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A BRANCHING LAW FROM Sp(n) TO Sp(q) $\times$ Sp(n-q) AND AN APPLICATION TO LAPLACE OPERATOR SPECTRA

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In this paper, we give a branching law from the group Sp(n) to the subgroup Sp(q) $\times$ Sp(n-q). We propose an application of this result to compute the Laplace spectrum on the forms of the manifold Sp(n)/Sp(q) $\times$ Sp(n-q), using the "identification" of the Laplace operator with the Casimir operator in symmetric spaces.

Key words : Branching law, Laplace spectrum, differential forms, representation theory, Casimir operator.

1. Introduction

Let (G, K) be a compact symmetric pair with a compact connected semisimple Lie group G and $M = G/K$. We suppose that the Riemannian metric on M is induced from the Killing form sign changed. This is a G-invariant Riemannian metric on M. We consider the Laplace operator Δ_p acting on the space of differential p-forms and its spectrum Spec$^p(M)$. The operator Δ_p is G-invariant when we consider the space of p-forms $C^\infty(\wedge^p M)$ as a G-module. Ikeda and Taniguchi [5] computed the spectrum on the forms for $M = S^n$ and $P^n(\mathbb{C})$ using representation theory. They showed that $\Delta_p = C$, the Casimir operator on G. On the other hand, Freudenthal's formula gives the eigenvalues of C on irreducible G-modules and Weyl's dimension formula gives their multiplicities. Then, it suffices to decompose $C^\infty(\wedge^p M)$ into irreducible G-submodules. Generally, this decomposition is not easy. Frobenius reciprocity law enables us to reduce the problem into the two followings: first, to give a branching law which consists to decompose an irreducible G-module (as a K-module by restriction) into irreducible K-submodules, second, to decompose the p-th exterior power of the adjoint representation of the group K into irreducible K-submodules. Tsukamoto [11] uses this method to compute the spectra of the spaces SO$(n+2)/$SO$(2) \times$ SO(n) and Sp$(n+1)/$Sp$(1) \times$ Sp(n). For the explicit spectrum computation in particular cases we can cite [7, 8]. In [9, 10], the authors compute the Laplace spectrum on functions for the manifolds SO$(2p+2q+1)/$SO$(2p) \times$ SO$(2q+1)$ and Sp$(n)/$SU(n). In [2] and [3], I generalized the result of [11] to calculate the spectrum on forms of Grassmann manifolds.

Another approach for the branching law is given by Kostant branching theorem (see for instance [4], page 371). In [6], Kostant gives a branching law from a simply-connected semisimple Lie group to a maximal compact subgroup.

This paper is organized as follow: In the second section, we give a branching law to decompose the restriction of any irreducible Sp(n)-module into a sum of irreducible Sp$(q) \times$ Sp$(n-q)$-modules. In section three, we decompose the pth exterior powers of the adjoint representation into irreducible Sp$(q) \times$ Sp$(n-q)$-modules.

2. Branching Law

Let $G =$ Sp(n) and $K =$ Sp$(q) \times$ Sp$(n-q)$. We denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$) the complexified Lie algebra of G (resp. K). Precisely,

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \ ; \ A, B, C \in M_n(\mathbb{C}), {}^tB = B, {}^tC = C \right\}$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & -{}^tA_1 & 0 \\ 0 & C_2 & 0 & -{}^tA_2 \end{pmatrix} \ ; \ \begin{array}{l} A_1, B_1, C_1 \in M_q(\mathbb{C}), \ A_2, B_2, C_2 \in M_{n-q}(\mathbb{C}), \\ {}^tB_i = B_i, {}^tC_i = C_i, i = 1, 2 \end{array} \right\}.$$

We choose the following Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$:

$$\mathfrak{t} = \{\text{diag}\,(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n); \ \lambda_j \in \mathbb{C}\}.$$

We regard λ_j as a form on $\mathfrak{t}$ giving the value of λ_j, then as an element of $\mathfrak{t}^*$.

