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NONPURE PIECEWISE-KOSZUL ALGEBRAS[1]

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In this paper, the notions of *nonpure piecewise-Koszul algebra* and *nonpure piecewise-Koszul module* are introduced, which are the "nonpure" version of piecewise-Koszul algebras and modules first introduced in [19]. Some criteria for a standard graded algebra to be nonpure piecewise-Koszul are given. We also discuss some basic properties of nonpure piecewise-Koszul modules. Further more, we give a sufficient condition for the questions raised in [20] to be true in terms of nonpure piecewise-Koszul modules.

Key words : Koszul algebras and modules, (nonpure) piecewise-Koszul algebras and modules.

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1. Introduction

It is well known that the noncommutative graded algebras play an important role in algebra, topology, and mathematical physics. Probably the most interesting class of such algebras is the class of Koszul algebras, which were first introduced by Priddy in 1970 (see [23]) and studied in detail in [22]. Later, motivated by the cubic Artin-Schelter regular algebras (see [1]), Berger first introduced the notion of *nonquadratic Koszul algebra* (see [4]) in 2001, which is a natural generalization of Koszul algebras and also admits a lot of important applications in other branches of mathematics, such as algebraic topology, algebraic geometry, quantum group, and Lie algebra and so on. Recently, many people have shown their interests on such algebras and many of them prefer the name "d-Koszul algebra" to "nonquadratic Koszul algebra" (see [5], [8], [10], [11], [13], [25] and [26], etc.), where $d \geq 2$ is a fixed integer. In 2007, the notion of *piecewise-Koszul algebra* was introduced, which unifies the notions of Koszul and d-Koszul algebras on one hand and has essential difference between Koszul and d-Koszul algebras on the other hand, we refer to [17], [19], [20] and [21] for the further details. Moreover, piecewise-Koszul algebras provide a negative answer to a question raised by Green and Marcos (see [9] in 2005 and we refer to [15], [18] and [21] for the further details.

It should be noted that Koszul modules, d-Koszul modules and piecewise-Koszul modules are "pure", i.e., each term in their minimal graded projective resolutions is generated in a single degree. A naive but interesting question is can we break through the "pure" restrict and generalize these notions to the "nonpure" case? Recently, some people have investigated the nonpure version of d-Koszul objects (see [5], [7] and [10], etc.) Motivated by these, the main purpose of the present paper is to generalize the notion of piecewise-Koszul objects to the nonpure case and the so-called *nonpure piecewise-Koszul algebra* and *nonpure piecewise-Koszul module* will be introduced and studied. More precisely, the whole paper is arranged as follows:

In Section 2, first we recall the definitions of piecewise-Koszul objects and give the definitions of nonpure piecewise-Koszul objects, then we will give some trivial examples of nonpure piecewise-Koszul objects.

In Section 3, we provide some criteria for a standard graded algebra to be nonpure piecewise-Koszul and Theorems 3.1 and 3.3 are the main results.

In Section 4, we give some basic properties of nonpure piecewise-Koszul modules. More precisely, we prove that the category of nonpure piecewise-Koszul modules is closed under extensions and cokernels of monomorphisms, all the syzygies of a piecewise-Koszul module are nonpure piecewise-Koszul under a proper shift, $J^iM[-i]$ is nonpure piecewise-Koszul for each $i \geq 0$ and $\mathcal{E}^{[0]}(M)$ is a finitely 0-generated graded $\mathrm{E}^{[0]}(A)$-module, where $\mathrm{E}^{[0]}(A) := \bigoplus_{i\geq 0} \mathrm{Ext}_A^{pi}(A_0, A_0)$, $\mathcal{E}^{[0]}(M) := \bigoplus_{i\geq 0} \mathrm{Ext}_A^{pi}(M, A_0)$, A is a piecewise-Koszul algebra with the graded Jacobson radical J and M is a nonpure piecewise-Koszul module.

In Section 5, as an application of nonpure piecewise-Koszul modules, we provide a sufficient condition for the questions raised in [20] to be true.

2. Notations, Definitions and Examples

Throughout, $\mathbb{k}$ denotes a fixed field, $\mathbb{N} = \{0, 1, 2, \cdots\}$ denotes the set of natural numbers and $\mathbb{Z}$ the set of integers. A positively graded $\mathbb{k}$-algebra $A = \bigoplus_{i\geq 0} A_i$ is called *standard* if the following three conditions are satisfied:

(a) $A_0 = \mathbb{k} \times \cdots \times \mathbb{k}$, a finite product of $\mathbb{k}$;

(b) $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$;

(c) $\dim_{\mathbb{k}} A_i < \infty$ for all $i \geq 0$.

Under the above assumptions, the graded Jacobson radical of A, denoted by J, is $\bigoplus_{i\geq 1} A_i$.

Let $Gr(A)$ denote the category of graded A-modules, and $gr(A)$ its full subcategory of finitely generated graded A-modules. The morphisms in these categories, denoted by $\mathrm{Hom}_{Gr(A)}(M, N)$ and $\mathrm{Hom}_{gr(A)}(M, N)$ respectively, are graded A-module maps of degree zero. We denote $Gr_s(A)$ and $gr_s(A)$ the full subcategories of $Gr(A)$ and $gr(A)$ whose objects are generated in a single degree s, respectively.

An object in $Gr_s(A)$ or $gr_s(A)$ is called a graded *pure* A-module and otherwise, we call it *nonpure*.

