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MULTI-FRAME VECTORS FOR UNITARY SYSTEMS

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In this paper, the set of all complete multi-normalized tight frame vectors $\mathcal{NF}^r(\mathcal{U})$ with multiplicity r and the set of all complete multi-frame vectors $\mathcal{F}^r(\mathcal{U})$ with multiplicity r for a system $\mathcal{U}$ of unitary operators acting on a separable Hilbert space are characterized in terms of co-isometries and surjective operators in $\mathcal{C}_{\Psi^r}(\mathcal{U})$, the set of all operators which locally commute with $\mathcal{U}$ at Ψ^r, a fixed complete wandering r-tuple for $\mathcal{U}$. Then we study the linear combinations of multi-frame vectors for $\mathcal{U}$ and establish some conditions under which these combinations are still the same type of multi-frame vectors for $\mathcal{U}$. Finally, we establish some interesting properties for multi-frame vectors when $\mathcal{U}$ is a unitary group. All these results have potential applications in the theory of multi-Gabor systems and multi-wavelet systems.

Key words : Multi-frame vector; multi-Riesz vector; complete wandering r-tuple; local commutant; unitary system.

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1. INTRODUCTION

In 1946, D. Gabor [1] introduced a fundamental approach to signal decomposition in terms of elementary signals. In 1952, Duffin and Schaeffer [2] abstracted Gabor's method to define frames in Hilbert space. Frame was reintroduced by Daubechies, Grossmann, and Meyer [3] in 1986. Today, frame theory is a central tool in many areas such as characterizing function spaces and signal analysis. We refer to [4-10] for an introduction to frame theory and its applications. The following are the standard definitions on frames in Hilbert space.

Let H be a separable complex Hilbert space. Let $B(H)$ denote the algebra of all bounded linear operators on H. A *frame* for H is a sequence of vectors $\{x_j : j \in J\}$ in H for which there are constants $A, B > 0$ satisfying:

$$A\|x\|^2 \leq \sum_j |\langle x, x_j\rangle|^2 \leq B\|x\|^2$$

for all $x \in H$. We call A, B the lower and upper frame bounds for the frame, respectively. If $A = B$, then the frame is called a *tight frame*. If $A = B = 1$, then the frame is called a *normalized tight frame*. We say that frames $\{x_j : j \in J\}$ and $\{y_j : j \in J\}$ on Hilbert space H and K, respectively, are *unitarily equivalent* if there is a unitary $U : H \to K$ such that $Ux_j = y_j$ for all $j \in J$. We will say that they are *similar* if there is a bounded linear invertible operator $T : H \to K$ such that $Tx_j = y_j$ for all $j \in J$. A sequence $\{y_j\}$ is a Riesz basis for H if and only if $\{y_j\}$ is similar to an orthonormal basis of H. Frame is a natural generalization of basis with less restriction. However frame preserves many good properties of basis, that's why frames are so useful in mathematics and applications.

Historically there are two basic methods for constructing frames for Hilbert space $L^2(R)$. The following are their definitions. For any $a \in R$ let E_a, T_a and D_a be the *modulation operator*, *translation operator* and *dilation operator* on $L^2(R)$ with parameter a, respectively. i.e., $E_af(x) = e^{2\pi iax}f(x)$, $T_af(x) = f(x-a)$ and $D_af(x) = |a|^{-1/2}f(x/a)$ for any $f(x) \in L^2(R)$. For $a > 1, b > 0$ and $g(x) \in L^2(R)$, if $\{D_{a^n}T_{mb}g\}_{m,n\in Z}$ generates a frame for $L^2(R)$, then we say that (g, a, b) generates a *wavelet frame* for $L^2(R)$. For $a > 0, b > 0$

and $g(x) \in Ł^2(R)$, if $\{E_{mb}T_{na}g\}_{m,n\in Z}$ forms a frame for $L^2(R)$, then we say that (g, a, b) generates a *Gabor frame* for $L^2(R)$. To construct frames with some good features such as compact support, vanishing moments, symmetry or anti-symmetry at the same time is very useful in applications. However for singly generated frames it is not easy to achieve them simultaneously. For example the Balian-Low Theorem tells us that we can't construct a Gabor system with well localization in both time and frequency when $ab = 1$. In order to overcome such shortcomings of singly generated frames, multi-frames are created. And there are tons of literatures on the theory and applications of multi-frames (see [14-19]). The following are the definitions. For $a > 1, b > 0$ and $\Psi = \{\psi_1, \psi_2, \cdots, \psi_L\} \subset L^2(R)$, if $\{D_{a^n}T_{mb}\psi_i : i = 1, 2, \cdots, L\}_{m,n\in Z}$ generates a frame for $L^2(R)$, then we say that (Ψ, a, b) generates a *multi-wavelet frame with multiplicity* L for $L^2(R)$ and we call Ψ a *mother multi-wavelet frame with multiplicity* L. For $a > 0, b > 0$ and $\Phi = \{\phi_1, \phi_2, \cdots, \phi_L\} \subset L^2(R)$, if $\{E_{mb}T_{na}\phi_i : i = 1, 2, \cdots, L\}_{m,n\in Z}$ forms a frame for $L^2(R)$, then we say that (Φ, a, b) generates a *multi-Gabor frame* for $L^2(R)$ and we call Φ a *mother multi-Gabor frame with multiplicity* L.

It is easy to see that the modulation operators, translation operators and dilation operators are all unitary operators. So we can view the mother multi-wavelet frame for $L^2(R)$ as a frame generator vector under the dilation-translation unitary system and view the mother multi-Gabor frame for $L^2(R)$ as a frame generator vector under the modulation-translation unitary system, which is indeed the motivation of this paper. And this abstract way helps us to understand multi-frames more deeply and leads to some interesting results on multi-frames.

This paper will be organized in the following way. In section 2, we introduce some definitions and preliminaries. In section 3, we consider the multi-frame vectors for unitary systems. In section 4, the linear combinations of multi-frame vectors for unitary systems are studied. In section 5, the multi-frame vectors for unitary groups are discussed. In section 6, some examples are provided to illustrate the applications of the main results.

2. DEFINITIONS AND PRELIMINARIES

In this section, we introduce the definitions and lemmas needed in the sequel.

Definition 2.1 — A *unitary system* $\mathcal{U}$ is a subset of the unitary operators acting on a separable Hilbert space H which contains the identity operator I.