We recall the following results:

• The roots of G:

$$\Delta_G = \{\pm\lambda_i \pm \lambda_j; \ 1 \leq i < j \leq n\} \cup \{\pm 2\lambda_i; \ 1 \leq i \leq n\}.$$

• The positive roots of G:

$$\Delta_G^+ = \{\lambda_i \pm \lambda_j; \ 1 \leq i < j \leq n\} \cup \{2\lambda_i; \ 1 \leq i \leq n\}.$$

• The simple roots of G:

$$\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2 - \lambda_3, ..., \alpha_{n-1} = \lambda_{n-1} - \lambda_n, \alpha_n = 2\lambda_n.$$

• Any dominant weight for $(\mathfrak{g}, \mathfrak{t})$ which corresponds to an irreducible representation of G has the form

$$\begin{cases} \Lambda = h_1\lambda_1 + h_2\lambda_2 + ... + h_n\lambda_n \\ h_i \in \mathbb{Z} \\ h_1 \geq h_2 \geq \cdots \geq h_n \geq 0. \end{cases} \tag{1}$$

• The Weyl group of G: $W_G = \{\phi = (\varepsilon_1, ..., \varepsilon_n, \sigma)/\ \varepsilon_i = \pm 1, \sigma \in S_n\}$, with $\phi(a_1\lambda_1 + ... + a_n\lambda_n) = \sum_{i=1}^{n} \varepsilon_i a_i \sigma(\lambda_i)$, $\det(\phi)$=sign(σ) and S_n is the group of all permutations of $\{1, ..., n\}$.

• The roots of K:

$$\Delta_K = \{\pm\lambda_i \pm \lambda_j;\ 1 \leq i < j \leq q \text{ or } q+1 \leq i < j \leq n\} \cup \{\pm 2\lambda_i;\ 1 \leq i \leq n\}.$$

• The positive roots of K:

$$\Delta_K^+ = \{\lambda_i \pm \lambda_j;\ 1 \leq i < j \leq q \text{ or } q+1 \leq i < j \leq n\} \cup \{2\lambda_i;\ 1 \leq i \leq n\}.$$

• The simple roots of K:

$$\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, ..., \lambda_{q-1} - \lambda_q, 2\lambda_q,$$
$$\lambda_{q+1} - \lambda_{q+2}, \lambda_{q+2} - \lambda_{q+3}, ..., \lambda_{n-1} - \lambda_n, 2\lambda_n.$$

• Any dominant weight for $(\mathfrak{k}, \mathfrak{t})$ which corresponds to an irreducible representation of K can be written:

$$\begin{cases} \Lambda' = k_1\lambda_1 + ... + k_q\lambda_q + k_{q+1}\lambda_{q+1} + ... + k_n\lambda_n \\ k_i \in \mathbb{Z} \quad \text{for all } \ 1 \leq i \leq n \\ k_1 \geq k_2 \geq ... \geq k_q \geq 0 \\ k_{q+1} \geq k_{q+2} \geq ... \geq k_n \geq 0. \end{cases} \tag{2}$$

• The Weyl group of K: $W_K = W_{\mathrm{Sp}(q)} \times W_{\mathrm{Sp}(n-q)}$.

Notation :

(i) We denote by:

$$e(\Lambda) = e^{2\pi i\Lambda},\ s(\Lambda) = e(\Lambda) - e(-\Lambda),$$
$$\alpha_{ij} = \frac{\lambda_i + \lambda_j}{2},\ \beta_{ij} = \frac{\lambda_i - \lambda_j}{2}.$$

(ii) For r and s integers such that $1 \leq r \leq s$, we designate by $[a_{ij}]_{r:s}$ a square matrix with i, j between r and s.

(iii) We denote by $\delta_G = \sum_{\alpha \in \Delta_G^+} \alpha/2$, the half sum of positive roots of G.

Definition 1 — Let $\Lambda \in \mathfrak{t}^*$ be a linear form on $\mathfrak{t}$. We introduce the alternate sum of Λ

$$\xi(\Lambda) : \mathfrak{t} \to \mathbb{C}, \quad \text{with} \quad \xi(\Lambda)(H) = \sum_{w \in W_G} \det(w).e(\Lambda(w(H))), \ \forall H \in \mathfrak{t}.$$

Here, $\det(w) \in \{-1, 1\}$, is the determinant of the linear automorphism w of $\mathfrak{t}$.

