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RANDERS CHANGE OF m^{th} ROOT METRIC

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The present paper deals with a Randers metric that has been derived after a particular $\beta-$change in the mth root metric. Various geometers such as [7], [9], [10] etc. have studied the mth root metric and its transformations. We have obtained some tensors and theorems holding the relation between the Finsler space equipped with the mth root metric and the one obtained after its Randers change.

Key words : Finsler space, Randers space, mth root metric and $\beta-$change.

1. INTRODUCTION

A Finsler space F^n is said to be equipped with (α, β) metric, if the metric function $L(x, y)$ is positively homogeneous of degree one in α and β. Matsumoto [4] in 1971 introduced a particular transformation of Finsler metric $\alpha(x, y)$ defined as:

$$L(x, y) = \alpha(x, y) + \beta(x, y), \tag{1.1}$$

where $\alpha(x,y)$ is the fundamental function of Finsler space F^n and $\beta(x,y) = b_i(x)\,y^i$ is a differential one form. Such a change of Finsler space is termed as $\beta-$change.

Randers [8] then introduced the Randers space with metric $L = \alpha + \beta$, where $\alpha = (a_{ij}(x)y^i\,y^j)^{1/2}$ is a Riemannian metric and β is as defined above. Later in 1978 Numata [6] introduced another metric $L = \mu+\beta$, where $\mu = (a_{ij}(y)y^i\,y^j)^{1/2}$ is a Minkowski metric and $\beta(x,y) = b_i(x)\,y^i$. This metric is similar to the Randers one but with different geometrical properties. In papers [1], [2], [4] etc. the corresponding $\beta-$change of various Finsler metrics have been studied from different viewpoints to obtain several theorems.

In 1979 Shimada [9] introduced the mth root metric on the differentiable manifold M^n defined as:

$$L = \sqrt[m]{a_{i_1 \ldots i_m} y^{i_1} \ldots y^{i_m}}, \tag{1.2}$$

where the coefficients $a_{i_1 \ldots i_m}$ are the components of symmetric covariant tensor field of order $(0, m)$ being the functions of positional co-ordinates only.

Since then various geometers such as [7], [10] etc. have explored the theory of mth root metric and studied its transformations.

In the present paper we have considered a transformation of the mth root metric such that it transforms to a similar metric as the Randers one defined in (1.1) in a way that the Riemannian metric α is replaced with the mth root metric λ defined in (1.2). Hence this paper deals with a Finsler space F^n equipped with the fundamental function $L(x,y)$ defined as:

$$L(x,y) = \lambda(x,y) + \beta(x,y), \tag{1.3}$$

where $\lambda = (a_{i_1 \ldots . i_m}(x) y^{i_1} \ldots . y^{i_m})^{1/m}$ is the metric function of Finsler space $\overline{F}^n$ and $\beta(x,y) = b_i(x)\,y^i$ is a differential one form. In this paper we shall study the relation between various tensors of Finsler space F^n with that of $\overline{F}^n$.

2. PRELIMINARIES

Let M^n be an $n-$dimensional smooth manifold and $F^n = (M^n, L)$ be an $n-$dimensional Finsler space equipped with the fundamental function $L(x, y)$ defined in (1.3). As we are concerned here with the transformation of mth root metric, we shall therefore restrict ourselves for $m > 2$ throughout the paper.

The normalized supporting element l_i, angular metric tensor h_{ij}, metric tensor g_{ij} and Cartan's $C-$tensor C_{ijk} [3] are defined respectively as:

$$l_i = \frac{\partial L}{\partial y^i}, \qquad h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j}, \qquad g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$$

and

$$C_{ijk} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}. \tag{2.1}$$

Dealing with the mth root metric we shall first introduce the tensors $a_i(x, y)$, $a_{ij}(x, y)$ and $a_{ijk}(x, y)$ as defined in [7] :

$$\begin{aligned} &(a) \qquad \alpha^{m-1} a_i = a_{ii_2 \ldots i_m} y^{i_2} \ldots y^{i_m}, \\ &(b) \qquad \alpha^{m-2} a_{ij} = a_{iji_3 \ldots i_m} y^{i_3} \ldots y^{i_m} \\ &(c) \qquad \alpha^{m-3} a_{ijk} = a_{ijki_4 \ldots i_m} y^{i_4} \ldots y^{i_m}. \end{aligned} \tag{2.2}$$

The differentiation of $a_{i_1 \cdots i_r}$ with respect to y^j is given by [10]:

$$\dot{\partial}_j a_{i_1 \ldots i_r} = \frac{(m - r)}{\lambda}(a_{i_1 \ldots i_{r,j}} - a_{i_1 \ldots i_r} a_j) \tag{2.3}$$

where $r = 1 \ldots n$.

