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ON EXISTENCE OF INVARIANT MEASURES

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Let G be a Lie group, $H \leq G$ a closed subgroup and $M \approx G/H$. In [14] André Weil gave a necessary and sufficient condition for the existence of invariant measures on homogeneous spaces of arbitrary locally compact groups. For Lie groups using the structure theory we give a neater necessary and sufficient condition for the existence of a G-invariant measure on M, cf. Theorems (2.1) and (3.2) in the introduction.

Key words : Invariant measures; homogeneous spaces.

1. INTRODUCTION

Let G be a Lie group, suppose G acts transitively on a C^∞-manifold M. Fix $p \in M$, and let H be the stabilizer subgroup at p. Since G acts transitively, we identify the quotient space G/H with M. It is known that G/H is admits a differential structure and M is diffeomorphic to G/H with respect to this structure (cf. [9], [6]). A G-invariant Borel measure μ on M is a measure such that

(i) for every non-empty open subset U of M, $\mu(U) > 0$.

(ii) for every compact subset K of M, $\mu(K) < \infty$.

(iii) for every measurable subset A of M, and for every element $g \in G$, we have $\mu(gA) = \mu(A)$.

It is not necessary for such a measure to exist on M. A criterion for the existence of G-invariant measures on a homogeneous space G/H was first found by Weil (cf. [14]). For a locally compact topological group G, Weil's criterion was in terms of the modular functions Δ of G and δ of H. Weil's theorem states that G/H admits a G-invariant measure if and only if $\Delta(h) = \delta(h)$, for every h in H (cf. [3], [8], [6]). For a Lie group, the criterion can be interpreted in terms of Lie algebras. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h} \leq \mathfrak{g}$ the Lie algebra of H. Then the tangent space at any $p \in M$ can be identified with the vector space $\mathfrak{g}/\mathfrak{h}$. Since H keeps p fixed we get the isotropy representation $\rho : H \longrightarrow Aut\,(T_p(M))$ of H. Also, for G/H we have the representation $\overline{Ad_H} : H \longrightarrow Gl(\mathfrak{g}/\mathfrak{h})$, induced by the representation $Ad|_H$. The action of G on M is equivalent to the action of G on G/H. Consequently the representations ρ and $\overline{Ad_H}$ are equivalent. And the differential representation $\rho_* : \mathfrak{h} \longrightarrow End\,(T_p(M))$ of ρ is equivalent to $\overline{ad_{\mathfrak{h}}} : \mathfrak{h} \longrightarrow End\,(\mathfrak{g}/\mathfrak{h})$ of $\overline{Ad_H}$. Let $R \leq H$ be the radical of H, i.e the maximal, solvable, normal subgroup of H and $N \leq H$ be the nilradical, i.e. the maximal, nilpotent, normal subgroup. Then in terms of the $\overline{ad_{\mathfrak{h}}}$-representation we prove the following:

***Theorem* 2.1** — *$M = G/H$, let $\mathfrak{g}$ = the Lie algebra of G, $\mathfrak{h}$ = the Lie algebra of H, $\mathfrak{r}$ = the radical of $\mathfrak{h}$, $\mathfrak{n}$ = the nilradical of $\mathfrak{h}$. Let $\mathfrak{a}$ = a complementary subspace to $\mathfrak{n}$ in $\mathfrak{r}$ and $D\mathfrak{r} = [\mathfrak{r}, \mathfrak{r}]$, then a necessary and sufficient condition for M to admit a G-invariant Borel measure is that for all $X \in \mathfrak{a}$,*

$$trace\, ad_{\mathfrak{g}}(X) = trace\, ad_{\mathfrak{g}}(X)|_{D\mathfrak{r}}.$$

A bi-invariant measure on a group G is a Borel measure μ which is invariant under both the left- and right-translations. More precisely we can say that μ is a bi-invariant measure if for every measurable set A,

$$\mu(A) = \mu(gAh^{-1}),\ \forall\, g, h \in G.$$

A group G admitting such a measure is called *unimodular*. The existence of a left-invariant measure on a locally compact, second countable topological group was first given by Haar (cf. [4]). John von Neumann in [11], proved that such a measure is unique up to multiplication by a positive real number. Similarly one sees that a locally compact topological group also admits a right-invariant Haar measure. As a corollary to Theorem (2.1), we get a necessary and sufficient condition for the existence of a bi-invariant measure on a Lie group. Let $R \leq G$ be the radical and $N \leq G$ the nilradical. Then we prove:

***Theorem* 3.2** — *Let* $\mathfrak{g}$ = *the Lie algebra of* G*,* $\mathfrak{r}$ = *the radical of* $\mathfrak{g}$*,* $\mathfrak{n}$ = *the nilradical of* $\mathfrak{g}$*. Let* $\mathfrak{a}$ = *a complementary subspace to* $\mathfrak{n}$ *in* $\mathfrak{r}$ *and* $D\mathfrak{r} = [\mathfrak{r}, \mathfrak{r}]$*, then a necessary and sufficient condition for* G *to admit a bi-invariant Borel measure is that for all* $X \in \mathfrak{a}$*,*

$$trace\, ad_{\mathfrak{g}}(X)|_{D\mathfrak{r}} = 0.$$

2. Invariant Measures on Homogeneous Spaces

Given a finite dimensional Lie algebra $\mathfrak{g}$ we know there exists a unique maximal solvable ideal $\mathfrak{r}$ called the *radical*. Similarly there exists a unique maximal nilpotent ideal $\mathfrak{n}$ called the nilradical. Also we have the Levi decomposition of $\mathfrak{g}$ (cf. [8]), which states that there exists a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} \approx \mathfrak{r} \rtimes \mathfrak{s}$$

By a theorem of Harish-Chandra (cf. [5]) $\mathfrak{s}$ is determined uniquely in the following sense, if $\mathfrak{r} \rtimes \mathfrak{s} = \mathfrak{r} \rtimes \mathfrak{s}'$ then there exists an automorphism σ of $\mathfrak{g}$ such that $\mathfrak{s}' = \sigma(\mathfrak{s})$. Moreover, $\sigma = exp(ad(N))$ where $N \in \mathfrak{n}$.

2.1 *Proof of Theorem 2.1* : Let G be a Lie group and H a closed subgroup. Let $M = G/H$, take $* = [H]$, the identity coset of H as the base-point. Let $V = T_*(M)$ and consider the isotropy representation.

$$\rho : H \longrightarrow Gl(V)$$

A necessary and sufficient condition for M to admit a G-invariant measure is that the image of H under the map ρ should lie in $Sl^{\pm}(V)$, where $Sl^{\pm}(V)$ is the group of all linear transformations of V which preserve the left-Haar measure on V (cf. [8], [6]). Note that the Haar measure on V is the same as the Lebesgue measure on $\mathbb{R}^n$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. Then, in terms of the differential isotropy representation, $\rho_* : \mathfrak{h} \longrightarrow gl(V)$, a necessary and sufficient condition for M to admit a G-invariant measure is that the image of $\mathfrak{h}$ under ρ_* must lie in $sl(V) = \{ A \in End(V) \,|\, trace\, A = 0 \}$. Now, the vector space V can be naturally identified with the vector space $\mathfrak{g}/\mathfrak{h}$. Under this identification the representation $\rho_* : \mathfrak{h} \longrightarrow gl(V)$ is isomorphic to the representation $\overline{ad_{\mathfrak{h}}} : \mathfrak{h} \longrightarrow End\,(\mathfrak{g}/\mathfrak{h})$. Thus, a necessary and sufficient condition for M to admit a G-invariant measure is that, for every $X \in \mathfrak{h}$,

$$trace\, \overline{ad_{\mathfrak{h}}}(X) = 0$$

$\overline{ad_{\mathfrak{h}}}$ is a map induced by the map $ad|_{\mathfrak{h}} = ad_{\mathfrak{h}} : \mathfrak{h} \longrightarrow End\,(\mathfrak{h})$, thus we get, for every $X \in \mathfrak{h}$,

$$trace\, \overline{ad_{\mathfrak{h}}}(X) = trace\, ad_{\mathfrak{g}}(X) - trace\, ad_{\mathfrak{h}}(X)$$

Therefore M admits a G-invariant measure if and only if $trace\, \overline{ad_{\mathfrak{h}}}(X) = 0$ if and only if

$$(*) \qquad trace\, ad_{\mathfrak{g}}(X) = trace\, ad_{\mathfrak{h}}(X), \;\; \forall\, X \in \mathfrak{h}$$

This is an analogue of Weil's criterion in terms of the modular functions of G and H. Moreover, let $\mathfrak{r}$ be the radical of $\mathfrak{h}$. Then we have the Levi decomposition $\mathfrak{h} = \mathfrak{r} \rtimes \mathfrak{s}$, where $\mathfrak{s}$ is a semi-simple subalgebra of $\mathfrak{h}$. Then every $X \in \mathfrak{h}$ can be written uniquely as

$$X = X_{\mathfrak{r}} + X_{\mathfrak{s}}, \;\; X_{\mathfrak{r}} \in \mathfrak{r},\, X_{\mathfrak{s}} \in \mathfrak{s}$$

And thus,

$$trace\, ad_{\mathfrak{h}}(X) = trace\, ad_{\mathfrak{h}}(X_{\mathfrak{r}}) + trace\, ad_{\mathfrak{h}}(X_{\mathfrak{s}}).$$

Lemma (2.1.1) — Let $\mathfrak{k}$ be a Lie algebra such that $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}]$. If $\varphi : \mathfrak{k} \longrightarrow \mathbb{R}$ is a Lie algebra homomorphism, then $\forall\, Z \in \mathfrak{k},\ \varphi(Z) = 0$.