Lemma 2 — Let $H_1, ..., H_n$ be integers satisfying $H_1 > ... > H_n > 0$. We have for all $q \in \{1, \ldots, n\}$:

$$\frac{\det[s(H_i \lambda_j)]_{1:n}}{\prod_{i=1}^{q} \prod_{j=i+1}^{n} s(\alpha_{ij}) s(\beta_{ij})} = \sum_{K_{1,v}} \cdots \sum_{K_{q,v}} \left\{ \prod_{r=1}^{q} \left(\prod_{s=r}^{n-1} \frac{s(l_{r,s}\lambda_r)}{s(\lambda_r)} \right) s(l_{r,n}\lambda_r) \right\} \det[s(K_{q,i}\lambda_j)]_{q+1:n},$$

where the summations are taken over all the sets of integers $K_{u,v}$ ($1 \leq u \leq q$ and $u+1 \leq v \leq n$) satisfying:

$$\begin{cases} K_{u-1,v+1} < K_{u,v} < K_{u-1,v-1} & \text{for} \quad u+1 \leq v \leq n-1 \\ K_{u,n} < K_{u-1,n-1} & \\ 0 < K_{u,n} < K_{u,n-1} < ... < K_{u,u+1}, & \end{cases} \tag{3}$$

$K_{0,v} = H_v$ and for all $1 \leq r \leq q$ and $r \leq s \leq n$, the integers $l_{r,s}$ are given by:

$$\begin{cases} l_{r,r} = K_{r-1,r} - \max(K_{r-1,r+1}, K_{r,r+1}) \\ l_{r,s} = \min(K_{r-1,s}, K_{r,s}) - \max(K_{r-1,s+1}, K_{r,s+1}) \quad \text{for} \quad r+1 \leq s \leq n-1 \\ l_{r,n} = \min(K_{r-1,n}, K_{r,n}). \end{cases}$$

PROOF : The case $q = 1$ is shown by Tsukamoto [11]. Using twice this case, we obtain:

$$\frac{\det[s(H_i\lambda_j)]_{1:n}}{\prod_{i=1}^{2}\prod_{j=i+1}^{n} s(\alpha_{ij})s(\beta_{ij})}$$

$$= \sum_{K_{1,v}}\sum_{K_{2,v}} \left\{ \left(\prod_{s=1}^{n-1} \frac{s(l_{1,s}\lambda_1)}{s(\lambda_1)}\right) \left(\prod_{s=2}^{n-1} \frac{s(l_{2,s}\lambda_2)}{s(\lambda_2)}\right) s(l_{1,n}\lambda_1)s(l_{2,n}\lambda_2) \right\}$$

$$\det[s(K_{2,i}\lambda_j)]_{3:n}.$$

The assertion is proved recursively on q. The details are similar to those of the Lemma 2.2.4 page 52 in [2]. □

***Theorem* 3** — *Let $V = V(\Lambda)$ be an irreducible G-module of highest weight $\Lambda = h_1\lambda_1 + ... + h_n\lambda_n$ satisfying (1). Then the irreducible decomposition of V as a K-module contains an irreducible K-submodule $V' = V'(\Lambda')$ with highest weight $\Lambda' = k_1\lambda_1 + ... + k_q\lambda_q + k_{q+1}\lambda_{q+1} + ... + k_n\lambda_n$ satisfying (2), if and only if:*

1.

$$\begin{cases} h_{i+q} \leq k_i \leq h_{i-q} & \text{for} \quad q+1 \leq i \leq n-q \\ k_i \leq h_{i-q} & \text{for} \quad n-q+1 \leq i \leq n. \end{cases} \tag{4}$$

2. *The multiplicity $m_{\Lambda'}$ of $V' = V'(\Lambda')$ in the decomposition is the coefficient, when it does not vanish, of $e((k_1+q)\lambda_1 + ... + (k_q+1)\lambda_q)$ in:*

$$\prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij}) \sum_{k_{1,v}} \cdots \sum_{k_{q-1,v}} \left\{ \prod_{r=1}^{q} \left(\prod_{s=r}^{n-1} \frac{s(l_{r,s}\lambda_r)}{s(\lambda_r)} \right) \right\}$$

$$s(l_{1,n}\lambda_1)...s(l_{q,n}\lambda_q). \tag{5}$$

where the summations are taken over all the sets of integers $k_{u,v}$, $1 \leq u \leq q-1$ and $u+1 \leq v \leq n$ such that:

- if $2u < 3q - n + 1$:

$$\begin{cases} \max(k_{u-1,v+1}, k_{q,v+q-u}) \leq k_{u,v} \leq k_{u-1,v-1} \\ \qquad\qquad\qquad \text{for} \quad u+1 \leq v \leq n-q+u \\ k_{u-1,v+1} \leq k_{u,v} \leq k_{u-1,v-1} \\ \qquad\qquad\qquad \text{for} \quad n-q+u+1 \leq v \leq 2q-u \\ k_{u-1,v+1} \leq k_{u,v} \leq \min(k_{u-1,v-1}, k_{q,v-q+u}) \\ \qquad\qquad\qquad \text{for} \quad 2q-u+1 \leq v \leq n-1 \\ k_{u,n} \leq \min(k_{u-1,n-1}, k_{q,n-q+u}) \\ 0 \leq k_{u,n} \leq \ldots \leq k_{u,u+1}, \end{cases} \tag{6}$$

- if $2u \geq 3q-n+1$:

$$\begin{cases} \max(k_{u-1,v+1}, k_{q,v+q-u}) \leq k_{u,v} \leq k_{u-1,v-1} \\ \qquad\qquad\qquad \text{for} \quad u+1 \leq v \leq 2q-u \\ \max(k_{u-1,v+1}, k_{q,v+q-u}) \leq k_{u,v} \leq \min(k_{u-1,v-1}, k_{q,v-q+u}) \\ \qquad\qquad\qquad \text{for} \quad 2q-u+1 \leq v \leq n-q+u \\ k_{u-1,v+1} \leq k_{u,v} \leq \min(k_{u-1,v-1}, k_{q,v-q+u}) \\ \qquad\qquad\qquad \text{for} \quad n-q+u+1 \leq v \leq n-1 \\ k_{u,n} \leq \min(k_{u-1,n-1}, k_{q,n-q+u}) \\ 0 \leq k_{u,n} \leq \ldots \leq k_{u,u+1}, \end{cases} \tag{7}$$

with $k_{0,v} = h_v$ and $k_{q,v} = k_v$. The integers $l_{r,s}$ are given by:

$$\begin{cases} l_{r,r} = k_{r-1,r} - \max(k_{r-1,r+1}, k_{r,r+1}) + 1 \\ l_{r,s} = \min(k_{r-1,s}, k_{r,s}) - \max(k_{r-1,s+1}, k_{r,s+1}) + 1 \\ \qquad\qquad\qquad \text{for} \quad r+1 \leq s \leq n-1 \\ l_{r,n} = \min(k_{r-1,n}, k_{r,n}) + 1. \end{cases} \tag{8}$$

PROOF : To decompose an irreducible G-module of highest weight Λ into irreducible K-modules, we will determine the set E of highest weights of K such that:

$$\chi_G(\Lambda) = \sum_{\Lambda' \in E} \chi_K(\Lambda'),$$

where $\chi_G(\Lambda)$ (resp. $\chi_K(\Lambda'))$ is the character of $V(\Lambda)$ (resp. $V'(\Lambda')$.

Using the Weyl character formula, we obtain:

$$\frac{\xi_G(\Lambda+\delta_G)}{\xi_G(\delta_G)} = \sum_{\Lambda'\in E} \frac{\xi_K(\Lambda'+\delta_K)}{\xi_K(\delta_K)}.$$

Then we have to determine the set E such that:

$$\frac{\xi_G(\Lambda+\delta_G)}{\xi_G(\delta_G)/\xi_K(\delta_K)} = \sum_{\Lambda'\in E} \xi_K(\Lambda'+\delta_K), \tag{9}$$

It is well known that (see for instance [1], page 242):

$$\xi_G(\delta_G) = \prod_{\alpha\in\Delta_G^+} (e(\alpha/2) - e(-\alpha/2))$$

and

$$\xi_K(\delta_K) = \prod_{\alpha\in\Delta_K^+} (e(\alpha/2) - e(-\alpha/2)),$$

then

$$\frac{\xi_G(\delta_G)}{\xi_K(\delta_K)} = \prod_{\alpha\in\Delta_G^+-\Delta_K^+} (e(\alpha/2) - e(-\alpha/2)).$$

Writing Λ in the form (1), we have $\Lambda+\delta_G = H_1\lambda_1 + H_2\lambda_2 + ... + H_n\lambda_n$, where $H_i = h_i + n - i + 1$ for all $1 \le i \le n$. The H_i are integers satisfying $H_1 > H_2 > ... > H_n > 0$.