3. FUNDAMENTAL TENSORS

We consider an $n-$dimensional Finsler space F^n with metric $L(x, y)$ defined in (1.3). The differentiation of (1.3) with respect to y^i yields the normalized supporting element l_i as:

$$l_i = a_i + b_i, \tag{3.1}$$

where l_i and a_i are defined in (2.1) and (2.2) respectively.

In view of (2.2) and (2.3) differentiation of (3.1) with respect to y^j yields:

$$h_{ij} = (m-1)\,\frac{L}{\lambda}\,(a_{ij} - a_i\,a_j). \tag{3.2}$$

The above equation can be re-written as:

$$\frac{h_{ij}}{L} = \frac{\lambda_{ij}}{\lambda}, \tag{3.3}$$

where λ_{ij} is the angular metric tensor of the Finsler space $\overline{F}^n$ equipped with metric λ.

On account of (2.1), (2.2) and (2.3) the fundamental tensor $g_{ij}(x,y)$ of Finsler space F^n is given as:

$$g_{ij} = (m-1)\tau a_{ij} + (a_i b_j + b_i a_j) + b_i b_j + (1 - \overline{m-1}\,\tau)\,a_i a_j, \tag{3.4}$$

where $\tau = L/\lambda$.

We may now rewrite the above expression as:

$$g_{ij} = p_0 a_{ij} + p_1(a_i b_j + b_i a_j) + p_2 b_i b_j + p_3 a_i a_j, \tag{3.5}$$

where $p_0 = (m-1)\,\tau$,

$p_1 = 1$,

$p_2 = 1$

and

$p_3 = 1 - (m-1)\tau$.

The tensor $a_{ij}(x, y)$ is called the basic tensor and is supposed to have a non-vanishing determinant.

In order to obtain the contravariant metric tensor $g^{ij}(x, y)$ we here assume the covariant entity H_{ij} defined as:

$$H_{ij} = p_0 a_{ij} + c_i c_j, \tag{3.6}$$

where $c_i = \pi b_i$ and p_0 is defined in (3.5).

The contravariant tensor $H^{ij}(x, y)$ is now defined as:

$$H^{ij} = \frac{1}{p_0}\left[a^{ij} - \frac{c^i c^j}{p_0 + c^2}\right], \tag{3.7}$$

where $c^i = g^{ij} c_j$ and c_i is already defined in (3.6).

In view of (3.5) the unknown quantity π is obtained using the following relations:

$\pi_{-1}^2 = p_3$,

(ii) $\pi_0 = \dfrac{p_2}{\pi_{-1}}$

and

(iii) $\pi^2 = p_1 - \pi_0^2$.

Substituting the above result in equation (3.7) we get:

$$H^{ij} = \frac{1}{(m-1)\tau}\left[a_{ij} + \frac{b_i b_j}{1 - (m-1)\tau - b^2}\right], \tag{3.8}$$

where $b^i = a^{ij} b_j$,

$$b^2 = g^{ij} b_i b_j = b_i b^i \tag{3.9}$$

and τ being defined in (3.4).

The metric tensor g_{ij} can now be written in the following form:

$$g_{ij} = H_{ij} + d_i d_j, \tag{3.10}$$

where $d_i = \pi_0 b_i + \pi_{-1} a_i = \dfrac{1}{\sqrt{1-(m-1)\tau}}\left[b_i + (1-\overline{m-1}\tau)a_i\right]$.

Then $g^{ij}(x,y)$ is defined as:

$$g^{ij}(x,y) = H^{ij} - \frac{d^i d^j}{1+d^2}, \tag{3.11}$$

where $d^i = H^{ij}d_j = \dfrac{\sqrt{1-\overline{m-1}\tau}}{(m-1)\tau}\left[\dfrac{(1+q)}{1-\overline{m-1}\tau-b^2}b^i + a^i\right]$,

$q = \dfrac{\beta}{\lambda} = b_i a^i = a_i b^i$,

$a^i = a^{ij}a_j$

and $d^2 = H^{ij}d_i d_j$

$$= \frac{1}{(m-1)(1-\overline{m-1}\tau-b^2)\tau}\left[(1-\overline{m-1}\tau-b^2)(q+1-\overline{m-1}\tau) + (1+q)\{q(1-\overline{m-1}\tau)+b^2\}\right].$$

