 + I_6.
\end{aligned} \tag{3.10}
$$

Now, recalling the relation $\Phi_t + f(\Phi)_x - \Phi_{xx} - \Phi_{xxt} = -F(\phi,\varphi)$ and $\phi_x = f(\phi)$, we further rewrite I_5 as

$$
I_5 = \int_0^\infty (-\tfrac{1}{2}w'f'(\Phi)\Phi_x - \tfrac{1}{2}w'\varphi_t - \tfrac{1}{2}w'F + \tfrac{1}{2}w'\varphi_{xxt})\tilde{v}^2 dx, \tag{3.11}
$$

and I_6 as

$$
\begin{aligned}
I_6 &= -\int_0^\infty \tfrac{1}{2}w''(\phi_x + \varphi_x + f(\Phi) - f(\phi+\varphi))\Phi_x\tilde{v}_x^2 dx \\
&= -\tfrac{1}{2}\int_0^\infty w''f(\Phi)\Phi_x\tilde{v}^2 dx - \tfrac{1}{2}\int_0^\infty w''\Phi_x\tilde{v}^2(f(\phi) - f(\phi+\varphi) + \varphi_x)dx \\
&= -\int_0^\infty \tfrac{1}{2}w''f(\Phi)\Phi_x\tilde{v}^2 dx + \int_0^\infty O(|\varphi| + |\varphi_x|)\Phi_x\tilde{v}^2 dx.
\end{aligned} \tag{3.12}
$$

Substituting (3.11) and (3.12) into (3.10), we have

$$
\begin{aligned}
&-\int_0^\infty (w(\Phi)\tilde{v})_{xx}\tilde{v}dx \\
&\quad = \int_0^\infty -\tfrac{1}{2}(w''f + w'f')(\Phi)\Phi_x\tilde{v}^2 dx + \int_0^\infty O(|\varphi| + |\varphi_x|)\Phi_x\tilde{v}^2 dx \\
&\qquad + \int_0^\infty (-\tfrac{1}{2}w'\varphi_t\tilde{v}^2 + w\tilde{v}_x^2 - \tfrac{1}{2}w'F\tilde{v}^2 + \tfrac{1}{2}w'\varphi_{xxt}\tilde{v}^2)dx.
\end{aligned} \tag{3.13}
$$

We also rewrite the fourth term on the left hand side of (3.8) as

$$
\begin{aligned}
&-\int_0^\infty (w(\Phi)\tilde{v})_{xxt}\tilde{v}dx = -(w(\Phi)\tilde{v})_{xt}\tilde{v}|_0^\infty + \int_0^\infty (w(\Phi)\tilde{v})_{xt}\tilde{v}_x dx \\
&= \int_0^\infty (w'\varphi_t\tilde{v} + w\tilde{v}_t)_x\tilde{v}_x dx = -\int_0^\infty (w'\varphi_t\tilde{v} + w\tilde{v}_t)\tilde{v}_{xx}dx \\
&= -\int_0^\infty w'\varphi_t\tilde{v}\tilde{v}_{xx}dx + \int_0^\infty w'\Phi_x\tilde{v}_t\tilde{v}_x dx + \frac{d}{dt}\int_0^\infty \tfrac{1}{2}w\tilde{v}_x^2 dx - \int_0^\infty \tfrac{1}{2}w'\varphi_t\tilde{v}_x^2 dx.
\end{aligned} \tag{3.14}
$$

Combining (3.8)-(3.14), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_0^\infty \tfrac{1}{2}w(\tilde{v}^2 + \tilde{v}_x^2)dx + \int_0^\infty \tfrac{1}{2}(wf'' - w''f)(\Phi)\Phi_x\tilde{v}^2 dx + \int_0^\infty w\tilde{v}_x^2 dx \\
&= \int_0^\infty (F\tilde{v} + \tfrac{1}{2}w'F\tilde{v}^2)dx - \int_0^\infty \tfrac{1}{2}w'\varphi_{xxt}\tilde{v}^2 dx - \int_0^\infty O(|\tilde{v}| + |\varphi| + |\varphi_x|)\tilde{v}^2\Phi_x dx \\
&\quad + \int_0^\infty (-w'\Phi_x\tilde{v}_t\tilde{v}_x + \tfrac{1}{2}w'\varphi_t\tilde{v}_x^2)dx + \int_0^\infty w'\varphi_t\tilde{v}\tilde{v}_{xx}dx.
\end{aligned} \tag{3.15}
$$