Let $M, N \in gr(A)$. Endowed with the Yoneda product, $\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(M,N)$ is a graded left $\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(N,N)$-module. In particular, $\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(A_0,A_0)$ is an associative graded algebra and $\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(M,A_0)$ is a graded left $\bigoplus_{i\geq 0}\mathrm{Ext}^i_A$ (A_0,A_0)-module. For simplicity, we write

$$E(A) = \bigoplus_{i\geq 0}\mathrm{Ext}^i_A(A_0,A_0) \text{ and } \mathcal{E}(M) = \bigoplus_{i\geq 0}\mathrm{Ext}^i_A(M,A_0),$$

and usually call $E(A)$ the *Yoneda algebra* of A and $\mathcal{E}(M)$ the *Ext-module* of M. Note that both $E(A)$ and $\mathcal{E}(M)$ are bigraded. More precisely, for any $i \in \mathbb{N}$, we have

$$\mathrm{Ext}^i_A(A_0,A_0) = \bigoplus_{j\in\mathbb{Z}}\mathrm{Ext}^i_{gr(A)}(A_0,A_0[j])$$

and

$$\mathrm{Ext}^i_A(M,A_0) = \bigoplus_{j\in\mathbb{Z}}\mathrm{Ext}^i_{gr(A)}(M,A_0[j]).$$

From [12], we know that standard graded algebras can be realized by finite quivers:

Lemma 2.1 — Let A be a standard graded algebra. Then there exists a finite quiver $\Gamma = (\Gamma_0, \Gamma_1)$ and a graded ideal I in $\Bbbk\Gamma$ with $I \subset \sum_{n\geq 2}(\Bbbk\Gamma)_n$ such that $A \cong \Bbbk\Gamma/I$ as graded algebras, where Γ_0 denotes the set of vertices of the quiver Γ and Γ_1 denotes the set of arrows of the quiver Γ.

Definition 2.2 ([19]) — Let $A = \bigoplus_{i\geq 0} A_i$ be a standard graded $\Bbbk$-algebra and $M = \bigoplus_{i\geq 0} M_i$ a finitely generated graded A-module. We call M a *piecewise-Koszul module* provided that M admits a minimal graded projective resolution

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

such that each Q_n is generated in degree $\delta_p^d(n)$ for all $n \in \mathbb{N}$, where d, p are fixed integers with $d \geq p \geq 2$, $k \in \mathbb{N}$ and the set function δ_p^d is defined as:

$$\delta_p^d(n) = \begin{cases} kd, & n = pk, \\ kd+1, & n = pk+1, \\ \cdots & \cdots \\ kd+p-2, & n = pk+p-2, \\ kd+p-1, & n = pk+p-1. \end{cases}$$

In particular, the standard graded algebra A will be called a *piecewise-Koszul algebra* if the trivial A-module A_0 is a piecewise-Koszul module.

The parameter p shows certain periodicity in the resolution above, and the parameter d is related to a gap between two segments. Clearly, Koszul algebras and d-Koszul algebras are piecewise-Koszul algebras, which take place of $d = p$ and $p = 2$ respectively. For the other examples of piecewise-Koszul objects, we refer to [15], [19] and [20].

Definition 2.3 — Let A be a standard graded algebra and $M = \bigoplus_{i \geq 0} M_i$ a finitely generated graded A-module. Let

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

be a minimal graded projective resolution of M. Then M is called a *nonpure piecewise-Koszul module* if for all $n \geq 0$, Q_n is generated in degrees in $\Delta_p^d(n)$, where d and p are fixed integers with $d \geq p \geq 2$, $k \in \mathbb{N}$ and the set function Δ_p^d is defined as:

$$\Delta_p^d(n) = \begin{cases} \{kd\}, & n = pk, \\ \{kd+1,\ kd+2,\ \cdots,\ kd+d-p+1\}, & n = pk+1, \\ \cdots & \cdots \\ \{kd+p-2,\ kd+p-1,\ \cdots,\ kd+d-2\}, & n = pk+p-2, \\ \{kd+p-1,\ kd+p,\ \cdots,\ kd+d-1\}, & n = pk+p-1. \end{cases}$$

In particular, the standard graded algebra A will be called a *nonpure piecewise-Koszul algebra* if the trivial A-module A_0 is a nonpure piecewise-Koszul module.

In fact, nonpure piecewise-Koszul objects exist naturally:

Example 2.4 : The following list some trivial examples of nonpure piecewise-Koszul objects:

(1) Koszul algebras/modules (see [22] and [23]) are special nonpure piecewise-Koszul algebras/modules.

(2) d-Koszul algebras/modules (see [4], [8], [11], [25] and [26], etc.) are special nonpure piecewise-Koszul algebras/modules.

(3) Piecewise-Koszul algebras/modules (see [19]) are special nonpure piecewise-Koszul algebras/modules.

Example 2.5 : Let Γ be the following quiver:

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 3 \xrightarrow{\epsilon} 4.$$

Let I be the graded homogeneous ideal generated by α^3 and $\alpha\beta\gamma$. Then the quotient algebra $A := \frac{\Bbbk\Gamma}{I}$ is a nonpure piecewise-Koszul algebra. In fact, it suffices to show that the trivial A-module $A_0 = \Bbbk^{\oplus 4}$ has a minimal graded projective resolution with the ith term is generated in degrees in $\Delta_p^d(i)$ for each $i \geq 0$. Now consider the minimal graded projective resolution of the simple module $S_1 = \Bbbk$ related to the vertex 1, under a routine computation, we obtain the following minimal graded projective resolution $\mathcal{P}_{\infty *} : \cdots \to (Ae_1 \oplus Ae_3)[6] \to Ae_1[4] \oplus Ae_3[5] \to (Ae_1 \oplus Ae_3)[3] \to (Ae_1 \oplus Ae_2)[1] \to Ae_1 \to S_1 \to 0$. Note that $\ker((Ae_1 \oplus Ae_3)[6] \to Ae_1[4] \oplus Ae_3[5]) = \ker((Ae_1 \oplus Ae_3)[3] \to (Ae_1 \oplus Ae_2)[1])[3] = (A\alpha \oplus A\beta\gamma)[6]$, thus we get a clear periodic minimal graded projective resolution of S_1. Similarly, we can get the minimal graded projective resolutions of the simple A-modules S_2, S_3 and S_4, which implies a minimal graded projective resolution of the trivial A-module $\Bbbk^{\oplus 4}$:

$$\mathcal{P}_* : \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \Bbbk^{\oplus 4} \longrightarrow 0$$

such that each P_n is generated in degrees in $\Delta_2^3(n)$ for all $n \geq 0$. Also note that $\mathcal{P}_*$ is not pure since $(\mathcal{P}_\infty)_*$ is not pure, thus A must not be a piecewise-Koszul (of course not d-Koszul) algebra and S_1 must not be a piecewise-Koszul (of course not d-Koszul) module. Therefore, $A := \frac{\Bbbk\Gamma}{I}$ is a nontrivial nonpure piecewise-Koszul algebra and S_1 is a nontrivial nonpure piecewise-Koszul module.