PROOF : If $Z \in \mathfrak{k}$ then $Z = \sum_{i=1}^{l} c_i[X_i, Y_i]$, where $X_i, Y_i \in \mathfrak{k}$. Therefore

$$\begin{aligned} \varphi(Z) &= \varphi(\sum_{i=1}^{l} c_i[X_i, Y_i]) \\ &= \sum_{i=1}^{l} c_i \varphi([X_i, Y_i]) \\ &= \sum_{i=1}^{l} c_i[\varphi(X_i), \varphi(Y_i)] \\ &= 0 \end{aligned}$$

Thus for every $Z \in \mathfrak{k},\ \varphi(Z) = 0$. q.e.d.

Since $\mathfrak{s}$ is semi-simple, $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$, and $trace\, ad_{\mathfrak{h}} : \mathfrak{s} \longrightarrow \mathbb{R}$ is a homomorphism. Thus for every $X_{\mathfrak{s}} \in \mathfrak{s}$, $trace\, ad_{\mathfrak{h}}(X_{\mathfrak{s}}) = 0$. Thus we get,

$$trace\, ad_{\mathfrak{h}}(X) = trace\, ad_{\mathfrak{h}}(X_{\mathfrak{r}})$$

Lemma (2.1.2) — Let $\mathfrak{l}$ be a Lie algebra and $\mathfrak{i}$ an ideal of $\mathfrak{l}$, then for all $X \in \mathfrak{i}$, $trace\, ad_{\mathfrak{l}}(X) = trace\, ad_{\mathfrak{i}}(X)$.

PROOF : Start with a basis of $\mathfrak{i}$ and extend it to a basis of $\mathfrak{l}$. For any $X \in \mathfrak{i}$, the matrix of $ad_{\mathfrak{l}}(X)$ looks like

$$\begin{pmatrix} A & * \\ 0 & 0 \end{pmatrix}$$

where, A is the matrix of the map $ad_{\mathfrak{l}}(X)|_{\mathfrak{i}} = ad_{\mathfrak{i}}(X)$. Thus we get that, $trace\, ad_{\mathfrak{l}}(X) = trace\, ad_{\mathfrak{i}}(X)$. q.e.d.

And since $\mathfrak{r}$ is an ideal of $\mathfrak{h}$ we get

$$trace\, ad_{\mathfrak{h}}(X_{\mathfrak{r}}) = trace\, ad_{\mathfrak{r}}(X_{\mathfrak{r}})$$

Thus M admits a G-invariant measure if and only if for all $X \in \mathfrak{r}$, $trace\, ad_{\mathfrak{g}}(X)$ $= trace\, ad_{\mathfrak{r}}(X)$. Thus to decide whether M admits a G-invariant measure we only need to look at $ad_{\mathfrak{r}}(X) : \mathfrak{r} \longrightarrow \mathfrak{r}$ where $X \in \mathfrak{r}$. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{h}$. Then $\mathfrak{n}$ is an ideal of $\mathfrak{h}$, and also of $\mathfrak{r}$. It is known that $D\mathfrak{r} = [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{n}$ (cf. [1], Chapter I, pp 49). Let $X_{\mathfrak{n}} \in \mathfrak{n}$ and $Z \in \mathfrak{r}$, $[\mathfrak{n}, \mathfrak{r}] \subseteq \mathfrak{n}$ gives us $[\mathfrak{n}, \cdots, [\mathfrak{n}, \mathfrak{r}]] \subseteq [\mathfrak{n}, \cdots, [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]]] = 0$. Thus, for some $k \in \mathbb{N}$, $ad_{\mathfrak{r}}(X_{\mathfrak{n}})^k(Z) = 0$ and $ad_{\mathfrak{r}}(X_{\mathfrak{n}})$ acts as a nilpotent endomorphism of $\mathfrak{r}$. Therefore,

$$\forall \;\; X_{\mathfrak{n}} \in \mathfrak{n}, \;\; trace\, ad_{\mathfrak{r}}(X_{\mathfrak{n}}) = 0$$