In the same way we have $\Lambda'+\delta_K = K_1\lambda_1 + ... + K_q\lambda_q + K_{q+1}\lambda_{q+1} + ... + K_n\lambda_n$, where $K_i = k_i + q - i + 1$ for all $1 \le i \le q$ and $K_i = k_i + n - i + 1$ for all $q+1 \le i \le n$. The K_i are integers satisfying:

$$K_1 > K_2 > ... > K_q > 0 \quad \text{and} \quad K_{q+1} > K_{q+2} > ... > K_n > 0.$$

Then we obtain:

$$\frac{\xi_G(\delta_G)}{\xi_K(\delta_K)}=\prod_{i=1}^{q}\prod_{j=q+1}^{n}s(\alpha_{ij})s(\beta_{ij}).$$

On the other hand, it is known that (see [1] or [2])

$$\xi_G(\Lambda+\delta_G)=\det\left[s(H_i\lambda_j)\right]_{1:n},$$

and

$$\xi_K(\Lambda'+\delta_K)=\det[s(K_i\lambda_j)]_{1:q}\times\det[s(K_i\lambda_j)]_{q+1:n}.$$

To determine the set E in the equality (9), it suffices to determine the integers $K_1,...,K_n$ such that

$$\frac{\det[s(H_i\lambda_j)]_{1:n}}{\prod_{i=1}^{q}\prod_{j=q+1}^{n}s(\alpha_{ij})s(\beta_{ij})}=\sum_{\substack{K_1>...>K_q>0\\K_{q+1}>...>K_n>0}}\det[s(K_i\lambda_j)]_{1:q}\times\det[s(K_i\lambda_j)]_{q+1:n}$$

Using Lemma 2, we have to determine the integers $K_1,...,K_n$ such that

$$\prod_{i=1}^{q-1}\prod_{j=i+1}^{q}s(\alpha_{ij})s(\beta_{ij})\quad\times\quad\sum_{K_{1,v}}\cdots\sum_{K_{q,v}}\left\{\prod_{r=1}^{q}\left(\prod_{s=r}^{n-1}\frac{s(l_{r,s}\lambda_r)}{s(\lambda_r)}\right)s(l_{r,n}\lambda_r)\right\}\det[s(K_{q,i}\lambda_j)]_{q+1:n}$$

$$=\sum_{\substack{K_1>...>K_q>0\\K_{q+1}>...>K_n>0}}\det[s(K_i\lambda_j)]_{1:q}\times\det[s(K_i\lambda_j)]_{q+1:n}.$$

We permute successively the summations on the $K_{1,v}, \ldots, K_{q,v}$ satisfying (3) to get the first one on $K_{q,v}$ which satisfies:

$$\begin{cases} H_{v+q} + q \le K_{q,v} \le H_{v-q} - q & \text{for} \quad q+1 \le v \le n-q \\ K_{q,v} \le H_{v-q} - q & \text{for} \quad n-q+1 \le v \le n \\ 0 < K_{q,n} < \ldots < K_{q,q+1}, & \end{cases}$$

and the other ones on $K_{1,v}, \ldots, K_{q-1,v}$ such that

- if $2u > n - q - 3$:

$$\begin{cases} a_{q,u,v} < K_{q-u-1,v} < K_{q-u-2,v-1} & \text{for} \quad q-u \le v \le n-u-1 \\ K_{q-u-2,v+1} < K_{q-u-1,v} < K_{q-u-2,v-1} & \text{for} \quad n-u \le v \le q+u+1 \\ K_{q-u-2,v+1} < K_{q-u-1,v} < b_{q,u,v} & \text{for} \quad q+u+2 \le v \le n-1 \\ K_{q-u-1,n} < b_{q,u,n} & \\ 0 < K_{q-u-1,n} < \ldots < K_{q-u-1,q-u}, & \end{cases} \tag{10}$$