3. Nonpure Piecewise-Koszul Algebras

In this section, we will give some criteria for a standard graded algebra to be nonpure piecewise-Koszul.

Motivated by some previous work on Koszul and d-Koszul objects of Green, Martínez-Villa and Marcos (see [9], [11] and [12], etc.), we get the following result:

***Theorem* 3.1** — *Let A be a standard graded algebra and $M = \bigoplus_{i\geq 0} M_i \in gr(A)$. Then*

(1) A is a nonpure piecewise-Koszul algebra if and only if for all $i \geq 0$, $\mathrm{Ext}^i_A(A_0, A_0)_{-j} = \mathrm{Ext}^i_{Gr(A)}(A_0, A_0[j]) = 0$ unless $j \in \Delta^d_p(i)$;

(2) M is a nonpure piecewise-Koszul module if and only if for all $i \geq 0$, $\mathrm{Ext}^i_A(M, A_0)_{-j} = \mathrm{Ext}^i_{Gr(A)}(M, A_0[j]) = 0$ unless $j \in \Delta^d_p(i)$.

Proof We only need to prove (2) since (1) is a special case of (2).

($\Rightarrow$) By the hypothesis, M admits a minimal graded projective resolution

$$\cdots \longrightarrow Q_i \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

such that each Q_i is generated in degrees in $\Delta^d_p(i)$ for all $i \geq 0$. Note that the resolution is minimal and A_0 is semisimple, thus for all $i \geq 0$, we have

$$\mathrm{Ext}^i_A(M, A_0)_{-j} = \mathrm{Ext}^i_{Gr(A)}(M, A_0[j]) = \mathrm{Hom}_{Gr(A)}(Q_i, A_0[j]).$$

Note also that each Q_i is generated in degrees in $\Delta^d_p(i)$ and $A_0[j]$ is concentrated in degree j, which implies that $\mathrm{Hom}_{Gr(A)}(Q_i, A_0[j]) = 0$ unless $j \in \Delta^d_p(i)$.

($\Leftarrow$) It suffices to prove that M admits a minimal graded projective resolution

$$\cdots \longrightarrow Q_i \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

with each Q_i being generated in degrees in $\Delta_p^d(i)$ for all $i \geq 0$. We do it by induction.

By the hypothesis, $\mathrm{Ext}_A^0(M, A_0)_{-j} = \mathrm{Hom}_{Gr(A)}(M, A_0[j]) = 0$ unless $j \in \Delta_p^d(0) = \{0\}$, which implies that M is generated in degree 0 as a graded A-module. Note that A is a standard graded algebra and M a bounded below graded module, it is easy to see that there exists a graded projective module Q_0 generated in degree 0 such that $Q_0 \to M \to 0$ is a graded projective cover. Let $K_1 := \ker(Q_0 \to M)$, it is trivial that K_1 is also a bounded below graded module. Since $\mathrm{Ext}_A^1(M, A_0)_{-j} = \mathrm{Hom}_{Gr(A)}(K_1, A_0[j]) = 0$ unless $j \in \Delta_p^d(1)$, we have K_1 is generated in degrees in $\Delta_p^d(1)$. Similarly, there exists a graded projective module Q_1 generated in degrees in $\Delta_p^d(1)$ such that $Q_1 \to K_1 \to 0$ is a graded projective cover. Now repeating the above arguments again and again, we can obtain the desired resolution $\mathcal{Q}_* \to M \to 0$, which completes the proof.

Lemma 3.2 ([27]) — Let A be a standard graded algebra and $A^e := A \otimes_{\Bbbk} A^{op}$ its enveloping algebra. Let $\mathbf{r}$ be the graded Jacobson radical of A^e and $f : P \to Q$ be a homomorphism of finitely generated A^e-projective modules. Then $Imf \subseteq \mathbf{rQ}$ if and only if for each simple A-module S, we have $Im(f \otimes_A 1_S) \subseteq J(Q \otimes_A S)$.

***Theorem* 3.3** — *Let A be a standard graded algebra. Using the notations of Lemma 3.2. Then*

(1) A is a nonpure piecewise-Koszul algebra if and only if, the opposite algebra of A, A^{opp} is a nonpure piecewise-Koszul algebra.

(2) A is a nonpure piecewise-Koszul algebra if and only if A is a nonpure piecewise-Koszul A^e-module.

PROOF : (1) It suffices to prove the necessity since $(A^{opp})^{opp} = A$. Let A be a nonpure piecewise Koszul algebra. Consider a minimal graded projective resolution

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

of the trivial A-module A_0 such that $P_i = A \cdot (\bigoplus_{j \in \Delta_p^d(i)} P_j)$ for all $i \geq 0$. Let

$V^* = \mathrm{Hom}_{A_0}(V, A_0)$ and $(V^\sharp)_i = (V_{-i})^*$. Thus, we get an injective resolution of A_0:

$$0 \longrightarrow A_0 \longrightarrow (P_0)^\sharp \longrightarrow (P_1)^\sharp \longrightarrow \cdots \longrightarrow (P_n)^\sharp \longrightarrow \cdots,$$

which implies that for all $i \geq 0$, $\mathrm{Ext}^i_{A^{opp}}(A_0, A_0) = \bigoplus_{j \in \Delta_p^d(i)} \mathrm{Ext}^i_{Gr(A^{opp})}(A_0, A_0[j])$. By Theorem 3.1, A^{opp} is a nonpure piecewise-Koszul algebra.

(2) Let

$$\mathcal{P}_* : \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be a minimal graded projective A^e-resolution of A. Then by Lemma 3.2, $\mathcal{P}_*$ is minimal if and only if $\mathcal{P}_* \otimes_A A_0$:

$$\cdots \longrightarrow P_n \otimes_A A_0 \longrightarrow \cdots \longrightarrow P_1 \otimes_A A_0$$

$$P_0 \otimes_A A_0 \longrightarrow A \otimes_A A_0 \cong A_0 \longrightarrow 0$$

is a minimal graded projective resolution of A_0. Further, for all $i \geq 0$, P_i is generated in degrees in $\{s_1, s_2, \cdots, s_n\}$ as a graded A^e-module if and only if $P_i \otimes_A A_0$ is generated in degrees in $\{s_1, s_2, \cdots, s_n\}$ as a graded A-module, which completes the proof. □

4. Nonpure Piecewise-Koszul Modules

In this section, we will give some basic properties of nonpure piecewise-Koszul modules.