Hence $g^{ij}(x,y)$ is given as:

$$g^{ij} = \frac{1}{(m-1)\tau}\left[a^{ij} - \frac{1}{(1+q)}(a^i b^j + b^i a^j) + \frac{(b^2+\overline{m-1}\tau-1)}{(1+q)^2}a^i a^j\right]. \tag{3.12}$$

Theorem **3.1** — *The metric tensors $g_{ij}(x,y)$ and $g^{ij}(x,y)$ of the Finsler space F^n equipped with the Randers metric obtained by transforming the mth root metric are given as:*

$$g_{ij} = (m-1)\tau a_{ij} + (a_i b_j + b_i a_j) + b_i b_j + (1-\overline{m-1}\,\tau)a_i a_j$$

and

$$g^{ij} = \frac{1}{(m-1)\tau}\left[a^{ij} - \frac{1}{(1+q)}(a^i b^j + b^i a^j) + \frac{(b^2+\overline{m-1}\tau-1)}{(1+q)^2}a^i a^j\right].$$

The differentiation of τ with respect to y^i is given as:

$$\frac{\partial\tau}{\partial y^i}=\frac{(1-\tau)a_i+b_i}{\lambda}. \tag{3.13}$$

In view of (2.2), (2.3) and (3.12) we differentiate (3.4) with respect to y^k to obtain the following result:

$$\begin{aligned}2\lambda C_{ijk} &= (m-1)[(m-2)\tau a_{ijk}+(1-\overline{m-1}\tau)(a_ia_{jk}+a_ja_{ik}+a_ka_{ij})\\ &+(a_{ij}b_k+a_{jk}b_i+a_{ki}b_j)-(a_ia_jb_k+a_ja_kb_i+a_ka_ib_j)\\ &+(\overline{2m-1}\tau-3)a_ia_ja_k].\end{aligned} \tag{3.14}$$

***Theorem* 3.2** — *The Cartan's $C-$tensor $C_{ijk}(x,y)$ under the Randers change of mth root metric takes the following form:*

$$\begin{aligned}2\lambda C_{ijk} &= (m-1)[(m-2)\tau a_{ijk}+(1-\overline{m-1}\tau)(a_ia_{jk}+a_ja_{ik}+a_ka_{ij})\\ &+(a_{ij}b_k+a_{jk}b_i+a_{ki}b_j)-(a_ia_jb_k+a_ja_kb_i+a_ka_ib_j)\\ &+(\overline{2m-1}\tau-3)a_ia_ja_k].\end{aligned}$$

Now, C_{ijk} is re-written as:

$$C_{ijk}=\tau\, p_{ijk}+(\lambda_{ij}m_k+\lambda_{jk}m_i+\lambda_{ik}m_j)/2\lambda, \tag{3.15}$$

where p_{ijk} is the (h)hv- torsion tensor of the Cartan's connection $C\Gamma$ of the mth root Finsler metric λ given as:

$$\begin{aligned}p_{ijk} &= \frac{(m-1)(m-2)}{2\lambda}\left[a_{ijk}-(a_ia_{jk}+a_ja_{ki}+a_ka_{ij})+2a_ia_ja_k\right],\\ \lambda_{ij} &= (m-1)(a_{ij}-a_ia_j)\\ &\quad \textit{and } m_i=b_i-\frac{\beta}{\lambda}a_i.\end{aligned} \tag{3.16}$$

***Theorem* 3.3** — *The (h)hv-torsion tensor $p_{ijk}(x,y)$ of the Cartan's connection $C\Gamma$ of the Randers metric obtained by β transformation of the mth root metric is given by:*

$$p_{ijk} = \frac{(m-1)(m-2)}{2\lambda}\left[a_{ijk} - (a_i a_{jk} + a_j a_{ki} + a_k a_{ij}) + 2a_i a_j a_k\right].$$

4. The V-Curvature Tensor of F^n

In view of (3.15) and definition of a^i we now have the following identities:

$$a^i a_i = 1, \qquad a_{ijk}a^i = a_{jk}, \qquad m_i a^i = 0, \qquad m_i b^i = b^2 - q^2,$$

$$\lambda_{ij}a^i = 0, \qquad \lambda_{ij}b^i = (m-1)m_i \quad \text{and} \quad p_{ijk}a^i = 0. \tag{4.1}$$

In order to find the v-curvature tensor of the Finsler space F^n it is necessary to obtain the (h)hv-torsion tensor $C^i_{jk} = g^{ir}C_{jrk}$.