- if $2u \le n - q - 3$:

$$\begin{cases} a_{q,u,v} < K_{q-u-1,v} < K_{q-u-2,v-1} & \text{for} \quad q-u \le v \le q+u+1 \\ a_{q,u,v} < K_{q-u-1,v} < b_{q,u,v} & \text{for} \quad q+u+2 \le v \le n-u-1 \\ K_{q-u-2,v+1} < K_{q-u-1,v} < b_{q,u,v} & \text{for} \quad n-u \le v \le n-1 \\ K_{q-u-1,n} < b_{q,u,n} & \\ 0 < K_{q-u-1,n} < \ldots < K_{q-u-1,q-u}, & \end{cases} \tag{11}$$

where

$$\begin{aligned} a_{q,u,v} &= \max(K_{q-u-2,v+1}, K_{q,v+u+1} + u) \\ b_{q,u,v} &= \min(K_{q-u-2,v-1}, K_{q,v-u-1} - u). \end{aligned}$$

Thus, we obtain:

$$\prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij})\times\sum_{K_{q,v}}\sum_{K_{1,v}}\dots\sum_{K_{q-1,v}}\left\{\prod_{r=1}^{q}\left(\prod_{i=s}^{n-1}\frac{s(l_{r,s}\lambda_r)}{s(\lambda_r)}\right)\right\}$$

$$s(l_{1,n}\lambda_1)\dots s(l_{q,n}\lambda_q)\det[s(K_{q,i}\lambda_j)]_{q+1:n}$$

$$=\sum_{\substack{K_1>\dots>K_q>0\\ K_{q+1}>\dots>K_n>0}}\det[s(K_i\lambda_j)]_{1:q}\times\det[s(K_i\lambda_j)]_{q+1:n}.$$

By identifying the left and right terms of the last equality, we get:
$K_i=K_{q,i}$ for all $q+1\leq i\leq n$ and

$$\sum_{K_1>\dots K_q>0}\det[s(K_i\lambda_j)]_{1:q}$$

$$=\prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij})\times\sum_{K_{1,v}}\dots\sum_{K_{q-1,v}}\left\{\prod_{r=1}^{q}\left(\prod_{s=r}^{n-1}\frac{s(l_{r,s}\lambda_r)}{s(\lambda_r)}\right)\right\}$$

$$s(l_{1,n}\lambda_1)\dots s(l_{q,n}\lambda_q),$$

where the conditions on $K_{u,v}$ for $1\leq u\leq q-1$, are (10) and (11). We find:

$$\begin{cases} h_{i+q}\leq k_i\leq h_{i-q} & \text{for}\quad q+1\leq i\leq n-q\\ k_i\leq h_{i-q} & \text{for}\quad n-q+1\leq i\leq n\\ 0\leq k_n\leq\dots\leq k_q. \end{cases}$$

If we denote by:

$$k_{u,v}=K_{u,v}-n+v-1,\quad \text{for all}\ \ 0\leq u\leq q-1\ \ \text{and}\ \ u+1\leq v\leq n,$$

we obtain the result. □

Remark : To understand the multiplicity $m_{\Lambda'}$ of $V'=V'(\Lambda')$ given by the previous theorem, we begin by remarking that for any integer r, we have $\dfrac{s(rx)}{s(x)}=$

$\sum_{k=0}^{r-1} e((2k-r+1)x)$. Then, the equation (5) can be expressed as a summation of terms $e(a_1\lambda_1 + ... + a_q\lambda_q)$ where $a_1, \ldots, a_q$ are integers. If the inequalities (4) are satisfied and the term $e((k_1+q)\lambda_1 + ... + (k_q+1)\lambda_q)$ appears in (5), we deduce that the irreducible decomposition of V as a K-module contains $V' = V'(\Lambda')$ and the multiplicity is the corresponding coefficient.

3. Decomposition of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$

We identify the complexified cotangent space of $M = G/K$ at $o = [K]$ with $(\mathfrak{g}/\mathfrak{k})^*$, the dual space of $\mathfrak{g}/\mathfrak{k}$.