Proposition 4.1 — Let A be a piecewise-Koszul algebra and $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $gr(A)$. Then

(1) M is nonpure piecewise-Koszul provided that K and N are nonpure piecewise-Koszul, i.e., the category of nonpure piecewise-Koszul modules is closed under extensions;

(2) N is nonpure piecewise-Koszul provided that K and M are nonpure piecewise-Koszul, i.e., the category of nonpure piecewise-Koszul modules is closed under cokernels of monomorphisms.

PROOF : (1) By the Long Exact Theorem, we get the following exact sequence

$$\mathrm{Ext}^i_{Gr(A)}(N, A_0) \to \mathrm{Ext}^i_{Gr(A)}(M, A_0) \to \mathrm{Ext}^i_{Gr(A)}(K, A_0)$$

for all $i \geq 0$. By hypothesis, K and M are nonpure piecewise-Koszul modules, by Theorem 3.1, we have

$$\mathrm{Ext}^i_A(N, A_0)_{-j} = \mathrm{Ext}^i_{Gr(A)}(N, A_0[j]) = 0, \quad (j \notin \Delta^d_p(i))$$

and

$$\mathrm{Ext}^i_A(K, A_0)_{-j} = \mathrm{Ext}^i_{Gr(A)}(K, A_0[j]) = 0, \quad (j \notin \Delta^d_p(i))$$

for all $i \geq 0$, which imply that

$$\mathrm{Ext}^i_A(M, A_0)_{-j} = \mathrm{Ext}^i_{Gr(A)}(M, A_0[j]) = 0, \quad (j \notin \Delta^d_p(i))$$

for all $i \geq 0$. By Theorem 3.1, M is a nonpure piecewise-Koszul module.

(2) Similarly, we have the long exact sequence

$$0 \to \mathrm{Hom}_{Gr(A)}(N, A_0) \to \mathrm{Hom}_{Gr(A)}(M, A_0) \to \mathrm{Hom}_{Gr(A)}(K, A_0) \to \mathrm{Ext}^1_{Gr(A)}(N, A_0) \to \mathrm{Ext}^1_{Gr(A)}(M, A_0) \to \mathrm{Ext}^1_{Gr(A)}(K, A_0) \to \mathrm{Ext}^2_{Gr(A)}(N, A_0) \to \mathrm{Ext}^2_{Gr(A)}(M, A_0) \to \mathrm{Ext}^2_{Gr(A)}(K, A_0) \to \cdots .$$

To finish the proof, we need to prove that

$$\mathrm{Ext}^i_A(N, A_0)_{-j} = \mathrm{Ext}^i_{Gr(A)}(N, A_0[j]) = 0, \quad (j \notin \Delta^d_p(i))$$

for all $i \geq 0$ and do it by induction.

For $i = 0$, it is obvious since M is a nonpure piecewise-Koszul module and generated in degree 0.

For $i = 1$, it suffices to prove that

$$\mathrm{Ext}^1_A(N, A_0)_{-j} = \mathrm{Ext}^1_{Gr(A)}(N, A_0[j]) = 0, \quad (j \notin \Delta^d_p(1) = \{1, 2, \cdots, d-p+1\}).$$

First we claim that $\mathrm{Ext}^1_A(N, A_0)_{-j} = \mathrm{Ext}^1_{Gr(A)}(N, A_0[j]) = 0$ for $j < 1$. In fact, let $\mathcal{P}_*$ and $\mathcal{Q}_*$ be the minimal graded projective resolutions of A_0 and N, respectively. Note that A is a piecewise-Koszul algebra, thus P_1 is supported in $\{i \geq \delta^d_p(1) = 1 | i \in \mathbb{N}\}$. By Lemma 3.2 of [11], we have that Q_1 is supported in $\{i \geq 1 | i \in \mathbb{N}\}$, which implies the claim.

Now note that K and M are nonpure piecewise-Koszul modules, which imply that

$$\mathrm{Ext}^1_A(K, A_0)_{-j} = \mathrm{Ext}^1_{Gr(A)}(K, A_0[j]) = 0, \quad (j \notin \Delta^d_p(1) = \{1, 2, \cdots, d-p+1\})$$

and

$$\mathrm{Ext}^1_A(N, A_0)_{-j} = \mathrm{Ext}^1_{Gr(A)}(N, A_0[j]) = 0, \quad (j \notin \Delta^d_p(1) = \{1, 2, \cdots, d-p+1\}).$$

Now it is easy to see that

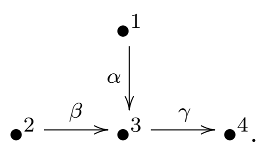
$$\mathrm{Ext}^1_A(N, A_0)_{-j} = \mathrm{Ext}^1_{Gr(A)}(N, A_0[j]) = 0, \quad (j \notin \Delta^d_p(1) = \{1, 2, \cdots, d-p+1\})$$

since the above long exact sequence.

For the cases of $i \geq 2$, we can prove them similarly. $\square$

Remark 4.2 : For an exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ in $gr(A)$, we can not get K is nonpure piecewise-Koszul even though M and N are nonpure piecewise-Koszul in general, i.e., the category of nonpure piecewise-Koszul modules does not preserve kernels of epimorphisms. The following is an easy counter example:

Let Γ be the following quiver:



Set

$$A = \frac{\Bbbk\Gamma}{\langle \beta\gamma \rangle}, \ K = \frac{Ae_1}{A\gamma}, \ M = \frac{Ae_1 \oplus Ae_2}{A(\alpha + \beta)}, N = \frac{Ae_2}{A\beta}.$$

Now it is easy to see that M and N are nonpure piecewise-Koszul modules but K is not.

However, we have a sufficient condition for the category of nonpure piecewise-Koszul modules to be closed under kernels of epimorphisms.