Definition 2.2 — A *wandering* r*-tuple* for $\mathcal{U}$ is a tuple $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ of elements in H such that $\mathcal{U}\Psi^r := \{U\psi_1, U\psi_2, \cdots, U\psi_r : U \in \mathcal{U}\}$ is an orthogonal set, that is $\langle U\psi_i, V\psi_j\rangle = 0$ if $U, V \in \mathcal{U}$ and $U \neq V$ or $i \neq j$. If $\mathcal{U}\Psi^r$ is an orthonormal basis for H, then Ψ^r is called a *complete wandering* r*-tuple* for $\mathcal{U}$. The set of all complete wandering r-tuple for $\mathcal{U}$ is denoted by $\mathcal{W}^r(\mathcal{U})$.

Definition 2.3 — A *multi-frame vector of multiplicity* r for $\mathcal{U}$ is a tuple $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ of elements in H such that $\mathcal{U}\Gamma^r := \{U\eta_1, U\eta_2, \cdots, U\eta_r : U \in \mathcal{U}\}$ is a frame for $\overline{Span}\{U\eta_1, U\eta_2, \cdots, U\eta_r : U \in \mathcal{U}\}$. If $\mathcal{U}\Gamma^r$ is a frame for H, then $\mathcal{U}\Gamma^r$ is called a *complete multi-frame vector of multiplicity* r for $\mathcal{U}$. The set of all complete multi-frame vectors of multiplicity r for $\mathcal{U}$ is denoted by $\mathcal{F}^r(\mathcal{U})$.

Remark 2.4 : *Complete multi-tight frame vector of multiplicity* r for $\mathcal{U}$, *Complete multi-normalized tight frame vector of multiplicity* r for $\mathcal{U}$ and *Complete Riesz vector of multiplicity* r for $\mathcal{U}$ can be defined similarly. $\mathcal{TF}^r(\mathcal{U})$, $\mathcal{NF}^r(\mathcal{U})$ and $\mathcal{R}^r(\mathcal{U})$ denote the set of all complete multi-tight frame vector of multiplicity r for $\mathcal{U}$, complete multi-normalized tight frame vector of multiplicity r for $\mathcal{U}$ and complete Riesz frame vector of multiplicity r for $\mathcal{U}$, respectively.

Definition 2.5 — Let $\mathcal{U}$ be a unitary system and $\Psi^r \in \mathcal{W}^r(\mathcal{U})$, the *local commutant* $\mathcal{C}_{\Psi^r}(\mathcal{U})$ at Ψ^r is defined by $\{T \in B(H) : (TU - UT)\psi_i = 0, i = 1, 2, \cdots, r, U \in \mathcal{U}\}$.

Remark 2.6 : Clearly $\mathcal{C}_{\Psi^r}(\mathcal{U})$ contains the commutant $\mathcal{U}'$ of $\mathcal{U}$ and $\mathcal{C}_{\Psi^r}(\mathcal{U}) = \cap_{i=1}^r \mathcal{C}_{\psi_i}(\mathcal{U}) \subseteq \mathcal{C}_{\psi_i}(\mathcal{U})$, for $i = 1, 2, \cdots, r$. It is obvious that $\mathcal{C}_{\Psi^r}(\mathcal{U})$ is a subspace of $B(H)$ and it is closed under strong operator topology and weak operator topology. When $\mathcal{U}$ is a unitary group, it is actually the commutant of $\mathcal{U}$ (see [11] Lemma 3.1). A useful result is the one to one correspondence between the complete wandering r-tuples and the unitary operators in $\mathcal{C}_{\Psi^r}(\mathcal{U})$. In particular, if $\Psi^r \in \mathcal{W}^r(\mathcal{U})$, then $\mathcal{W}^r(\mathcal{U}) = U(\mathcal{C}_{\Psi^r}(\mathcal{U})\Psi^r = \{T\psi_i : T \in U(\mathcal{C}_{\Psi^r}(\mathcal{U}), i = 1, 2, \cdots, r\}$, where $U(\mathcal{S})$ denotes the set of all unitary operators in $\mathcal{S}$ for any subset $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$(For more details, please see [12] Proposition 1.3 and [11] Theorem 3.5). It is also known that Ψ^r separates $\mathcal{C}_{\Psi^r}(\mathcal{U})$ in the sense that the mapping $A \to (A\psi_1, A\psi_2, \cdots, A\psi_r)$ from $\mathcal{C}_{\Psi^r}(\mathcal{U})$ to H^r is injective (see [11] Lemma 3.1). In [13], it is pointed out that the set of all complete Riesz vectors for $\mathcal{U}$ is in one to one correspondence with the set of all invertible operators in $\mathcal{C}_{\psi}(\mathcal{U})$, where ψ is a fixed complete wandering vector for $\mathcal{U}$. And it is easy to be generalized to the case of complete multi-Riesz vectors. In fact, suppose $\{\eta_1, \eta_2, \cdots, \eta_r\}$ is a complete multi-Riesz vector for $\mathcal{U}$, then there is an invertible operator $T \in B(H)$ such that $U\eta_i = TU\psi_i$ for $i = 1, 2, \cdots, r$. In particular, let $U = I$ we have $\eta_i = T\psi_i$ for $i = 1, 2, \cdots, r$ and $UT\psi_i = TU\psi_i$ for $i = 1, 2, \cdots, r$. So $T \in \mathcal{C}_{\Psi^r}(\mathcal{U})$. Conversely, if there is an invertible operator $T \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $\eta_i = T\psi_i$ for $i = 1, 2, \cdots, r$, then $U\eta_i = UT\psi_i = TU\psi_i$ for all $U \in \mathcal{U}$ and $i = 1, 2, \cdots, r$. So $\{\eta_1, \eta_2, \cdots, \eta_r\}$ is a complete multi-Riesz vector for $\mathcal{U}$.

Lemma 2.7 — ([20]) Suppose that $T : \mathcal{K} \to \mathcal{H}$ is a bounded surjective operator. Then there exists a bounded operator (called the *pseudo-inverse* of T) $T^\dagger : H \to \mathcal{K}$ for which

$$TT^\dagger f = f, \forall f \in H.$$

3. Local Commutant and Multi-frame Vectors for Unitary Systems

In this section, we firstly characterize (complete) multi-frame vectors for a unitary system $\mathcal{U}$ in terms of local commutant of $\mathcal{U}$, which play key roles in the sequel. Then we consider a lifting property of multi-frame vectors in the sense that the similar or equivalent relation between two multi-frame vectors will be lifted to the same relations between their generators.