Using the Sobolev's inequality and Lemma 2.3 easily gets

$$|\int_0^\infty O(|\tilde{v}|+|\varphi|+|\varphi_x|)\Phi_x\tilde{v}^2dx| \le C(N(T)+h_1(u_+))\int_0^\infty \Phi_x\tilde{v}^2dx. \tag{3.16}$$

By the Sobolev's inequality and Young's inequality, one obtains

$$\begin{aligned} |\int_0^\infty (F\tilde{v}+\tfrac{1}{2}w'F\tilde{v}^2)dx| &\le C\int_0^\infty |\tilde{v}||F|dx \le C\|\tilde{v}\|^{\frac{1}{2}}\|\tilde{v}_x^2\|^{\frac{1}{2}}\|F\|_{L^1} \\ &\le \tfrac{1}{8}\int_0^\infty w\tilde{v}_x^2dx + C\|F\|_{L^1}^{\frac{4}{3}}. \end{aligned} \tag{3.17}$$

Due to Lemma 2.1 and Lemma 2.3, we can estimate the last three terms on the right hand of (3.15) as follows, respectively.

$$\begin{aligned} |-\int_0^\infty \tfrac{1}{2}w'\varphi_{xxt}\tilde{v}^2dx| &= |\int_0^\infty \tfrac{1}{2}w'\tfrac{\varphi_{xxt}}{\sqrt{\varphi_x}}\sqrt{\varphi_x}\tilde{v}^2dx| \\ &\le \tfrac{\varepsilon_0}{8}\int_0^\infty \varphi_x\tilde{v}^2dx + Ch_1(u_+)(1+t)^{-1-\frac{1}{q}}, \end{aligned} \tag{3.18}$$

$$|\int_0^\infty(-w'\Phi_x\tilde{v}_t\tilde{v}_x+\tfrac{1}{2}w'\varphi_t\tilde{v}_x^2)dx| \le \tfrac{1}{8}\int_0^\infty w\tilde{v}_x^2dx + Cd_0(\int_0^\infty w\tilde{v}_t^2dx+\int_0^\infty w\tilde{v}_x^2dx), \tag{3.19}$$

$$\int_0^\infty w'\varphi_t\tilde{v}\tilde{v}_{xx}dx \le \varepsilon_2\int_0^\infty w\tilde{v}_{xx}^2dx + Ch_1(u_+)\int_0^\infty \Phi_x\tilde{v}^2dx, \tag{3.20}$$

where $\varepsilon_0, \varepsilon$ are suitably small positive constant.

Substituting (3.16)-(3.20) into (3.15) and making use of (3.1), (3.6) obtain

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty \tfrac{1}{2}w(\tilde{v}^2+\tilde{v}_x^2)dx + \nu\int_0^\infty \Phi_x\tilde{v}^2dx + \tfrac{3}{4}\int_0^\infty w\tilde{v}_x^2dx \\ &\le C(N(T)+h_1(u_+))\int_0^\infty \Phi_x\tilde{v}^2dx + \tfrac{\varepsilon_1}{8}\int_0^\infty \varphi_x\tilde{v}^2dx + Ch_1(u_+)(1+t)^{-1-\frac{1}{q}} \\ &\quad + C\|F\|_{L^1}^{\frac{4}{3}} + \varepsilon_2\int_0^\infty w\tilde{v}_{xx}^2dx + Cd_0(\int_0^\infty w\tilde{v}_t^2dx+\int_0^\infty w\tilde{v}_x^2dx), \end{aligned} \tag{3.21}$$

where ν is a positive constant. Noticing (3.5), (3.6) and the fact

$$\|v_x\|^2 = \|w_x\tilde{v}+w\tilde{v}_x\|^2 \le C(\|\sqrt{\Phi_x}\tilde{v}\|^2+\|\tilde{v}_x\|^2)$$

by taking $N(T)+d_0$ suitably small and integrating (3.21) with respect to t over $[0,t]$, we get the desired inequality (3.7).

Thus the proof of Lemma 3.4 is completed.