Proposition 4.3 — Let A be a standard graded algebra and $0 \longrightarrow KM \longrightarrow N \longrightarrow 0$ be an exact sequence in $gr(A)$ with $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$. Then M is nonpure piecewise-Koszul if and only if K and N are both nonpure piecewise-Koszul.

PROOF : We only need to prove the necessity since the sufficiency has been proved in Proposition 4.1. Note that for all $i \geq 0$, we have $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$, which implies the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}_* & \longrightarrow & \mathcal{L}_* & \longrightarrow & \mathcal{Q}_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{P}_*$, $\mathcal{L}_*$ and $\mathcal{Q}_*$ are the minimal graded projective resolutions of K, M and N, respectively. Therefore, we have $L_n = P_n \oplus Q_n$ for all $n \geq 0$. By hypothesis, M is nonpure piecewise-Koszul, thus L_n is generated in degrees in $\Delta_p^d(n)$ for each $n \geq 0$, which implies that P_n and Q_n are both generated in degrees in $\Delta_p^d(n)$ for each $n \geq 0$.

Proposition 4.4 — Let A be a piecewise-Koszul algebra and M be a piecewise-Koszul A-module. Then

(1) all the syzygies of M, $\Omega^i(M)[-\delta_p^d(i)]$ $(\forall i \geq 0)$, are nonpure piecewise-Koszul modules,

(2) all $J^i M[-i]$ $(\forall i \geq 0)$ are nonpure piecewise-Koszul modules.

PROOF : We only prove (1) since (2) is a special case of Theorem 4.5.

By the hypothesis, M is a piecewise-Koszul module, then M has a minimal graded projective resolution

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

such that each Q_n is generated in degree $\delta_p^d(n)$ for all $n \geq 0$, which implies the following long exact sequence

$$\cdots \longrightarrow Q_n[-\delta_p^d(i)] \longrightarrow \cdots \longrightarrow Q_{i+1}[-\delta_p^d(i)]$$

$$\longrightarrow Q_i[-\delta_p^d(i)] \longrightarrow \Omega^i(M)[-\delta_p^d(i)] \longrightarrow 0.$$

Now set $L_j := Q_{i+j}[-\delta_p^d(i)]$. Thus $\Omega^i(M)[-\delta_p^d(i)]$ admits the following minimal graded projective resolution

$$\cdots \longrightarrow L_n \longrightarrow \cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow \Omega^i(M)[-\delta_p^d(i)] \longrightarrow 0$$

such that L_j is generated in degree $\delta_p^d(i+j) - \delta_p^d(i)$ for all $j \geq 0$.

To complete the proof, we consider the following cases:

1. Let $i = pn$ and $n \in \mathbb{N}$. Then for all $j \geq 0$, it is clear that L_j is generated in degree $\delta_p^d(i+j) - \delta_p^d(i) = \delta_p^d(pn) + \delta_p^d(j) - \delta_p^d(pn) = \delta_p^d(j) \in \Delta_p^d(j)$.
2. Let $i = pn + 1$ and $n \in \mathbb{N}$.
   (a) If $j = pk$ ($k \in \mathbb{N}$), then it is trivial that L_j is generated in degree $\delta_p^d(i+j) - \delta_p^d(i) = \delta_p^d(pn+1+pk) - \delta_p^d(pn+1) = (n+k)d + 1 - (nd+1) = kd \in \Delta_p^d(j)$.
   (b) If $j = pk + 1$ ($k \in \mathbb{N}$), then it is trivial that L_j is generated in degree $\delta_p^d(i+j) - \delta_p^d(i) = \delta_p^d(pn+1+pk+1) - \delta_p^d(pn+1) = (n+k)d + 2 - (nd+1) = kd + 1 \in \Delta_p^d(j)$.
   (c) $\cdots$
   (d) If $j = pk + p - 2$ ($k \in \mathbb{N}$), then it is trivial that L_j is generated in degree $\delta_p^d(i+j) - \delta_p^d(i) = \delta_p^d(pn+1+pk+p-2) - \delta_p^d(pn+1) = (n+k)d + p - 1 - (nd+1) = kd + p - 2 \in \Delta_p^d(j)$.

(e) If $j = pk + p - 1$ ($k \in \mathbb{N}$), then it is trivial that L_j is generated in degrees in $\delta_p^d(i+j) - \delta_p^d(i) = \delta_p^d(pn+1+pk+p-1) - \delta_p^d(pn+1) = (n+k+1)d - (nd+1) = kd + d - 1 \in \Delta_p^d(j)$.

Now repeat the above steps, we can prove the cases of $i = pn+2, \cdots, i = pn+p-2$ and $i = pn+p-1$. Therefore, L_j is generated in degrees in $\Delta_p^d(j)$ for all $j \geq 0$, which implies that all the syzygies of the piecewise-Koszul module M, $\Omega^i(M)[-\delta_p^d(i)]$ ($\forall i \geq 0$), are nonpure piecewise-Koszul modules. □

***Theorem* 4.5** — *Let A be a piecewise-Koszul algebra and M a nonpure piecewise-Koszul A-module. Then $J^i M[-i]$ is nonpure piecewise-Koszul for all $i \geq 0$.*

PROOF : We only need to prove that $JM[-1]$ is a nonpure piecewise-Koszul module since other cases can be proved inductively. Note that $M = \bigoplus_{i \geq 0} M_i$ is nonpure piecewise-Koszul, hence it is generated in degree 0, which implies that $JM[-1] = \bigoplus_{i \geq 1} M_i[-1]$ is also generated in degree 0. Thus to complete the proof, it suffices to show that $JM[-1]$ admits a minimal graded projective resolution

$$\cdots \longrightarrow L_n \longrightarrow \cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow JM[-1] \longrightarrow 0,$$

such that for each $n \geq 0$, L_n is generated in degrees in $\Delta_p^d(n)$. By Theorem 3.1, it is equivalent to show that for all $n \geq 0$, we have the following equations:

$$\mathrm{Ext}^{pn}_{Gr(A)}(JM[-1], A_0[j]) = \mathrm{Ext}^{pn}_{Gr(A)}(JM, A_0[j+1]) = 0 \text{ unless } j = nd;$$