***Theorem* 3.1** — *Suppose that $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ is a complete wandering r-tuple for a unitary system $\mathcal{U}$. Then* *(i) an r-tuple of vectors* $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ *is a multi-normalized tight frame vector of multiplicity* r *for* $\mathcal{U}$ *if and only if there is a (unique) partial isometry* $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ *such that* $A\Psi^r = \Gamma^r$*, i.e.,* $A\psi_i = \eta_i$ *for* $i = 1, 2, \cdots, r$*.*

(ii) an r-tuple of vectors $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\} \in \mathcal{NF}^r(\mathcal{U})$ *if and only if there is a (unique) co-isometry* $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ *such that* $A\Psi^r = \Gamma^r$*, i.e.,* $A\psi_i = \eta_i$ *for* $i = 1, 2, \cdots, r$*.*

PROOF : The uniqueness follows from the fact that Ψ^r separates $\mathcal{C}_{\Psi^r}(\mathcal{U})$. First, let's show $(i) \Rightarrow (ii)$. If r-tuple of vectors $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\} \in \mathcal{NF}^r(\mathcal{U})$, then Γ^r is a multi-normalized tight frame vector of multiplicity r for $\mathcal{U}$ for $\overline{Span}\{U\eta_i : U \in \mathcal{U}, i = 1, 2, \cdots, r\} = H$. So by (i), there is a (unique) partial isometry $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $A\Psi^r = \Gamma^r$, i.e., $A\psi_i = \eta_i$ for $i = 1, 2, \cdots, r$. It is sufficient to show that A is actually an isometry. Suppose that $A^*x = 0$, then

$$
\begin{aligned}
0 = \|A^*x\|^2 = \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle A^*x, U\psi_i\rangle|^2 & \\
= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, AU\psi_i\rangle|^2 & \\
= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, UA\psi_i\rangle|^2 & \\
= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, U\eta_i\rangle|^2 = \|x\|^2. &
\end{aligned}
$$

Hence $x = 0$, so A is a isometry in $\mathcal{C}_{\Psi^r}(\mathcal{U})$. Conversely, if there is a (unique) co-isometry $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $A\Psi^r = \Gamma^r$, i.e., $A\psi_i = \eta_i$ for $i = 1, 2, \cdots, r$, then by (i), Γ^r is a multi-normalized tight frame vector of multiplicity r for $\mathcal{U}$ for $\overline{Span}\{U\eta_i : U \in \mathcal{U}, i = 1, 2, \cdots, r\}$, i.e., for any $x \in \overline{Span}\{U\eta_i : U \in \mathcal{U}, i = 1, 2, \cdots, r\}$, we have $\|x\|^2 = \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, U\eta_i\rangle|^2$. So it is sufficient to show that $\overline{Span}\{U\eta_i : U \in \mathcal{U}, i = 1, 2, \cdots, r\} = H$. If $x \in \overline{Span}\{U\eta_i : U \in \mathcal{U}, i =$

$1, 2, \cdots, r\}^{\perp}$, then

$$\begin{aligned} 0 &= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} |\langle x, U\eta_i \rangle|^2 \\ &= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} |\langle x, UA\psi_i \rangle|^2 \\ &= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} |\langle x, AU\psi_i \rangle|^2 \\ &= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} |\langle A^* x, U\psi_i \rangle|^2 = \|A^* x\|^2. \end{aligned}$$

Since A^* is isometry, so $x = 0$. So $\overline{Span}\{U\eta_i : U \in \mathcal{U}, i = 1, 2, \cdots, r\} = H$ and thus $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\} \in \mathcal{NF}^r(\mathcal{U})$. Now we only need to prove (i).

Suppose that $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a multi-normalized tight frame vector of multiplicity r for $\mathcal{U}$. Define a linear operator T by

$$Tx = \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} \langle x, U\eta_i \rangle U\psi_i$$

for $x \in \overline{Span}\{\mathcal{U}\Gamma^r\}$, and $Tx = 0$ when $x \perp \overline{Span}\{\mathcal{U}\Gamma^r\}$. Since $\mathcal{U}\Gamma^r = \{U\eta_1, U\eta_2, \cdots, U\eta_r : U \in \mathcal{U}\}$ is a normalized tight frame and $\mathcal{U}\Psi^r = \{U\psi_1, U\psi_2, \cdots, U\psi_r : U \in \mathcal{U}\}$ is an orthonormal basis, it implies that for any $x \in \overline{Span}\{\mathcal{U}\Gamma^r\}$

$$\|Tx\|^2 = \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} |\langle x, U\eta_i \rangle|^2 = \|x\|^2.$$

So T is isometric on $\overline{Span}\{\mathcal{U}\Gamma^r\}$. Thus T is a partial isometry with closed range TH. Let P be the orthogonal projection onto TH, and let $A = T^*P$. Then for any $x \in H, U \in \mathcal{U}, i = 1, 2, \cdots, r$, we have

$$\begin{aligned} &\langle x, T^*PU\psi_i \rangle = \langle Tx, PU\psi_i \rangle \\ = \; &\langle Tx, U\psi_i \rangle = \langle x, U\eta_i \rangle, \end{aligned}$$

where the last equality comes from the definition of T. So $AU\psi_i = T^*PU\psi_i = U\eta_i$ for any $U \in \mathcal{U}$ and $i = 1, 2, \cdots, r$. In particular, if we choose $U = I \in \mathcal{U}$

then we obtain that $\eta_i = A\psi_i$ for $i = 1, 2, \cdots, r$ and $AU\psi_i = U\eta_i = UA\psi_i$ for $i = 1, 2, \cdots, r$, i.e., $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$.

Note that T is a partial isometry with initial space $\overline{Span}\{\mathcal{U}\Gamma^r\}$ and final space $PH(= TH)$, so T^* is a partial isometry with initial space PH and final space $\overline{Span}\{\mathcal{U}\Gamma^r\}$. Thus $A = T^*P$ is isometric on PH and takes the value zero on $P^{\perp}H$, which implies that A is a partial isometry.

Conversely, let A be a partial isometry in $\mathcal{C}_{\Psi^r}(\mathcal{U})$ and let $\eta_i = A\psi_i$ for $i = 1, 2, \cdots, r$. Since the final space of A is AH, hence A^* is a partial isometry with initial space AH, which implies that A^* is isometric on AH. Thus for any $x \in AH$, we have

$$\begin{aligned}\|x\|^2 = \|A^*x\|^2 &= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle A^*x, U\psi_i\rangle|^2 \\ &= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, AU\psi_i\rangle|^2 = \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, UA\psi_i\rangle|^2 \\ &= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, U\eta_i\rangle|^2.\end{aligned}$$

Thus $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a multi-normalized tight frame vector of multiplicity r for $\mathcal{U}$ on AH. But $AH = \overline{Span}\{A\mathcal{U}\Psi^r\} = \overline{Span}\{\mathcal{U}A\Psi^r\} = \overline{Span}\{\mathcal{U}\Gamma^r\}$. This implies that $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a multi-normalized tight frame vector of multiplicity r for $\mathcal{U}$ on $\overline{Span}\{\mathcal{U}\Gamma^r\}$. □

More generally, we have the following result.