Let $\mathfrak{a} \subset \mathfrak{r}$ be a *subspace* of $\mathfrak{r}$ complementary to $\mathfrak{n}$. Now since $\mathfrak{r} = \mathfrak{n} \oplus \mathfrak{a}$ as a vector space given any $X \in \mathfrak{r}$ we can write $X = X_{\mathfrak{n}} + X_{\mathfrak{a}}$ where $X_{\mathfrak{n}} \in \mathfrak{n}$ and $X_{\mathfrak{a}} \in \mathfrak{a}$. Thus $ad_{\mathfrak{r}}(X) = ad_{\mathfrak{r}}(X_{\mathfrak{n}}) + ad_{\mathfrak{r}}(X_{\mathfrak{a}})$. But since $D\mathfrak{r} = [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{n}$, and $trace\, ad_{\mathfrak{r}}(X_{\mathfrak{n}}) = 0$ we get

$$trace\, ad_{\mathfrak{r}}(X) = trace\, ad_{\mathfrak{r}}(X_{\mathfrak{a}})$$

$D\mathfrak{r}$ is an ideal of both $\mathfrak{r}$ and $\mathfrak{n}$. We start with a basis of $D\mathfrak{r}$ and extend it to a basis of $\mathfrak{r}$. We know that for any $Z \in \mathfrak{r}$, $ad_{\mathfrak{r}}(X_{\mathfrak{a}})(Z) \in D\mathfrak{r}$. Thus, with respect to the basis above the matrix of $ad_{\mathfrak{r}}(X_{\mathfrak{a}})$ looks like,

$$\begin{pmatrix} A & * \\ 0 & 0 \end{pmatrix}$$

where A is the matrix of $ad_{\mathfrak{r}}(X_{\mathfrak{a}})|_{D\mathfrak{r}}$ which is the same as $ad_{\mathfrak{g}}(X_{\mathfrak{a}})|_{D\mathfrak{r}}$. Thus

$$trace\, ad_{\mathfrak{r}}(X_{\mathfrak{a}}) = trace\, ad_{\mathfrak{g}}(X_{\mathfrak{a}})|_{D\mathfrak{r}}$$

Thus we see that M admits a G-invariant measure if and only if

$$\forall\, X \in \mathfrak{a}, \;\; trace\, ad_{\mathfrak{g}}(X) = trace\, ad_{\mathfrak{g}}(X)|_{D\mathfrak{r}}$$

q.e.d.

3. Bi-invariant Measures on G

For Lie groups the existence of the left- and right-invariant measures are given in terms of a nowhere vanishing top degree differential form which is either left- or right-invariant. Recall that G is called *unimodular* if G admits a measure μ that is bi-invariant, that is both left- and right-invariant. We get a neater condition for G to admit a bi-invariant measure by considering G as a homogeneous space of $G \times G$.

3.1 *G as a homogeneous space of $G \times G$*

Let $\Gamma = G \times G$, Γ acts on G as follows, given $(g, h) \in \Gamma$ and $x \in G$

$$(g, h) \cdot x := gxh^{-1}.$$

This action is transitive and the stabilizer at the identity $e \in G$ is the group

$$\Delta(G) = \{ (g, g) \mid g \in G \} \approx G.$$

Thus $\Gamma/\Delta(G) \approx G$, thus we can ask if G admits a Γ-invariant measure. Say μ is a Γ-invariant measure on G, then for any measurable subset A,

$$\begin{aligned} \mu(A) &= \mu((g, h) \cdot A) \\ &= \mu(gAh^{-1}). \end{aligned}$$

Thus a Γ-invariant measure on G is the same as a bi-invariant measure on G.

3.2. *Γ-invariant measure on G*

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{G} \approx \mathfrak{g} \oplus \mathfrak{g}$ be the Lie algebra of Γ. The Lie algebra of $\Delta(G)$ is $\Delta(\mathfrak{g}) = \{ (X, X) \mid X \in \mathfrak{g} \}$. Now given $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$,

$$trace\, ad_{\mathfrak{g} \oplus \mathfrak{g}}((X, Y)) = trace\, ad_{\mathfrak{g}}(X) + trace\, ad_{\mathfrak{g}}(Y)$$

If $\mathfrak{r}$(resp. $\mathfrak{n}$) is the radical(resp. nilradical) of $\mathfrak{g}$ then $\Delta(\mathfrak{r})$(resp. $\Delta(\mathfrak{n})$) is the radical(resp. nilradical) of $\Delta(\mathfrak{g})$. If $\mathfrak{a}$ is a subspace of $\mathfrak{r}$ complementary to $\mathfrak{n}$, then

$\Delta(\mathfrak{a})$ is a subspace of $\Delta(\mathfrak{r})$ complementary to $\Delta(\mathfrak{n})$. Theorem (2.1) says G admits a Γ-invariant measure if and only if $\forall\,(X,X) \in \Delta(\mathfrak{a})$

$$trace\, ad_{\mathfrak{g}\oplus\mathfrak{g}}((X,X)) = trace\, ad_{\mathfrak{g}\oplus\mathfrak{g}}((X,X))|_{\Delta(D\mathfrak{r})}$$