$$\mathrm{Ext}^{pn+1}_{Gr(A)}(JM[-1], A_0[j]) = \mathrm{Ext}^{pn+1}_{Gr(A)}(JM, A_0[j+1]) = 0 \text{ unless } j \in \Delta_p^d(pn+1);$$

$$\vdots$$

$$\mathrm{Ext}^{pn+p-2}_{Gr(A)}(JM[-1], A_0[j]) = \mathrm{Ext}^{pn+p-2}_{Gr(A)}(JM, A_0[j+1])$$
$$= 0 \text{ unless } j \in \Delta_p^d(pn+p-2);$$

$$\begin{aligned}\mathrm{Ext}^{pn+p-1}_{Gr(A)}(JM[-1], A_0[j]) &= \mathrm{Ext}^{pn+p-1}_{Gr(A)}(JM, A_0[j+1]) \\ &= 0 \text{ unless } j \in \Delta^d_p(pn+p-1).\end{aligned}$$

Now consider the graded exact sequence

$$0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0,$$

which implies the following long exact sequence as vector spaces

$\cdots \to \mathrm{Ext}^{pn}_{Gr(A)}(M, A_0[j+1]) \to \mathrm{Ext}^{pn}_{Gr(A)}(JM, A_0[j+1]) \to \mathrm{Ext}^{pn+1}_{Gr(A)}(M/JM, A_0[j+1]) \to \mathrm{Ext}^{pn+1}_{Gr(A)}(M, A_0[j+1]) \to \mathrm{Ext}^{pn+1}_{Gr(A)}(JM, A_0[j+1]) \to \mathrm{Ext}^{pn+2}_{Gr(A)}(M/JM, A_0[j+1]) \to \mathrm{Ext}^{pn+2}_{Gr(A)}(M, A_0[j+1]) \to \mathrm{Ext}^{pn+2}_{Gr(A)}(JM, A_0[j+1]) \to \mathrm{Ext}^{pn+3}_{Gr(A)}(M/JM, A_0[j+1]) \to \cdots \to \mathrm{Ext}^{pn+p-2}_{Gr(A)}(M, A_0[j+1]) \to \mathrm{Ext}^{pn+p-2}_{Gr(A)}(JM, A_0[j+1]) \to \mathrm{Ext}^{pn+p-1}_{Gr(A)}(M/JM, A_0[j+1]) \to \mathrm{Ext}^{pn+p-1}_{Gr(A)}(M, A_0[j+1]) \to \mathrm{Ext}^{pn+p-1}_{Gr(A)}(JM, A_0[j+1]) \to \mathrm{Ext}^{pn+p}_{Gr(A)}(M/JM, A_0[j+1]) \to \cdots$.

Let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

be a minimal graded projective resolution of the trivial A-module A_0. Note that A is a piecewise-Koszul module, thus P_{pn} is generated in degree nd for each $n \geq 0$, which implies that P_{pn} is supported in $\{i \in \mathbb{N} | i \geq nd\}$ for each $n \geq 0$. By Lemma 3.2 of [11], we know that L_{pn} is supported in $\{i \in \mathbb{N} | i \geq nd\}$ for each $n \geq 0$, which implies that for each $n \geq 0$, we have

$$\mathrm{Ext}^{pn}_{Gr(A)}(JM, A_0[j+1]) = 0, \quad (j < nd).$$

Note again that A is a piecewise-Koszul algebra, so M/JM is a piecewise-Koszul module, which implies that for each $n \geq 0$,

$$\mathrm{Ext}^{pn+1}_{Gr(A)}(M/JM, A_0[j+1]) = 0, \quad (j \neq nd).$$

Therefore, if $j \neq nd$, then we have the following exact sequence

$$\mathrm{Ext}^{pn}_{Gr(A)}(M, A_0[j+1]) \to \mathrm{Ext}^{pn}_{Gr(A)}(JM, A_0[j+1]) \to 0.$$

Recall that M is a nonpure piecewise-Koszul module, thus

$$\mathrm{Ext}^{pn}_{Gr(A)}(M, A_0[j+1]) = 0, \ \ (j \neq nd-1).$$

Therefore, for the case of $j \neq nd$: if moreover $j \neq nd-1$, then $\mathrm{Ext}^{pn}_{Gr(A)}(JM, A_0[j+1]) = 0$; if $j = nd-1$, we already have known that $\mathrm{Ext}^{pn}_{Gr(A)}(JM, A_0[j+1]) = 0$ since $nd-1 < nd$.

Now we will claim $\mathrm{Ext}^{pn+1}_{Gr(A)}(JM[-1], A_0[j]) = \mathrm{Ext}^{pn+1}_{Gr(A)}(JM, A_0[j+1]) = 0$ unless $j \in \Delta^d_p(pn+1)$. In fact, similar to the proof of the case of pn, by Lemma 3.2 of [11], we can prove that

$$\mathrm{Ext}^{pn+1}_{Gr(A)}(JM, A_0[j+1]) = 0, \ \ (j < nd+1).$$

Similarly, M/JM is a piecewise-Koszul module, which implies that for each $n \geq 0$,

$$\mathrm{Ext}^{pn+2}_{Gr(A)}(M/JM, A_0[j+1]) = 0, \ \ (j \neq nd+1).$$

Also note that M is a nonpure piecewise-Koszul module, which implies that

$$\mathrm{Ext}^{pn+1}_{Gr(A)}(M, A_0[j+1]) = 0 \text{ unless } j+1 \in \Delta^d_p(pn+1).$$

Now let $j \notin \Delta^d_p(pn+1) = \{nd+1, \ nd+2, \ \cdots, \ nd+d-p+1\}$.

Then we have the following exact sequence

$$\mathrm{Ext}^{pn+1}_{Gr(A)}(M, A_0[j+1]) \to \mathrm{Ext}^{pn+1}_{Gr(A)}(JM, A_0[j+1])$$

$$\to \mathrm{Ext}^{pn+2}_{Gr(A)}(M/JM, A_0[j+1]) = 0.$$

Therefore, if $j \notin \{nd, \ nd+1, \ \cdots, \ nd+d-p\}$, then $\mathrm{Ext}^{pn+1}_{Gr(A)}(JM, A_0[j+1]) = 0$ since $\mathrm{Ext}^{pn+1}_{Gr(A)}(M, A_0[j+1]) = 0$; and if $j \in \{nd, \ nd+1, \ \cdots, \ nd+d-p\}$, then $j = nd < nd+1$, which we have already known that $\mathrm{Ext}^{pn+1}_{Gr(A)}(JM, A_0[j+1]) = 0$.