***Theorem* 3.2** — *Suppose that $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ is a complete wandering r-tuple for a unitary system $\mathcal{U}$. Then an r-tuple of vectors $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a multi- frame vector of multiplicity r for $\mathcal{U}$ with frame bounds a and b if and only if there is a (unique) operator $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $A\Psi^r = \Gamma^r$, i.e., $A\psi_i = \eta_i$ for $i = 1, 2, \cdots, r$ and $aP \leq AA^* \leq bP$ for some orthogonal projection P.*

PROOF : The uniqueness follows from the fact that Ψ^r separates $\mathcal{C}_{\Psi^r}(\mathcal{U})$. Suppose that $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a multi-frame vector of multiplicity r for $\mathcal{U}$.

Define a linear operator B by

$$Bx = \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} \langle x, U\eta_i \rangle U\psi_i$$

for $x \in \overline{Span}\{\mathcal{U}\Gamma^r\}$. Since $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a multi-frame vector with frame bounds a and b on $\overline{Span}\{\mathcal{U}\Gamma^r\}$ and $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ is an orthonormal basis for H, we have

$$a\|x\|^2 \leq \|Bx\|^2 = \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} |\langle x, U\eta_i \rangle|^2 \leq b\|x\|^2$$

for any $x \in \overline{Span}\{\mathcal{U}\Gamma^r\}$. So B is a bounded operator which is bounded below and with closed range BH. So $B : H \rightarrow BH$ is an invertible operator. Let Q be the orthogonal projection from H onto BH. Then for all $U, V \in \mathcal{U}$ and $i, j = 1, 2, \cdots, r$, we have

$$\langle B^*QU\psi_j, V\eta_i \rangle = \langle QU\psi_j, BV\eta_i \rangle = \langle U\psi_j, BV\eta_i \rangle = \langle U\eta_j, V\eta_i \rangle.$$

So $B^*QU\psi_j = U\eta_j$ for any $U \in \mathcal{U}$ and $j = 1, 2, \cdots, r$. Let $A = B^*Q$. Then $AU\psi_j = U\eta_j$ for any $U \in \mathcal{U}$ and $j = 1, 2, \cdots, r$. In particular, we have $A\psi_j = \eta_j$ for $j = 1, 2, \cdots, r$. So $AU\psi_j = UA\psi_j$, for any $U \in \mathcal{U}$ and $j = 1, 2, \cdots, r$, i.e., $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$. Since

$$\langle AA^*x, x \rangle = \langle A^*x, A^*x \rangle = \|A^*x\|^2 = \|QBx\|^2 = \|Bx\|^2$$

for any $x \in H$. But for any $x \in \overline{Span}\{\mathcal{U}\Gamma^r\}$ we have

$$a\|x\|^2 \leq \|Bx\|^2 \leq b\|x\|^2$$

So if we let P be the orthogonal projection from H onto $\overline{Span}\{\mathcal{U}\Gamma^r\}$, then we have

$$aP \leq AA^* \leq bP.$$

Conversely, let $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $A\Psi^r = \Gamma^r$, i.e., $A\psi_i = \eta_i$ for $i = 1, 2, \cdots, r$ and $aP \leq AA^* \leq bP$ for some orthogonal projection P. Then

$$\begin{aligned}\|A^*x\|^2 &= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle A^*x, U\psi_i\rangle|^2 \\ &= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, AU\psi_i\rangle|^2 = \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, UA\psi_i\rangle|^2 \\ &= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, U\eta_i\rangle|^2.\end{aligned}$$

Since $aP \leq AA^* \leq bP$ for some orthogonal projection P, so for any $x \in PH$, we have

$$a\|x\|^2 \leq \|A^*x\|^2 \leq b\|x\|^2.$$

Thus for any $x \in PH$, we have

$$a\|x\|^2 \leq \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} |\langle x, U\eta_i\rangle|^2 \leq b\|x\|^2.$$

Hence $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a multi- frame vector of multiplicity r for $\mathcal{U}$ with frame bounds a and b on PH.

From the above arguments, it is easy to get the following corollary.

Corollary 3.3 — Suppose that $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ is a complete wandering r-tuple for a unitary system $\mathcal{U}$. Then an r-tuple of vectors $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a complete multi- frame vector of multiplicity r for $\mathcal{U}$ with frame bounds a and b if and only if there is a (unique) operator $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $A\Psi^r = \Gamma^r$, i.e., $A\psi_i = \eta_i$ for $i = 1, 2, \cdots, r$ and $aI \leq AA^* \leq bI$.

The following is an useful result about complete multi-frames.

***Theorem* 3.4** — *Suppose that $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ is a complete wandering r-tuple for a unitary system $\mathcal{U}$. Then an r-tuple of vectors $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a complete multi- frame vector of multiplicity r for $\mathcal{U}$ if and only if there is a* *(unique) surjective operator* $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ *such that* $A\Psi^r = \Gamma^r$*, i.e.,* $A\psi_i = \eta_i$ *for* $i = 1, 2, \cdots, r$.

PROOF : Suppose that $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a complete multi- frame vector of multiplicity r for $\mathcal{U}$. Then by Corollary 3.3, there is a (unique) operator $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $A\Psi^r = \Gamma^r$, i.e., $A\psi_i = \eta_i$ for $i = 1, 2, \cdots, r$ and $aI \leq AA^* \leq bI$, which implies that AA^* is invertible. So there exists an operator $T \in B(H)$ such that $A(A^*T) = I$, which means that A is right invertible. So A is surjective.

Conversely, suppose there is a (unique) surjective operator $A \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $A\Psi^r = \Gamma^r$, i.e., $A\psi_i = \eta_i$ for $i = 1, 2, \cdots, r$. Since A is surjective, so by Lemma 2.7, there exists a bounded operator $A^\dagger$ such that $AA^\dagger = I$. So $(A^\dagger)^*A^* = I$. So for any $x \in H$, we have $\|x\| = \|(A^\dagger)^*A^*x\| \leq \|(A^\dagger)^*\|\|A^*x\|$. Thus $\|A^*x\|^2 \geq \frac{1}{\|(A^\dagger)^*\|^2}\|x\|^2$. Since for any $x \in H$ we have

$$\sum_{i=1}^{r} \sum_{U \in \mathcal{U}} |\langle x, U\eta_i \rangle|^2 = \|A^*x\|^2.$$

And we also have $\frac{1}{\|(A^\dagger)^*\|^2}\|x\|^2 \leq \|A^*x\|^2 \leq \|A^*\|^2\|x\|^2$, so $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ is a complete multi- frame vector of multiplicity r for $\mathcal{U}$.