Also note that $trace\, ad_{\mathfrak{g}\oplus\mathfrak{g}}((X,X)) = 2trace\, ad_{\mathfrak{g}}(X)$. And $\overline{ad_{\mathfrak{h}}}$ in this case is just the $ad_{\mathfrak{g}}$-representation of $\mathfrak{g}$. Thus $trace\, \overline{ad_{\mathfrak{h}}} \equiv 0$ implies $trace\, ad_{\mathfrak{g}} \equiv 0$. Thus we get $0 = trace\, ad_{\mathfrak{g}\oplus\mathfrak{g}}((X,X)) = 2trace\, ad_{\mathfrak{g}}(X)$. Therefore, since

$$trace\, ad_{\mathfrak{g}\oplus\mathfrak{g}}((X,X))|_{D\mathfrak{r}\oplus D\mathfrak{r}} = 2trace\, ad_{\mathfrak{g}}(X)|_{D\mathfrak{r}}$$

we see that G admits a Γ-invariant measure, that is a bi-invariant measure if and only if

$$\forall\ X \in \mathfrak{a}, \quad trace\, ad_{\mathfrak{g}}(X)|_{D\mathfrak{r}} = 0$$

q.e.d.

Thus as a corollary to Theorem 2.1 we have proved Theorem 3.2.

We get some interesting corollaries to Theorem 2.1 and Theorem 3.2, which are useful in applications.

Corollary (3.2.1) — Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{r}$ its radical. Then $\mathfrak{g}$ is unimodular if and only if $\mathfrak{r}$ is unimodular.

The proof follows from Lemma(2.1.2) and Theorem(3.2).

Corollary (3.2.2) — Let $\mathfrak{g}$ be a Lie algebra such that the radical $\mathfrak{r}$ is nilpotent, then $\mathfrak{g}$ is unimodular.

The proof follows from Theorem 3.2.

Condition $(*)$ in the proof of Theorem(2.1) above also gives us the following

Corollary (3.2.3) — Let G be a unimodular Lie group and H a closed subgroup. Then G/H admits a G-invariant measure if and only if H is unimodular.

PROOF : Condition $(*)$ says G/H admits a G-invariant measure if and only if for every $X \in \mathfrak{h}$, $trace\, ad_{\mathfrak{h}}(X) = trace\, ad_{\mathfrak{g}}(X)$. G is unimodular if and only if for every $X \in \mathfrak{g}$, $trace\, ad_{\mathfrak{g}}(X) = 0$. Combining the two we get G/H admits a bi-invariant measure if and only if for all $X \in \mathfrak{h}$, $trace\, ad_{\mathfrak{h}}(X) = 0$. Thus, G/H admits a G-invariant measure if and only if H is unimodular q.e.d.

Corollary (3.2.4) — Let G be a unimodular Lie group and $H \leq G$ a closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. Let the radical $\mathfrak{r}$ of $\mathfrak{h}$ be nilpotent (i.e. if $\mathfrak{r} = \mathfrak{n}$), then G/H admits a G-invariant measure.

PROOF : Since $\mathfrak{r} = \mathfrak{n}$, the subspace $\mathfrak{a}$ is trivial. Therefore using the Levi decomposition for every $X \in \mathfrak{h}$, $trace\, ad_{\mathfrak{h}}(X) = 0 = trace\, ad_{\mathfrak{g}}(X)$. Thus by Theorem(2.1) G/H admits a G-invariant measure q.e.d.

As a special case we get the following

Corollary 3.2.5 — Let G be a unimodular Lie group and H a closed, 1-parameter subgroup of G. Then G/H admits a G-invariant measure.

4. EXAMPLES

4.1. Aff$(\mathbf{n}, \mathbb{R})$

Let $Aff(n, \mathbb{R}) =$ the group of all affine transformations of $\mathbb{A}^n$, the n-dimensional affine plane over $\mathbb{R}$. Choosing affine coordinates we get the following representation,

$$Aff(n, \mathbb{R}) = \{\, \vec{x} \mapsto A\vec{x} + \vec{b} \,|\, A \in Gl_n(\mathbb{R}),\ \vec{x}, \vec{b} \in \mathbb{R}^n \,\}$$

Let $\mathfrak{aff}_{\mathfrak{n}}$ be the Lie algebra of $Aff(n, \mathbb{R})$, then we see that,

$$\mathfrak{aff}_{\mathfrak{n}} \approx (\mathbb{R}^{\mathfrak{n}} \rtimes \mathbb{R}) \rtimes \mathfrak{sl}_{\mathfrak{n}}(\mathbb{R})$$

The above description is the Levi decomposition for $\mathfrak{aff}_{\mathfrak{n}}$, with $\mathfrak{r} \approx \mathbb{R}^n \rtimes \mathbb{R}$ and $\mathfrak{s} \approx \mathfrak{sl}_{\mathfrak{n}}(\mathbb{R})$. For $\mathfrak{r} \approx \mathbb{R}^n \rtimes \mathbb{R}$, the action of $\mathbb{R}$ on $\mathbb{R}^n$ is just the linear action of scalar matrices. That is, $\lambda \in \mathbb{R}$ corresponds to the matrix λI_n, where I_n is the

$n \times n$ identity matrix. Thus we can show that the nilradical $\mathfrak{n}$ is isomorphic to $\mathbb{R}^n$. Therefore $\mathfrak{a}$, the subspace complementary to $\mathfrak{n}$ in $\mathfrak{r}$, is the subspace of all scalar matrices. That is, $\mathfrak{a} \approx \mathbb{R}$.