Thus we have proved the claim $\mathrm{Ext}^{pn+1}_{Gr(A)}(JM[-1], A_0[j]) = \mathrm{Ext}^{pn+1}_{Gr(A)}(JM, A_0[j+1]) = 0$ unless $j \in \Delta^d_p(pn+1)$.

For the other equations, repeating the above arguments, we can prove them inductively. □

Set

$$\mathrm{E}^{[l]}(A) := \bigoplus_{i\geq 0} \mathrm{Ext}_A^{pi+l}(A_0, A_0), \quad (l = 0,\ 1,\ \cdots,\ p-1)$$

and

$$\mathcal{E}^{[l]}(M) := \bigoplus_{i\geq 0} \mathrm{Ext}_A^{pi+l}(M, A_0), \quad (l = 0,\ 1,\ \cdots,\ p-1),$$

where $M \in gr(A)$.

***Theorem* 4.6** — *Let M be a nonpure piecewise-Koszul module in $gr(A)$. Using the above notations. Then we have the following statements:*

1. we have $\mathbb{k}$-isomorphisms for all $n \geq 1$:

$$\mathrm{Ext}_{Gr(A)}^{pn-1}(JM, A_0[nd]) \cong \mathrm{Ext}_{Gr(A)}^{pn-1}(J^2M, A_0[nd])$$
$$\cong \cdots \cong \mathrm{Ext}_{Gr(A)}^{pn-1}(J^{d-p+1}M, A_0[nd]);$$

2. $\mathcal{E}^{[0]}(M) \in Gr_0(\mathrm{E}^{[0]}(A))$;
3. $\mathcal{E}^{[0]}(M) \in gr(\mathrm{E}^{[0]}(A))$.

PROOF : (1) It suffices to prove the first isomorphism since others can be proved similarly. Consider the exact sequence

$$0 \longrightarrow J^2M \longrightarrow JM \longrightarrow JM/J^2M \longrightarrow 0,$$

which implies the following exact sequence

$$\mathrm{Ext}_A^{pn-1}(JM/J^2M, A_0) \to \mathrm{Ext}_A^{pn-1}(JM, A_0) \to \mathrm{Ext}_A^{pn-1}$$
$$(J^2M, A_0) \to \mathrm{Ext}_A^{pn}(JM/J^2M, A_0).$$

Note that A is a piecewise-Koszul algebra, thus $JM/J^2(M)[-1]$ is a piecewise-Koszul module, which implies that

$$\mathrm{Ext}_{Gr(A)}^{pn-1}(JM/J^2M, A_0[j]) = 0, \quad (j \neq nd + p - d)$$

and

$$\mathrm{Ext}^{pn}_{Gr(A)}(JM/J^2M, A_0[j]) = 0, \quad (j \neq nd+1).$$

Therefore, when $j = nd$, we have the exact sequence

$$0 \to \mathrm{Ext}^{pn-1}_{Gr(A)}(JM, A_0[nd]) \to \mathrm{Ext}^{pn-1}_{Gr(A)}(J^2M, A_0[nd]) \to 0,$$

thus $\mathrm{Ext}^{pn-1}_{Gr(A)}(JM, A_0[nd]) \cong \mathrm{Ext}^{pn-1}_{Gr(A)}(J^2M, A_0[nd])$, as desired.

(2) Consider the exact sequence

$$0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0.$$

Similarly, we obtain the following exact sequence,

$$\mathrm{Ext}^{pn-1}_{A}(M, A_0) \to \mathrm{Ext}^{pn-1}_{A}(JM, A_0) \to$$

$$\mathrm{Ext}^{pn}_{A}(M/JM, A_0) \to \mathrm{Ext}^{pn}_{A}(M, A_0) \to \mathrm{Ext}^{pn}_{A}(JM, A_0).$$

By Theorem 4.5, we know that $JM[-1]$ is a nonpure piecewise-Koszul module. By Theorem 3.1, we have

$$\mathrm{Ext}^{pn}_{Gr(A)}(JM, A_0[j]) = \mathrm{Ext}^{pn}_{Gr(A)}(JM[-1], A_0[j-1]) = 0, \quad (j \neq nd+1)$$

and

$$\mathrm{Ext}^{pn-1}_{Gr(A)}(M, A_0[j]) = 0, \quad (j \notin \Delta^d_p(pn-1)$$
$$= \{nd-d+p-1,\ nd-d+p,\ \cdots,\ nd-1\})$$

since M is also a nonpure piecewise-Koszul module.

Therefore, when $j = nd$, we have the exact sequence

$$0 \to \mathrm{Ext}^{pn-1}_{Gr(A)}(JM, A_0[nd]) \to \mathrm{Ext}^{pn}_{Gr(A)}(M/JM, A_0[nd]) \to$$
$$\mathrm{Ext}^{pn}_{Gr(A)}(M, A_0[nd]) \to 0.$$

Observe that M/JM is a piecewise-Koszul module and M is a nonpure piecewise-Koszul module, we have

$$\mathcal{E}^{[0]}(M/JM) = \bigoplus_{i \geq 0} \mathrm{Ext}^{pi}_{A}(M/JM, A_0) = \bigoplus_{i \geq 0} \mathrm{Ext}^{pi}_{A}(M/JM, A_0[nd])$$

and

$$\mathcal{E}^{[0]}(M) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^{pi}(M, A_0) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^{pi}(M, A_0[nd]).$$

Now apply the exact functor "$\bigoplus_{n\geq 0}$" to the above exact sequence, we get the short exact sequence

$$0 \to \bigoplus_{n\geq 0} \mathrm{Ext}_{Gr(A)}^{pn-1}(JM, A_0[nd]) \to \mathcal{E}^{[0]}(M/JM) \to \mathcal{E}^{[0]}(M) \to 0.$$

It is clear that $\bigoplus_{n\geq 0} \mathrm{Ext}_{Gr(A)}^{pn-1}(JM, A_0[nd])$ can be viewed as a graded $\mathrm{E}^{[0]}(A)$-module under the grading $(\bigoplus_{n\geq 0} \mathrm{Ext}_{Gr(A)}^{pn-1}(JM, A_0[nd]))_n = \mathrm{Ext}_{Gr(A)}^{pn-1}$ $(JM, A_0[nd])$.