Suppose that $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}$ and $\Omega^r = \{\omega_1, \omega_2, \cdots, \omega_r\}$ are complete multi-frame vectors for unitary system $\mathcal{U}$ with multiplicity r. If Γ^r and Ω^r are similar (or unitarily equivalent), then there is an invertible operator (or unitary operator) W satisfying $WU\eta_i = U\omega_i$ for any $U \in \mathcal{U}$ and $i = 1, 2, \cdots, r$. In particular, we have $W\eta_i = \omega_i$ for any $i = 1, 2, \cdots, r$ and $W \in \mathcal{C}_{\Gamma^r}(\mathcal{U})$. So we have the following relation lifting property, which lifts the similar or equivalent relations between two complete multi-frame vectors to the similar or equivalent relations between their generators. Based on these observations, we have the following fact.

***Theorem* 3.5** — *Suppose that* $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\}, \Omega^r = \{\omega_1, \omega_2, \cdots, \omega_r\} \in \mathcal{F}^r(\mathcal{U})$. *Then* Γ^r *and* Ω^r *are similar (or unitarily equivalent) if and only if there is an invertible operator (or unitary operator)* $T \in \mathcal{C}_{\Gamma^r}(\mathcal{U})$ *such that* $T\Gamma^r = \Omega^r$*, i.e.,* $T\eta_i = \omega_i$ *for* $i = 1, 2, \cdots, r$.

4. Linear Combinations of Multi-frame Vectors for Unitary Systems

The linear combinations of multi-frame vectors for a unitary system $\mathcal{U}$ may not be a multi-frame vector for $\mathcal{U}$. To find the conditions such that the linear combinations of multi-frame vectors for a unitary system $\mathcal{U}$ are still a multi-frame vector for $\mathcal{U}$ is an interesting and useful problem, since it helps us to obtain new multi-frame vectors from the known ones. In this section we discuss such problem and some results are established by using the characterizations in section 3.

***Theorem* 4.1** — *Suppose that* $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ *is a complete wandering r-tuple for a unitary system* $\mathcal{U}$. *Suppose* $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\} \in \mathcal{NF}^r(\mathcal{U})$, $\Omega^r = \{\omega_1, \omega_2, \cdots, \omega_r\} \in \mathcal{NF}^r(\mathcal{U})$ *and* A_1, A_2 *are co-isometries in* $\mathcal{C}_{\Psi^r}(\mathcal{U})$ *such that* $\Gamma^r = A_1\Psi^r, \Omega^r = A_2\Psi^r$ *and* $A_1A_2^* = 0$. *Let* $B_1, B_2 \in \mathcal{U}'$, *then* $B_1\Gamma^r + B_2\Omega^r \in \mathcal{NF}^r(\mathcal{U})$ *if and only if* $B_1B_1^* + B_2B_2^* = I$.

Proof : Since $\Gamma^r = A_1\Psi^r, \Omega^r = A_2\Psi^r$, so $B_1\Gamma^r + B_2\Omega^r = B_1A_1\Psi^r + B_2A_2\Psi^r = (B_1A_1 + B_2A_2)\Psi^r$. Since $\mathcal{C}_{\Psi^r}(\mathcal{U})$ is a left-module of $\mathcal{U}'$ (see [11] Lemma 3.1(viii)), so $B_1A_1 + B_2A_2 \in \mathcal{C}_{\Psi^r}(\mathcal{U})$. By Theorem 3.1, $B_1\Gamma^r + B_2\Omega^r \in \mathcal{NF}^r(\mathcal{U})$ if and only if $B_1A_1+B_2A_2$ is a co-isometry. Since $(B_1A_1+B_2A_2)(B_1A_1 + B_2A_2)^* = B_1A_1A_1^*B_1^* + B_1A_1A_2^*B_2^* + B_2A_2A_1^*B_1^* + B_2A_2A_2^*B_2^*$ and $A_1A_2^* = 0, A_1A_1^* = A_2A_2^* = I$, so $(B_1A_1 + B_2A_2)(B_1A_1 + B_2A_2)^* = B_1B_1^* + B_2B_2^*$. Hence $B_1A_1 + B_2A_2$ is a co-isometry if and only if $B_1B_1^* + B_2B_2^* = I$. This finishes the proof.

In particular, let $B_1 = mI, B_2 = nI$, where m, n are scalars and I is the identity operator, then we have

Corollary 4.2 — Suppose that $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ is a complete wandering r-tuple for a unitary system $\mathcal{U}$. Suppose $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\} \in \mathcal{NF}^r(\mathcal{U})$, $\Omega^r = \{\omega_1, \omega_2, \cdots, \omega_r\} \in \mathcal{NF}^r(\mathcal{U})$ and A_1, A_2 are co-isometries in $\mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $\Gamma^r = A_1\Psi^r, \Omega^r = A_2\Psi^r$ and $A_1A_2^* = 0$. Then $m\Gamma^r + n\Omega^r \in \mathcal{NF}^r(\mathcal{U})$ if and only if $|m|^2 + |n|^2 = 1$, where m, n are scalars.

The above results are easy to be generalized to the case for any finite number of complete multi-normalized tight frame vectors, which we write down as follows, omitting the proof.

***Theorem* 4.3** — *Suppose that $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ is a complete wandering r-tuple for a unitary system $\mathcal{U}$. Suppose for $i = 1, 2, \cdots, n$, $\Gamma_i^r = \{\eta_1^{(i)}, \eta_2^{(i)}, \cdots, \eta_r^{(i)}\} \in \mathcal{NF}^r(\mathcal{U})$, A_i are co-isometries in $\mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $\Gamma_i^r = A_i\Psi^r$ and $A_iA_j^* = 0$ for any $i \neq j$. Let $\{B_i\}_{i=1}^n \subseteq \mathcal{U}'$, then $\sum_{i=1}^n B_i\Gamma_i^r \in \mathcal{NF}^r(\mathcal{U})$ if and only if $\sum_{i=1}^n B_iB_i^* = I$. In particular, $\sum_{i=1}^n a_i\Gamma_i^r \in \mathcal{NF}^r(\mathcal{U})$ if and only if $\sum_{i=1}^n |a_i|^2 = 1$, where a_i are scalars for $i = 1, 2, \cdots, n$.*

For general complete multi-frame vectors, we have the following sufficient conditions for their linear combinations keep to be complete multi-frame vectors.