Let X be a non-trivial element in $\mathfrak{a}$. X corresponds to the matrix λI_n, with $\lambda \neq 0$. Thus,

$$trace\, ad_{\mathfrak{aff_n}}(X)|_{\mathfrak{n}} = trace\, ad_{\mathfrak{aff_n}}(X)|_{D\mathfrak{r}} = trace\, \lambda I_n = n\lambda \neq 0$$

Thus, by Theorem (3.2), $Aff(n, \mathbb{R})$ does not admit a bi-invariant measure.

4.2. $\mathbf{SAff}(\mathbf{n}, \mathbb{R})$

Let $SAff(n, \mathbb{R})$ be the group of all measure preserving affine transformations. We have the following representation,

$$SAff(n, \mathbb{R}) = \{\, \vec{x} \mapsto A\vec{x} + \vec{b} \,|\, A \in Sl_n^{\pm}(\mathbb{R}),\, \vec{x}, \vec{b} \in \mathbb{R}^n \,\}$$

where $Sl_n^{\pm}(\mathbb{R}) = \{\, A \in Gl_n(\mathbb{R}) \,|\, det\, A = \pm 1 \,\}$. If $\mathfrak{saff_n}$ is the Lie algebra of $SAff(n, \mathbb{R})$, then $\mathfrak{saff_n} \approx \mathbb{R}^{\mathfrak{n}} \rtimes \mathfrak{sl_n}(\mathbb{R})$. This is the Levi decomposition for $\mathfrak{saff_n}$, and we see that $\mathfrak{r} \approx \mathbb{R}^n$. Thus the radical is nilpotent. Therefore, by the Corollary(3.2.2) above, $SAff(n, \mathbb{R})$ admits a bi-invariant measure.

4.3. *Isometries of* $\mathbb{E}^n$

Let $\mathcal{E}(n)$ be the group of all isometries of the Euclidean n-space, $\mathbb{E}^n$. We have the following representation,

$$\mathcal{E}(n) = \{\, \vec{x} \mapsto A\vec{x} + \vec{b} \,|\, A \in O(n),\, \vec{x}, \vec{b} \in \mathbb{R}^n \,\}$$

Let $\mathfrak{e_n}$ be the Lie algebra of $\mathcal{E}(n)$, then

$$\mathfrak{e_n} = \{\, \vec{x} \mapsto A\vec{x} + \vec{b} \,|\, A \in O(n),\, \vec{x}, \vec{b} \in \mathbb{R}^n \,\}$$

where $o(n)$ is the Lie algebra of $O(n)$. We divide the analysis into the following two cases, 4.3.1. *Case (i) :* $n \geq 3$

For $n \geq 3$, the Levi decomposition of $\mathfrak{e}_\mathfrak{n}$ is given by

$$\mathfrak{e}_\mathfrak{n} \approx \mathbb{R}^n \rtimes o(n)$$

Here, $o(n)$, which is the Lie algebra of all skew-symmetric matrices is semi-simple. Thus $\mathfrak{r} \approx \mathbb{R}^n$. That is the radical is nilpotent. Thus by the Corollary (3.2.2) above, $\mathcal{E}(n)$ admits a bi-invariant measure for $n \geq 3$.

4.3.2. Case (ii) : $n = 2$

The group $\mathcal{E}(2)$ can be described in terms of complex coordinates as

$$\mathcal{E}(2) = \{\, z \mapsto az + b \,|\, a, b \in \mathbb{C},\ |a| = 1 \,\}.$$

And thus can be realised as a subgroup of the solvable group of all bi-holomorphic maps of $\mathbb{C}$, $Aut(\mathbb{C})$. Thus, $\mathcal{E}(2)$ is a solvable group. Therefore the Lie algebra $\mathfrak{e}_2$ is solvable, that is $\mathfrak{r} = \mathfrak{e}_2$. The nilradical $\mathfrak{n}$ is isomorphic to $\mathbb{R}^2$ and the complementary subspace, $\mathfrak{a}$, is isomorphic to $\mathbb{R}$. More precisely, $\mathfrak{a} \approx o(2)$. Also the derived ideal, $D\mathfrak{r}$, of $\mathfrak{r}$ is also isomorphic to $\mathbb{R}^2$. $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a generator for $\mathfrak{a}$. That is, if $Y \in \mathfrak{a}$, then $Y = tX$ for some $t \in \mathbb{R}$. Thus, for all $Y \in \mathfrak{a}$, $trace\, ad_{\mathfrak{e}_2}(Y)|_{D\mathfrak{r}} = t(trace\, ad_{\mathfrak{e}_2}(X)|_{D\mathfrak{r}})$. But,

$$trace\, ad_{\mathfrak{e}_2}(X)|_{D\mathfrak{r}} = trace \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0.$$