Hence transvection of (3.13) with g^{ir} yields:

$$C^i_{jk} = \frac{1}{(m-1)}p^i_{jk} + \frac{1}{2L(m-1)}(\lambda^i_j m_k + \lambda^i_k m_j + \lambda_{jk}m^i)$$
$$-\frac{a^i}{L(1+q)}\left[m_j m_k + \frac{(b^2-q^2)}{2(m-1)}\lambda_{jk}\right] - \frac{1}{(m-1)(1+q)}a^i p_{jrk}b^r, \tag{4.2}$$

where

$$\begin{aligned} p^i_{jk} &= p_{jrk}a^{ir} = \frac{(m-1)(m-2)}{2\lambda}\left[a^i_{jk} - (\delta^i_j a_k + \delta^i_k a_j + a^i a_{jk}) + 2a^i a_j a_k\right], \\ \lambda^i_j &= \lambda_{jr}a^{ir} = (m-1)(\delta^i_j - a^i a_j), \\ & m^i = m_r a^{ir} = b^i - q a^i \\ & \text{and} \quad a^i_{jk} = a^{ir}a_{jrk}. \end{aligned} \tag{4.3}$$

From (4.1) and (4.3) we now have another set of identities as following:

$$\begin{aligned} p_{ijr}\lambda^r_h &= p^r_{ij}\lambda_{rh} = (m-1)p_{ijh}, \qquad p_{ijr}m^r = p_{ijr}b^r, \qquad m_r\lambda^r_i = (m-1)m_i, \\ & m_im^i = b^2-q^2, \qquad \lambda_{ir}\lambda^r_j = (m-1)\lambda_{ij} \\ & \text{and} \qquad \lambda_{ir}m^r = (m-1)m_i. \end{aligned} \tag{4.4}$$

In view of (3.13), (4.1), (4.2) and (4.3) we have:

$$\begin{aligned} C_{ijr}C^r_{hk} &= \frac{\tau}{(m-1)}p_{ijr}p^r_{hk} + \frac{1}{2\lambda(m-1)}(p_{ijr}\lambda_{hk} + p_{hkr}\lambda_{ij})b^r \\ &+\frac{1}{4\lambda L}(2m_hm_k\lambda_{ij} + m_im_k\lambda_{hj} + m_im_h\lambda_{jk} + 2m_im_j\lambda_{hk} \\ &+m_jm_k\lambda_{ih} + m_jm_h\lambda_{ik}) \\ &+\frac{1}{2\lambda}(p_{ijh}m_k + p_{ijk}m_h + p_{jkh}m_i + p_{ihk}m_j) \\ &+\frac{1}{4\lambda L}(b^2-q^2)\lambda_{ij}\lambda_{hk}. \end{aligned} \tag{4.5}$$

The v-curvature tensor S_{hijk} is given by [3]:

$$S_{hijk} = C_{ijr}C^r_{hk} - C_{ikr}C^r_{hj}. \tag{4.6}$$

Now, interchanging indices j and k in (4.4) we get:

$$\begin{aligned} C_{ikr}C^r_{hj} &= \frac{\tau}{(m-1)}p_{ikr}p^r_{hj} + \frac{1}{2\lambda(m-1)}(p_{ikr}\lambda_{hj} + p_{hjr}\lambda_{ik})b^r \\ &+\frac{1}{4\lambda L}(2m_hm_j\lambda_{ik} + m_im_j\lambda_{hk} + m_im_h\lambda_{jk} + 2m_im_k\lambda_{hj} \\ &+m_jm_k\lambda_{ih} + m_km_h\lambda_{ij}) \\ &+\frac{1}{2\lambda}(p_{ikh}m_j + p_{ijk}m_h + p_{jkh}m_i + p_{ihj}m_k) \\ &+\frac{1}{4\lambda L}(b^2-q^2)\lambda_{ik}\lambda_{hj}. \end{aligned} \tag{4.7}$$

In view of (4.6) subtracting (4.7) from (4.5) we get:

$$S_{hijk} = Q_{(jk)} \left\{ \frac{\tau}{(m-1)} p_{ijr} p^r_{hk} + \lambda_{hk} m_{ij} + \lambda_{ij} m_{hk} \right\}, \tag{4.8}$$

where

$$m_{ij} = \frac{1}{2\lambda(m-1)} \left\{ p_{ijr} b^r + \frac{(m-1)}{2L} m_i m_j + \frac{1}{4L}(b^2 - q^2)\lambda_{ij} \right\}$$

and the symbol $Q_{(jk)}$ denotes the interchange of indices j and k followed by subtraction.