***Theorem* 4.4** — *Suppose that $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ is a complete wandering r-tuple for a unitary system $\mathcal{U}$. Suppose $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\} \in \mathcal{F}^r(\mathcal{U})$, $\Omega^r = \{\omega_1, \omega_2, \cdots, \omega_r\} \in \mathcal{F}^r(\mathcal{U})$ and A_1, A_2 are surjective operators in $\mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $\Gamma^r = A_1\Psi^r, \Omega^r = A_2\Psi^r$ and $A_1A_2^* = 0$. If $B_1, B_2 \in \mathcal{U}'$ and B_1 or B_2 is surjective, then $B_1\Gamma^r + B_2\Omega^r \in \mathcal{F}^r(\mathcal{U})$. In particular, $a\Gamma^r + b\Omega^r \in \mathcal{F}^r(\mathcal{U})$ for any scalars a, b such that they are not both zeros.*

PROOF : Since $\Gamma^r = A_1\Psi^r, \Omega^r = A_2\Psi^r$, so $B_1\Gamma^r + B_2\Omega^r = B_1A_1\Psi^r + B_2A_2\Psi^r = (B_1A_1 + B_2A_2)\Psi^r$. By Lemma 3.1(viii) in [11], we know that $\mathcal{C}_{\Psi^r}(\mathcal{U})$ is left module of $\mathcal{U}'$. Since $A_1, A_2 \in \mathcal{C}_{\Psi^r}(\mathcal{U})$ and $B_1, B_2 \in \mathcal{U}'$, so $B_1A_1 + B_2A_2 \in \mathcal{C}_{\Psi^r}(\mathcal{U})$. If we can prove that $B_1A_1 + B_2A_2$ is surjective, then by Corollary 3.4, the result is proved. Without loss of generality, suppose B_1 is surjective. Since $A_2A_1^* = 0$, so $(B_1A_1 + B_2A_2)A_1^* = B_1A_1A_1^*$. From the proof of Theorem 3.2, it is easy to see that $A_1A_1^*$ is invertible, so $B_1A_1A_1^*$ is surjective. Thus for any $z \in H$, there exists $y \in H$ such that $B_1A_1A_1^*(y) = z$. Let $x = A_1^*y$, then $(B_1A_1 + B_2A_2)x = (B_1A_1 + B_2A_2)A_1^*y = B_1A_1A_1^*(y) = z$. Hence $B_1A_1 + B_2A_2$ is surjective. This finishes the proof.

In general, we have the following theorem.

***Theorem* 4.5** — *Suppose that $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\}$ is a complete wandering r-tuple for a unitary system $\mathcal{U}$. Suppose for $i = 1, 2, \cdots, n$, $\Gamma_i^r = \{\eta_1^{(i)}, \eta_2^{(i)}, \cdots, \eta_r^{(i)}\}$*

$\in \mathcal{F}^r(\mathcal{U})$, A_i are surjective operators in $\mathcal{C}_{\Psi^r}(\mathcal{U})$ such that $\Gamma_i^r = A_i\Psi^r$ and $A_iA_j^* = 0$ for any $i \neq j$. If $\{B_i\}_{i=1}^n \subseteq \mathcal{U}'$ and there exist certain i such that B_i is surjective, then $\sum_{i=1}^n B_i\Gamma_i^r \in \mathcal{F}^r(\mathcal{U})$. In particular, $\sum_{i=1}^n a_i\Gamma_i^r \in \mathcal{F}^r(\mathcal{U})$ for any set of scalars $\{a_i\}_{i=1}^n$ such that they are not all zeros.

5. MULTI-FRAME VECTORS FOR UNITARY GROUPS

In the case that the unitary system is actually a group, some interesting results about the multi-frame vectors will be established in this section.

***Theorem* 5.1** — *Let $\mathcal{S}$ be a unital semigroup of unitaries in $B(H)$. If $\mathcal{NF}^r(\mathcal{S}) \neq \emptyset$, then $\mathcal{S}$ is a group.*

PROOF : Let $U \in \mathcal{S}$. Let $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\} \in \mathcal{NF}^r(\mathcal{S})$. Then for any $x \in H$, we have

$$\|U^*x\|^2 = \sum_{i=1}^r \sum_{S\in\mathcal{S}} |\langle U^*x, S\eta_i\rangle|^2 = \sum_{i=1}^r \sum_{S\in\mathcal{S}} |\langle x, US\eta_i\rangle|^2.$$

On the other hand,

$$\|U^*x\|^2 = \|x\|^2 = \sum_{i=1}^r \sum_{S\in\mathcal{S}} |\langle x, S\eta_i\rangle|^2.$$

Since $U\mathcal{S} \subset \mathcal{S}$, so $\sum_{i=1}^r \sum_{S\notin U\mathcal{S}} |\langle x, S\eta_i\rangle|^2 = 0$. If $U^{-1} \notin \mathcal{S}$, then $I \notin U\mathcal{S}$. Thus $\langle x, \eta_i\rangle = 0$ for $i = 1, 2, \cdots, r$. Let $x = \eta_i$, then $\eta_i = 0$ for $i = 1, 2, \cdots, r$. This contradicts to the fact that $\{\eta_1, \eta_2, \cdots, \eta_r\}$ is a multi-frame vector. So $\mathcal{S}$ is a group. □

To characterize the unitary systems such that there exist complete wandering vectors for them is a difficult problem. Next result shows that if the system is actually a group then this problem is related with the problem to characterize such systems so that there exist complete multi-frame vectors for them.