Therefore, for all $Y \in \mathfrak{a}$, $trace\, ad_{\mathfrak{e}_2}(Y)|_{D\mathfrak{r}} = 0$. Thus, $\mathcal{E}(2)$ admits a bi-invariant measure.

We have shown that for all $n \in \mathbb{N}$, $\mathcal{E}(n)$ admits a bi-invariant measure.

4.4. *The Space of* k*-planes in* $\mathbb{E}^n$

Let $\mathbb{E}^n$ be the n-dimensional Euclidean space. Let $\Pi^k_{\mathbb{E}^n}$ be the space of all k-planes in $\mathbb{E}^n$. The isometry group, $\mathcal{E}(n)$ acts transitively on $\Pi^k_{\mathbb{E}^n}$, thus it is a homogeneous space of $\mathcal{E}(n)$. Let $\pi \in \Pi_{\mathbb{E}^n}$ be a k-plane, and τ be an isometry of $\mathbb{E}^n$ fixing π. Observe that the isometry group $\mathcal{E}(j)$, for j less than equal to n, is a subgroup of $\mathcal{E}(n)$.

Since τ fixes π, $\tau|_\pi$ is an isometry. Thus, $\tau|_\pi$ is an isometry of $\mathbb{E}^k$, that is $\tau|_\pi$ belongs to $\mathcal{E}(k)$. On the complement τ is still an isometry. But since it must be a translation along a vector parallel to π, τ must be an $(n-k)$-rotatory map on the complement. Thus τ belongs to $\mathcal{E}(k) \times O(n-k)$. That is the stabilizer subgroup at π is isomorphic to $\mathcal{E}(k) \times O(n-k)$. Thus as a homogeneous space,

$$\Pi^k_{\mathbb{E}^n} \approx \mathcal{E}(n)/(\mathcal{E}(k) \times O(n-k))$$

We have already seen that $\mathcal{E}(j)$ is a unimodular Lie group, for all $j \in \mathbb{N}$. And $O(n-k)$, being compact, is also unimodular. Thus $\mathcal{E}(k) \times O(n-k)$ is unimodular. Therefore, by Corollary(3.2.3) $\Pi^k_{\mathbb{E}^n}$ admits an $\mathcal{E}(n)$-invariant Borel measure.

4.5. *The Space of k-planes in $\mathbb{H}^n$*

Let $\mathbb{H}^n$ be the n-dimensional hyperbolic space. A submanifold N of $\mathbb{H}^n$ is said to be totally geodesic if for any two points P and Q in N, the geodesic joining P and Q lies completely in N. Equivalently, for any P and any $\vec{v} \in T_P(N)$, the geodesic through P with $\vec{v}$ as the tangent vector at P lies completely in N. By a k-plane in $\mathbb{H}^n$, we mean a k-dimensional, totally geodesic submanifold of $\mathbb{H}^n$.

Let $\Pi^k_{\mathbb{H}^n}$ be the space of all k-planes in $\mathbb{H}^n$. The full group of isometries of $\mathbb{H}^n$ acts transitively on $\Pi^k_{\mathbb{H}^n}$. Via the hyperboloid model of the hyperbolic n-space, we see that the isometry group of $\mathbb{H}^n$ is $O_0(n,1)$, the subgroup of $O(n,1)$ which preserves one sheet of the hyperboloid. Thus, $\Pi^k_{\mathbb{H}^n}$ is a homogeneous space of $O_0(n,1)$. Let π be a k-plane in $\mathbb{H}^n$ and φ an isometry fixing π. Observe that π is isometric to the hyperbolic k-space $\mathbb{H}^k$, and $\varphi|_\pi$ is an isometry. Thus φ_π is an element of $O_0(k,1)$ the isometry group of $\mathbb{H}^k$. On the complement φ acts as an $(n-k)$-rotatory map. Thus, the stabilizer at π is isomorphic to $O_0(k,1) \times O(n-k)$. Therefore,

$$\Pi^k_{\mathbb{H}^n} \approx O_0(n,1)/(O_0(k,1) \times O(n-k))$$

Both $O_0(n,1)$ and $O_0(k,1)\times O(n-k)$ are unimodular. Thus, by Corollary (3.2.3) $\Pi^k_{\mathbb{H}^n}$ admits a $O_0(n,1)$-invariant measure.