Lemma 3.5 — Under the assumption of Proposition 3.2, it holds

$$\begin{aligned}&\|v_x(t)\|_1^2+\int_0^t\|v_{xx}(\tau)\|^2d\tau\\&\le C(\|v_0\|_2^2+\int_0^t(\|F\|_{L^1}^{\frac{4}{3}}+\|F\|^2)d\tau+d_0+d_0\int_0^\infty\|v_t\|^2d\tau).\end{aligned}\tag{3.22}$$

PROOF : Multiplying $(2.6)_1$ by $-v_{xx}$ and integrating the result over $[0,\infty)$, we get

$$\begin{aligned}&\frac{d}{dt}\int_0^\infty\frac{1}{2}(v_x^2+v_{xx}^2)dx+\int_0^\infty v_{xx}^2dx\\&=-\int_0^\infty Fv_{xx}dx+\int_0^\infty v_{xx}(f(v+\Phi)-f(\Phi))_xdx.\end{aligned}\tag{3.23}$$

We estimate each term on the right hand side of (3.23) as

$$|-\int_0^\infty Fv_{xx}dx|\le\frac{1}{8}\|v_{xx}\|^2+C\|F\|^2,\tag{3.24}$$

$$\int_0^\infty v_{xx}(f(v+\Phi)-f(\Phi))_xdx\le\frac{1}{8}\|v_{xx}\|^2+C(\|\sqrt{\Phi_x}v\|^2+\|v_x\|^2).\tag{3.25}$$

Substituting (3.24) and (3.25) into (3.23), we obtain

$$\frac{d}{dt}\int_0^\infty\frac{1}{2}(v_x^2+v_{xx}^2)dx+\frac{3}{4}\int_0^\infty v_{xx}^2dx\le C(\|F\|^2+\|\sqrt{\Phi_x}v\|^2+\|v_x\|^2).\tag{3.26}$$

Integrating (3.26) with respect to t over $[0,t]$ and combining the result with the estimate (3.7), we can complete the proof of Lemma 3.5.

Lemma 3.6 — Under the assumption of Proposition 3.2, it holds

$$\begin{aligned}&\|v(t)\|_2^2+\int_0^t(\|\sqrt{\Phi_x}v(\tau)\|^2+\|v_x(\tau)\|_1^2d\tau\\&\le C(\|v_0\|_2^2+h_1(u_+)^{\frac{1}{6}}+d_0+d_0\int_0^t\|v_t(\tau)\|^2d\tau).\end{aligned}\tag{3.27}$$

PROOF: Due to Lemma 3.4 and Lemma 3.5, we obtain

$$\begin{aligned}&\|v(t)\|_2^2+\int_0^t(\|\sqrt{\Phi_x}v(\tau)\|^2+\|v_x(\tau)\|_1^2)d\tau\\&\le C(\|v_0\|_2^2+\int_0^t(\|F\|_{L^1}^{\frac{4}{3}}+\|F\|^2)d\tau+d_0+d_0\int_0^\infty\|v_t\|^2d\tau).\end{aligned}\tag{3.28}$$

Now, following the arguments in [13], we estimate the terms $\|F\|_{L^1}^{\frac{4}{3}}$ and $\|F\|^2$ on the right hand side of (3.28). By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}\|\phi_x\varphi\|_{L^1} &\le C\int_0^t \frac{\varphi}{(1+x)^2}dx + C\int_t^\infty \frac{\varphi}{(1+x)^2}dx\\ &\le C(\frac{-\varphi}{(1+x)}|_0^t + \int_0^t \frac{\varphi_x}{(1+x)}dx) + C\|\varphi\|_{L^\infty}\int_t^\infty \frac{dx}{(1+x)^2}\\ &\le C\|\varphi_x\|_{L^\infty}^{\frac{1}{8}}\|\varphi_x\|_{L^\infty}^{\frac{7}{8}} log(2+t) + Ch_1(u_+)^{\frac{1}{8}}h_1(u_+)^{\frac{7}{8}}(1+t)^{-1}\\ &\le Ch_1(u_+)^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}log(2+t).\end{aligned}$$

Similarly, we get

$$\|\phi\varphi_x\|_{L^1} + \|\varphi_{xx}\|_{L^1} + \|\varphi_{xxt}\|_{L^1} \le Ch_1(u_+)^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}log(2+t),$$

$$\|\phi_x\varphi\|^2 + \|\phi\varphi_x\|^2 + \|\varphi_{xx}\|^2 + \|\varphi_{xxt}\|^2 \quad \le Ch_1(u_+)^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}.$$

Hence, from the above estimates and the following inequality

$$|F(\Phi)| \le C(|\phi\varphi_x| + |\phi_x\varphi| + |\varphi_{xx}| + |\varphi_{xxt}|,$$

it follows that

$$\|F\|_{L^1}^{\frac{4}{3}} \le Ch_1(u_+)^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}log(2+t), \tag{3.29}$$

$$\|F\|^2 \le Ch_1(u_+)^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}. \tag{3.30}$$

Substituting (3.29) and (3.30) into (3.28) obtains the desired estimate (3.27).