***Theorem* 5.2** — *Suppose that $\mathcal{U}$ is a unitary group on H such that $\mathcal{F}^r(\mathcal{U}) \neq \emptyset$. Then there exists a Hilbert space K and a unitary group $\mathcal{V}$ on K such that $\mathcal{W}(\mathcal{V}) \neq$*

$\emptyset$ *and there is a bounded operator* $T \in B(H, K)$ *which is bounded below and with closed range such that* $\mathcal{U} = T^*\mathcal{V}(T^*)^\dagger$*, where* $(T^*)^\dagger$ *is the pseudo-inverse of* T^**.*

PROOF : Let $K = l^2(\mathcal{U})$, and for each $U \in \mathcal{U}$, let l_U be the left regular representation defined by $l_U(\chi_V) = \chi_{UV}$, $V \in \mathcal{U}$, where χ_V is the characteristic function at the single point set $\{V\}$. Consider the unitary group $\mathcal{V} = \{l_U : U \in \mathcal{U}\}$. Then $\chi_V \in \mathcal{W}(\mathcal{V})$ for all $V \in \mathcal{U}$, hence $\mathcal{W}(\mathcal{V}) \neq \emptyset$. Choose any $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\} \in \mathcal{F}^r(\mathcal{U})$ and define $T : H \to K$ by

$$T(x) = \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} \langle x, U\eta_i \rangle \chi_U.$$

Since χ_I is a complete wandering vector for $\mathcal{V}$, so $\{\chi_U : U \in \mathcal{U}\} = \{l_U(\chi_I) : U \in \mathcal{U}\}$ is a orthonormal basis for K. Hence

$$\|Tx\|^2 = \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} |\langle x, U\eta_i \rangle|^2.$$

Since $\{\eta_1, \eta_2, \cdots, \eta_r\}$ is a complete frame vector for $\mathcal{U}$, so there exist positive constants a, b such that

$$a\|x\|^2 \leq \|Tx\|^2 \leq b\|x\|^2.$$

So $T \in B(H, K)$ and T is bounded below and with closed range. So T^* is surjective. Let $(T^*)^\dagger$ denote the pseudo-inverse of T^*, then $T^*(T^*)^\dagger = I$, where I is the identity operator on H. For any $U, V \in \mathcal{U}$ and $i = 1, 2, \cdots, r$ we have

$$\begin{aligned} l_U T(V\eta_i) &= l_U(\sum_{j=1}^{r} \sum_{S \in \mathcal{U}} \langle V\eta_i, S\eta_j \rangle \chi_S) \\ &= \sum_{j=1}^{r} \sum_{S \in \mathcal{U}} \langle V\eta_i, S\eta_j \rangle \chi_{US} \\ &= \sum_{j=1}^{r} \sum_{S \in \mathcal{U}} \langle V\eta_i, U^*S\eta_j \rangle \chi_S \\ &= \sum_{j=1}^{r} \sum_{S \in \mathcal{U}} \langle UV\eta_i, S\eta_j \rangle \chi_S \\ &= TU(V\eta_i). \end{aligned}$$

Thus $l_U T = TU$ on H since $\{V\eta_i : U \in \mathcal{U}, i = 1, 2, \cdots, r\}$ is complete multi-frame vector for $\mathcal{U}$. Hence $T^* l_U = UT^*$ for any $U \in \mathcal{U}$, which implies that $\mathcal{U} = T^*\mathcal{V}(T^*)^\dagger$.

In [4], the authors establish the dilation property for the complete normalized tight frame vectors and complete multi-normalized tight frame vectors under the assumption that the unitary system is a unitary group. The following shows that we also have the dilation property for the general complete multi-frame vectors (then of course for the complete frame vectors).

***Theorem* 5.3** — *Suppose $\mathcal{U}$ is a unitary group with $\mathcal{W}^r(\mathcal{U}) \neq \emptyset$, and $\Gamma^r = \{\eta_1, \eta_2, \cdots, \eta_r\} \in \mathcal{F}^r(\mathcal{U})$. Then there is a multi-normalized tight frame vector $\Omega^r = \{\omega_1, \omega_2, \cdots, \omega_r\}$ for $\mathcal{U}$ such that $\{\Gamma^r \oplus \Omega^r\}$ is a complete multi-Riesz vector for $H \oplus [\mathcal{U}\Omega^r]$.*

PROOF : Fix $\Psi^r = \{\psi_1, \psi_2, \cdots, \psi_r\} \in \mathcal{W}^r(\mathcal{U})$. Define the map $\theta : H \to H$ by

$$\theta(x) = \sum_{i=1}^{r} \sum_{U\in\mathcal{U}} \langle x, U\eta_i\rangle U\psi_i.$$

Then θ is a bounded operator with closed range $\theta(H)$. Let P be the orthogonal projection onto $\theta(H)$, then $\theta^* PU\psi_i = U\eta_i$ for all $U \in \mathcal{U}$ and $i = 1, 2, \cdots, r$. Since for any $V \in \mathcal{U}$, we have

$$\begin{aligned} V\theta(x) &= \sum_{i=1}^{r} \sum_{U\in\mathcal{U}} \langle x, U\eta_i\rangle VU\psi_i \\ &= \sum_{i=1}^{r} \sum_{U\in\mathcal{U}} \langle Vx, VU\eta_i\rangle VU\psi_i. \end{aligned}$$

Since $\mathcal{U}$ is a group, so we have

$$V\theta(x) = \sum_{i=1}^{r} \sum_{S\in\mathcal{U}} \langle Vx, S\eta_i\rangle S\psi_i = \theta(Vx) \in \theta(H).$$

So $\theta(H)$ is invariant under any $V \in \mathcal{U}$. So $\theta(H)$ is a reduced subspace of any $V \in \mathcal{U}$. Thus $P \in \mathcal{U}'$. So $U\eta_i = \theta^* PU\psi_i = \theta^* UP\psi_i$ for all $U \in \mathcal{U}$ and

$i = 1, 2, \cdots, r$. By Theorem 3.1, we know that $P\Psi^r$ is a multi-normalized tight frame vector for $\mathcal{U}$. Similarly, since $P^\perp \in \mathcal{U}'$, so $P^\perp\Psi^r$ is a multi-normalized tight frame vector for $\mathcal{U}$. Let $\Omega^r = P^\perp\Psi^r$, then $\{UP\Psi^r \oplus U\Omega^r : U \in \mathcal{U}\}$= $\{PU\Psi^r \oplus P^\perp U\Psi^r : U \in \mathcal{U}\}$ is an orthonormal set. Since

$$\{U\Gamma^r \oplus U\Omega^r : U \in \mathcal{U}\} = \{(\theta^* \oplus I)(PU\Psi^r \oplus P^\perp U\Psi^r) : U \in \mathcal{U}\}.$$

and θ^* is an invertible operator from $\theta(H)$ to H, so $\{U\Gamma^r \oplus U\Omega^r : U \in \mathcal{U}\}$ is a complete multi-Riesz basis with multiplicity r for $\mathcal{U}$ on $H \oplus [\mathcal{U}\Omega^r]$. □