Therefore each term of the above short exact sequence is an $\mathrm{E}^{[0]}(A)$-module. Note that M/JM is a piecewise-Koszul module, by [19], we have that $\mathcal{E}^{[0]}(M/JM)$ is a Koszul module over the Koszul algebra $\mathrm{E}^{[0]}(A)$, thus $\mathcal{E}^{[0]}(M/JM)$ can be generated in degree 0 as a graded $\mathrm{E}^{[0]}(A)$-module, which implies that $\mathcal{E}^{[0]}(M) \in Gr_0(\mathrm{E}^{[0]}(A))$, as desired.

(3) By (2), we have $\mathcal{E}^{[0]}(M) \in Gr_0(\mathrm{E}^{[0]}(A))$. Observing that M/JM is a finitely generated $A/J \cong A_0$-module since M is a finitely generated graded module. Clearly, M/JM is a finite dimensional $\Bbbk$-space since A_0 is a finite dimensional semisimple $\Bbbk$-algebra, which implies that $\mathrm{Hom}_A(M/JM, A_0)$ is a finite dimensional $\Bbbk$-space. Note that

$$(\mathcal{E}^{[0]}(M/JM))_0 = \mathrm{Hom}_A(M/JM, A_0),$$

we obtain that $\mathcal{E}^{[0]}(M/JM)$ is a finitely generated graded $\mathrm{E}^{[0]}(A)$-module. Therefore, we get that $\mathcal{E}^{[0]}(M)$ is a finitely generated graded $\mathrm{E}^{[0]}(A)$-module since the epimorphism $\mathcal{E}^{[0]}(M/JM) \to \mathcal{E}^{[0]}(M) \to 0$. □

5. An Application of Nonpure Piecewise-Koszul Modules

In this section, we give a sufficient condition for the questions raised in [19] to be true.

Firstly, let's introduce the motivations and recall the questions:

Let A be a piecewise-Koszul algebra and M a piecewise-Koszul module. Using the notations of Section 4. Then we have the following statements, which can be found in [19] and [20]:

- $\mathrm{E}^{[0]}(A)$ is a Koszul algebra;
- $\mathcal{E}^{[0]}(M)$ is a Koszul module over $\mathrm{E}^{[0]}(A)$;
- all the graded $\mathbb{k}$-spaces $\mathcal{E}^{[l]}(M)$ $(l = 0,\ 1,\ \cdots,\ p-1)$ can be viewed as graded $\mathrm{E}^{[0]}(A)$-modules. Moreover, all $\mathcal{E}^{[l]}(M)$ $(l = 0,\ 1,\ \cdots,\ p-1)$ are generated in degree 0 as $\mathrm{E}^{[0]}(A)$-modules.

Now motivated by the above, the authors raised the following natural question:

- Are these $\mathrm{E}^{[0]}(A)$-modules $\mathcal{E}^{[l]}(M)$ $(l = 1,\ 2,\ \cdots,\ p-1)$ Koszul?

Lemma 5.1 — Let A be a standard graded algebra and M a finitely generated graded A-module. Then we have the isomorphisms

$$\mathcal{E}^{[l]}(M) \cong \mathcal{E}^{[0]}(\Omega^l(M))\ (l = 0,\ 1,\ \cdots,\ p-1)$$

as graded $\mathrm{E}^{[0]}$-modules.

PROOF : Let

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

be a minimal graded projective resolution of M. Then $\Omega^l(M)$ has the following minimal graded projective resolution

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_{l+1} \longrightarrow Q_l \longrightarrow \Omega^l(M) \longrightarrow 0.$$

Note that A_0 is semisimple, we have

$$\mathcal{E}^{[l]}(M) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^{pi+l}(M, A_0) = \bigoplus_{i\geq 0} \mathrm{Hom}_A(Q_{pi+l}, A_0)$$

and

$$\mathcal{E}^{[0]}(\Omega^l(M)) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^{pi}(\Omega^l(M), A_0) = \bigoplus_{i\geq 0} \mathrm{Hom}_A(Q_{pi+l}, A_0),$$

which complete the proof.

***Theorem* 5.2** — *Let A be a piecewise-Koszul algebra and X be any nonpure piecewise-Koszul module in $gr(A)$. Suppose that $\mathcal{E}^{[0]}(X)$ is a Koszul $\mathrm{E}^{[0]}(A)$-module. Then all $\mathrm{E}^{[0]}(A)$-modules $\mathcal{E}^{[l]}(M),\ (l = 1,\ 2,\ \cdots,\ p-1)$ are Koszul, where M is a piecewise-Koszul module.*

PROOF : By Proposition 4.4, $\Omega^i(M)[-\delta_p^d(i)]$ is a nonpure piecewise-Koszul module for all $i \geq 0$ since M is a piecewise-Koszul module. By hypothesis, $\mathcal{E}^{[0]}(\Omega^i(M)[-\delta_p^d(i)])$ is a Koszul $\mathrm{E}^{[0]}(A)$-module for all $i \geq 0$. By Lemma 5.1, we have $\mathcal{E}^{[l]}(M) \cong \mathcal{E}^{[0]}(\Omega^l(M)) = \mathcal{E}^{[0]}(\Omega^l(M)[-\delta_p^d(l)])$ for $l = 1,\ 2,\ \cdots,\ p-1$. Therefore, all $\mathcal{E}^{[l]}(M),\ (l = 1,\ 2,\ \cdots,\ p-1)$ are Koszul $\mathrm{E}^{[0]}(A)$-modules.

Remark 5.3 : Let A be a piecewise-Koszul algebra and M a nonpure piecewise-Koszul A-module. Then do we always have that $\mathcal{E}^{[0]}(M)$ is a Koszul $\mathrm{E}^{[0]}(A)$-module?

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