***Theorem* 4.1** — *The v-curvature tensor $S_{hijk}(x,y)$ under the Randers change of mth root metric takes the following form:*

$$S_{hijk} = Q_{(jk)} \left\{ \frac{\tau}{(m-1)} p_{ijr} p^r_{hk} + \lambda_{hk} m_{ij} + \lambda_{ij} m_{hk} \right\}.$$

The reciprocal $\overline{g}^{ij}(x,y)$ of metric tensor $\overline{g}_{ij}(x,y)$ of the Finsler space $\overline{F}^n$ equipped with mth root metric is given by [7]:

$$\overline{g}^{ij} = \frac{1}{(m-1)} \left\{ a^{ij} + (m-2) a^i a^j \right\}. \tag{4.9}$$

In order to relate the v-curvature tensor of F^n with the mth root Finsler space of $\overline{F}^n$ we may now find the (h)hv-torsion tensor of $\overline{F}^n$ given as:

$$\overline{p}^i_{jk} = \overline{g}^{ir} p_{jrk} = \frac{1}{(m-1)} p^i_{jk}, \tag{4.10}$$

where $\overline{g}^{ij}$ is defined in (4.9).

Thus, the v-curvature tensor $\overline{S}_{hijk}$ of mth root Finsler space $\overline{F}^n$ is given as:

$$\overline{S}_{hijk} = p_{ijr} \overline{p}^r_{hk} - p_{ikr} \overline{p}^r_{hj} = \frac{1}{(m-1)} (p_{ijr} p^r_{hk} - p_{ikr} p^r_{hj}). \tag{4.11}$$

In view of (4.11) S_{hijk} may now be re-written as:

$$S_{hijk} = \tau \overline{S}_{hijk} + Q_{(jk)}(\lambda_{ij} m_{hk} + \lambda_{hk} m_{ij}), \tag{4.12}$$

where λ_{ij}, m_{ij} and $\overline{S}_{hijk}$ are defined in (3.3), (4.7) and (4.10) respectively.

From the definition of $S4-$like Finsler space we know that the v-curvature tensor of the four dimensional Finsler space may be written as [3]:

$$L^2 S_{hijk} = Q_{(jk)}(h_{hj} K_{ik} + h_{ik} K_{hj}), \tag{4.13}$$

where K_{ij} is a symmetric tensor field of type (0, 2) such that $K_{ij}\, y^j = 0$.

Hence, if the v-curvature tensor of the Finsler space F^n vanishes, then the Finsler space $\overline{F}^n$ is $S4-$like.

***Theorem* 4.2** — *The Finsler space $\overline{F}^n$ equipped with the mth root metric is $S4-$like, if the v-curvature tensor $S_{hijk}(x, y)$ of the Finsler space F^n obtained by the Randers change of mth root metric vanishes.*

If the v-curvature tensor of the Finsler space $\overline{F}^n$ vanishes then (4.12) reduces to:

$$S_{hijk} = \lambda_{ij} m_{hk} + \lambda_{hk} m_{ij} - \lambda_{hj} m_{ik} - \lambda_{ik} m_{hj}. \tag{4.14}$$

Contraction of (4.14) with g^{hj} yields the Ricci tensor $S_{ik} = g^{hj} S_{hijk}$ given as:

$$S_{ik} = -\frac{1}{(m-1)\tau}[m\lambda_{ik} + (m-1)(n-3)m_{ik}], \tag{4.15}$$

where $m = m_{ij}\, a^{ij}$.

Using the definition of m_{ij} the above expression may be re-written as:

$$S_{ik} + H_1 \lambda_{ik} + H_2 p_{ikr} b^r = H_3 m_i m_k, \tag{4.16}$$

where

$$H_1 = \frac{m}{(m-1)\,\tau} + \frac{(n-3)}{8(m-1)\,L^2}(b^2 - q^2),$$

$$H_2 = \frac{(n-3)}{2L(m-1)}$$

and

$$H_3 = -\frac{(n-3)}{4L^2}.$$

***Theorem* 4.3** — *If the v-curvature tensor $\overline{S}_{hijk}$ of the Finsler space $\overline{F}^n$ equipped with the mth root metric vanishes then there exist scalars H_1 and H_2 in the Finsler space F^n $(n \geq 4)$ obtained under the Randers change of mth root metric, such that the matrix $\|S_{ik} + H_1\lambda_{ik} + H_2\, p_{ikr}b^r\|$ is of rank less than two.*

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