6. EXAMPLES

Example 6.1 : Let $\{e_n\}_{n=-\infty}^{+\infty}$ be an orthonormal basis for Hilbert space H, and let $S \in B(H)$ be the bilateral shift of multiplicity one, i.e., $Se_n = e_{n+1}$ for any n. Let $\mathcal{U} = \{S^{3n} : n \in Z\}$ be the group generated by S^3. Then $\mathcal{U}$ is a unitary system on H, and it is easy to see that $\Psi^3 = (e_0, e_1, e_3)$ is a complete wandering tuple with multiplicity 3 for H, i.e., $\Psi^3 \in \mathcal{W}^3(\mathcal{U})$. Since $\mathcal{U}$ is a unitary group, the commutant and local commutant of $\mathcal{U}$ are same by Lemma 3.1 in [11]. Then by Theorem 3.1 and Theorem 3.4, we know that

$$\begin{aligned}\mathcal{NF}^3(\mathcal{U}) &= \{V\Psi^3 : V \in Cois(\{S^3\}')\} = \{(Ve_0, Ve_1, Ve_3) : \\ &\quad V \in Cois(\{S^3\}')\}, \\ \mathcal{F}^3(\mathcal{U}) &= \{L\Psi^3 : L \in Sur(\{S^3\}')\} = \{(Le_0, Le_1, Le_3) : \\ &\quad V \in Cois(\{S^3\}')\},\end{aligned}$$

where $Cois(A)$ denotes the set of all co-isometries in operator set A, and $Sur(A)$ denotes the set of all surjective operators in operator set A.

Example 6.2 : Let $H = L^2([0,1]), \mathcal{U} = \{M_{e^{2\pi int}} : n \in Z\}$, where $M_{e^{2\pi int}}$ denotes the multiplication operator on $L^2([0,1])$ with symbol $e^{2\pi int}$, then $\mathcal{U}$ is a unitary system on H. Let $\Gamma^3 = \{\eta_1(t), \eta_2(t), \eta_3(t)\}$, where $\eta_1(t) = \chi_[0, \frac{1}{3}](t)$, $\eta_2(t) = \chi_{(\frac{1}{3}, \frac{2}{3}]}(t)$, $\eta_3(t) = \chi_{(\frac{2}{3}, 1]}(t)$. Then it is easy to see that Γ^3 is a complete normalized multi-frame vector with multiplicity 3, i.e., $\Gamma^3 \in \mathcal{NF}^3(\mathcal{U})$. Similarly, $\Psi^3 = \{\psi_1(t), \psi_2(t), \psi_3(t)\} \in \mathcal{NF}^3(\mathcal{U})$, where $\psi_1(t) = \chi_{(\frac{1}{3}, \frac{2}{3}]}(t)$,

$\psi_2(t) = \chi_{(\frac{2}{3},1]}(t)$, $\psi_3(t) = \chi_{[0,\frac{1}{3}]}(t)$. And it is easy to check that their corresponding co-isometries A_1 and A_2 satisfy $A_1 A_2^* = 0$, so by Corollary **??** we know that for any $\alpha \in R$, $\sin\alpha \cdot \Gamma^3 + \cos\alpha \cdot \Psi^3 \in \mathcal{NF}^3(\mathcal{U})$, i.e.,

$$\{\sin\alpha \cdot \chi_{[0,\frac{1}{3}]}(t) + \cos\alpha \cdot \chi_{(\frac{1}{3},\frac{2}{3}]}(t), \sin\alpha \cdot \chi_{(\frac{1}{3},\frac{2}{3}]}(t) + \cos\alpha \cdot \chi_{(\frac{2}{3},1]}(t), \\ \sin\alpha \cdot \chi_{(\frac{2}{3},1]}(t) + \cos\alpha \cdot \chi_{[0,\frac{1}{3}]}(t)\} \in \mathcal{NF}^3(\mathcal{U}).$$

REFERENCES

1. D. Gabor, Theory of communications, *Jour. Inst. Elec. Eng.*, **93** (1946), 429-457.
2. R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Transactions of AMS*, **72**, 341-366.
3. I. Daubechies, A. Grossmann and Y. Meyer, Painless nonorthognal expansions, *J. Math. Phys.*, **27** (1986), 1271-1283.
4. D. Han and D. R. Larson, Frames, basis and group representations, *Mem. Amer. Math. Soc.*, **147**, (2000).
5. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.
6. I. Daubchies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.
7. C. Heil and D. Walnut, Continuous and discrete wavelet transforms, *SIAM Rev.*, **31** (1989), 628-666.
8. H. G. Feichtinger and T. Strohmer, *Gabor Analysis and Algorithms: Theory and Applications*, Birkhäuser, Boston, MA, 1998.
9. M. Frazier and B. Jawerth, Decompositions of Besov spaces, *Iniana Univ. Math. J.*, **34** (1985), 777-799.
10. K. Grochenig, Describing functions: Atomic decompositions versus frames, *Monatshefte fur Math.*, **112** (1991), 1-41.
11. X. Guo, Wandering r-tuples for unitary systems, *J. Math. Anal. Appl.*, **374** (2011), 722-728.

12. X. Dai and D. Larson, Wandering vectors for unitary systems and orthogonal wavelets, *Mem. Amer. Math. Soc.*, **134** (1998).

13. D. Larson, Von Neumann algebras and wavelets, Proceedings, NATO Adv. Studies Institute on Operator Algebras and Applications, **495** (1997), 267-312.

14. D. Han, tight frame approximation for multi-frames and supper-frames, *Journal of Approximation Theory*, **129** (2004), 78-93.

15. Liu Q. S., Jin H. L., Tang X. O., *et al.*, *Boosting multi-Gabor subspaces for face recognition*, Navayanan P. J., *et al.*, eds., Computer Vision-ACCV 2006, 539-548.

16. Zibulski M. and Zeevi Y., Discrete multiwindow Gabor-type transforms, *IEE Trans. Signal Processing*, **45** (1997), 1428-1442.

17. Hernandez E., Labate D. and Weiss G. A., Unified characrization of reproducing systems generated by a finite family, II, *J. Geom Anal.*, **12** (2002), 615-662.

18. Dorfler M., Feichtinger H., Quilted Gabor families I: Reduced multi-Gabor frames, *Appl. Comput. Harmon. Anal.*, **356** (2004), 2001-2023.

19. Lu DaYong and FAN QiBin, Gabor frames generated by multiple generators, (in Chinese), *Sci. Sin. Math.*, **40**(7) 2010, 693-708.

20. Ole Christensen and T. K. Jensen, *An introduction to the theory of bases, frames, and wavelets*, Technical University of Denmark, 1999.