4.6. *The space of decompositions*

Let V be a real vector space of dimension n. Let π be a partition of the natural number n, $\pi : n = n_1 + n_2 + \cdots + n_r$. A *decomposition of type* π is expressing V as a direct sum of subspaces $V_1, V_2, \cdots, V_r$, such that $dim\, V_i = n_i$. Let $\mathcal{D}_\pi$ be the set of all decompositions of type π. Then $GL(V)$ acts transitively on $\mathcal{D}_\pi$. So $\mathcal{D}_\pi$ has a natural structure of a differentiable manifold. Now, the decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

is stabilized by exactly those elements of $Gl(V)$ that leave each V_i invariant. Thus the stabilizer at each decomposition is isomorphic to $Gl(V_1)\times Gl(V_2)\times\cdots\times Gl(V_r)$. That is, an isomorphism of T of V which leaves the decomposition $(V_1\oplus \cdots\oplus V_r)$ invariant is of the form $T = T_1\oplus\cdots\oplus T_r$, where $T_i = T|_{V_i}$. Thus we can identify the stabilizer with the subgroup

$$H := \left\{ \begin{pmatrix} Gl(V_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Gl(V_r) \end{pmatrix} \right\}.$$

And H is isomorphic to $Gl(V_1)\times Gl(V_2)\times\cdots\times Gl(V_r)$. Therefore we have the identification,

$$\mathcal{D}_\pi \approx GL(V)/(Gl(V_1)\times Gl(V_2)\times\cdots\times Gl(V_r))$$

Thus we get $\mathfrak{g} = End\,(V) \approx M_n(\mathbb{R})$ and $\mathfrak{h} \approx M_{n_1}(\mathbb{R})\times M_{n_2}(\mathbb{R})\times\cdots\times M_{n_r}(\mathbb{R})$. Since $Gl(V)$ and $Gl(V_i)$ both admit bi-invariant measures, we get that G and H both admit bi-invariant measures. Thus, by Corollary (3.2.3) $\mathcal{D}_\pi$ admits a G-invariant measure.

4.7. *Unimodular Lie algebras in dimension 3*

Let $\mathfrak{g}$ be a 3-dimensional Lie algebra. It is known that the only semi-simple, in fact simple, Lie algebras in dimension 3 are $sl_2(\mathbb{R})$ and $so(3)$ and the rest are of the form $\mathfrak{g} \approx \mathbb{R}^2 \rtimes \mathbb{R}$. If $< X > \approx \mathbb{R}$, then its action is given by $ad(X)$ which is a linear map of $\mathbb{R}^2$. Moreover, $\mathfrak{g}$ is unimodular if and only if $trace\, ad_{\mathfrak{g}}(X)|_{\mathbb{R}^2} = 0$. Up to conjugacy and using the fact that X may be replaced by any tX, where $t \in \mathbb{R}^*$, we get the following possibilities,

1. $ad_{\mathfrak{g}}(X)|_{\mathbb{R}^2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, in which case $\mathfrak{g}$ corresponds to the abelian Lie algebra $\mathbb{R}^3$.
2. $ad_{\mathfrak{g}}(X)|_{\mathbb{R}^2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, in which case $\mathfrak{g}$ corresponds to the Lie algebra $\mathfrak{e}_2$ of the group of all Euclidean isometries of $\mathbb{E}^2$.
3. $ad_{\mathfrak{g}}(X)|_{\mathbb{R}^2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, in which case $\mathfrak{g}$ corresponds to the 3-dimensional Heisenberg algebra $\mathfrak{H}$.
4. $ad_{\mathfrak{g}}(X)|_{\mathbb{R}^2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, in which case $\mathfrak{g}$ corresponds to the Lie algebra $\mathfrak{Sol}_3$, which is the Lie algebra corresponding to the 3-dimensional Riemannian, solve-geometry of Thurston (cf. [10]). It may also be interpreted as the Lie algebra of the group of all isometries of the Minkowski plane $\mathbb{E}^{1,1}$.

Therefore, up to isomorphism there are six unimodular Lie algebras in dimension 3, $sl_2(\mathbb{R})$, $so(3)$, $\mathfrak{e}_2$, $\mathfrak{H}$, $\mathfrak{Sol}_3$ and $\mathbb{R}^3$.

Theorem 2.1 and Theorem 3.2 give us necessary and sufficient conditions for the existence of invariant measures on spaces. We have listed a few examples where such measures exist. A theorem of S. S. Chern's (cf. [2]) gives one possible way of computing such measures. In a subsequent paper we give another interpretation of Chern's theorem. We also compute invariant measures for certain spaces. The computation is also closely related to the classical Cauchy-Crofton formula of integral geometry.

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