Lemma 3.7 — Under the assumption of Proposition 3.3, it holds

$$\|v_x(t)\|^2 + \int_0^t \|v_t(\tau)\|^2 d\tau + \int_0^t \|v_{xt}(\tau)\|^2 d\tau \le C(\|v_0\|_1^2 + d_0^{\frac{1}{6}}). \tag{3.31}$$

PROOF : Multiplying $(2.6)_1$ by v_t and integrating the result over $[0,\infty)$, we have

$$\begin{aligned}&\frac{d}{dt}\int_0^\infty \frac{1}{2}v_x^2 dx + \int_0^\infty v_{xt}^2 dx + \int_0^\infty v_t^2 dx\\ &= -\int_0^\infty v_t(f(v+\Phi) - f(\Phi))_x dx + \int_0^\infty F v_t dx.\end{aligned} \tag{3.32}$$

The first term on the right hand side of (3.32) can be estimated as

$$\begin{aligned} |-\int_0^\infty v_t(f(v+\Phi)-f(\Phi))_x dx| &\le \int_0^\infty C(|v_x|+|v|\Phi_x)|v_t|dx \\ &\le \tfrac{1}{4}\|v_t\|^2 + C(\|v_x\|^2 + \|\sqrt{\Phi_x}v\|^2). \end{aligned} \tag{3.33}$$

By (3.30), we estimate the second term on the right hand side of (3.32) as

$$\begin{aligned} |\int_0^\infty F v_t dx| &\le \tfrac{1}{4}\|v_t\|^2 + \|F\|^2 \\ &\le \tfrac{1}{4}\|v_t\|^2 + Ch_1(u_+)^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{3.34}$$

Substituting (3.33) and (3.34) into (3.32), we have

$$\begin{aligned} &\tfrac{d}{dt}\int_0^\infty \tfrac{1}{2}v_x^2 dx + \int_0^\infty v_{xt}^2 dx + \int_0^\infty v_t^2 dx \\ &\le C(\|v_x\|^2 + \|\sqrt{\Phi_x}v\|^2 + h_1(u_+)^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{3.35}$$

Integrating (3.35) with respect to t over $[0,t]$ and combining the result with the estimate (3.27), we obtain (3.31) for sufficiently small $\|v_0\|_{H^2}^2 + d_0$. This completes the proof of Lemma 3.7.

Now putting the estimates (3.27) and (3.31) together, we can get (3.2). This completes the proof of Proposition 3.3.

In what follows, we prove Theorem 2.4.

PROOF : To prove that the problem (2.6) has a unique global solution $v(x,t) \in X(0,\infty)$, we only need to verified the a priori estimate (3.1) holds, once we have Proposition 3.2, we can take a suitably small constant η such that

$$C\eta \le \frac{\delta^2}{2}.$$

By Proposition 3.2, we have

$$\|v(t)\|_2^2 \le C\|v_0\|_2^2 \le C\eta \le \frac{\delta^2}{2}.$$

This shows that the a priori assumption (3.1) is reasonable. So we have proved that (2.6) has a unique global solution.

In what follows, we shall prove $\lim_{t\to\infty}\sup_{x\in R^+}|\frac{\partial^i v(x,t)}{\partial x^i}| = 0 \ \ (i = 0, 1)$.

Put $g(t) = \|v_x(t)\|^2$. Then the inequality (3.2) yields $g \in L^1(0,\infty)$ and $\frac{dg}{dt} \in L^1(0,\infty)$, from which it follows that $\lim_{t\to\infty}\|v_x(t)\|^2 = \lim_{t\to\infty} g(t) = 0$. Therefore, by applying the Sobolev's inequality, we get

$$\sup_{x\in R^+}|v(x,t)| \leq 2^{\frac{1}{2}}\|v(t)\|^{\frac{1}{2}}\|v_x(t)\|^{\frac{1}{2}} \to 0 \ \ (t\to\infty),$$

$$\sup_{x\in R^+}|v_x(x,t)| \leq 2^{\frac{1}{2}}\|v_x(t)\|^{\frac{1}{2}}\|v_{xx}(t)\|^{\frac{1}{2}} \to 0 \ \ (t\to\infty).$$

Thus the proof of Theorem 2.4 is completed.

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