The K-module $(\mathfrak{g}/\mathfrak{k})^*$ is irreducible of highest weight $\lambda_1 + \lambda_{q+1}$.

Notation : Let H and L be two groups, V a H-module and W a L-module. The space $V \otimes W$ has a structure of $H \times L$-module by the action of H on V and L on W. We denote by $V \boxtimes W$ the obtained $H \times L$-module.

Thus, the $\mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$-module $(\mathfrak{g}/\mathfrak{k})^*$ is isomorphic to $V(\lambda_1) \boxtimes V(\lambda_{q+1})$.

3.1. *Particular Case* $K = Sp(2) \times Sp(n-2)$

Let H be the subgroup $T \times T$ of $\mathrm{Sp}(2)$ where T is a torus of $\mathrm{Sp}(1)$. We begin by decomposing the restriction of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ to $H \times \mathrm{Sp}(n-2)$. To restrict $(\mathfrak{g}/\mathfrak{k})^*$, i.e. $V(\lambda_1) \boxtimes V(\lambda_3)$, to $H \times \mathrm{Sp}(n-2)$, we restrict the $\mathrm{Sp}(2)$-module $V(\lambda_1)$ to H.

The decomposition of the $\mathrm{Sp}(2)$-module $V(\lambda_1)$ into irreducible H-submodules:

$$V(\lambda_1)|_H \cong V(\lambda_1) \oplus V(-\lambda_1) \oplus V(\lambda_2) \oplus V(-\lambda_2).$$

We denote by $V_1 = V(\lambda_1) \boxtimes V(\lambda_3)$, $V_2 = V(-\lambda_1) \boxtimes V(\lambda_3)$, $V_3 = V(\lambda_2) \boxtimes V(\lambda_3)$ and $V_4 = V(-\lambda_2) \boxtimes V(\lambda_3)$. Then

$$(\mathfrak{g}/\mathfrak{k})^* \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4 \quad \text{(irreducible } H \times \mathrm{Sp}(n-2)\text{-modules)}.$$

Using the notation $\wedge^{a,b,c,d} = \wedge^a V_1 \otimes \wedge^b V_2 \otimes \wedge^c V_3 \otimes \wedge^d V_4$ ($H \times$ Sp$(n-2)$-module), we get the $H \times$ Sp$(n-2)$-decomposition

$$\wedge^p (\mathfrak{g}/\mathfrak{k})^* \cong \sum \wedge^{a,b,c,d} \qquad \text{with } a+b+c+d = p. \tag{12}$$

On the other hand, the restriction to Sp$(n-2)$ of V_1, V_2, V_3 or V_4 is isomorphic to $V = V(\lambda_3)$. Also, the $H \times$ Sp$(n-2)$-module, $\wedge^{a,b,c,d}$, is isomorphic to:

$$V((a-b)\lambda_1) \boxtimes V((c-d)\lambda_2) \boxtimes (\wedge^a V \otimes \wedge^b V \otimes \wedge^c V \otimes \wedge^d V). \tag{13}$$

It means that it suffices to decompose the Sp$(n-2)$-module $(\wedge^a V \otimes \wedge^b V \otimes \wedge^c V \otimes \wedge^d V)$ into irreducible Sp$(n-2)$-submodules to obtain the decomposition of the $H \times$ Sp$(n-2)$-module $\wedge^{a,b,c,d}$. We suppose that:

$$\wedge^a V \otimes \wedge^b V \otimes \wedge^c V \otimes \wedge^d V \cong \sum V(\mu), \quad \text{(irreducible Sp}(n-2)\text{-modules)}. \tag{14}$$

We obtain:

$$\wedge^{a,b,c,d} \cong \sum_{\mu} V((a-b)\lambda_1) \boxtimes V((c-d)\lambda_2) \boxtimes V(\mu), \quad (H \times \text{Sp}(n-2)\text{-modules}). \tag{15}$$

Notation : We set $\gamma_{j-2} = \lambda_j$ for $3 \leq j \leq n$, and:

$$\begin{array}{rcl} \Gamma_0 & = & 0 \\ \Gamma_j & = & \gamma_1 + ... + \gamma_j \quad \text{for} \quad 1 \leq j \leq n-2 \end{array}$$

The Γ_j for $1 \leq j \leq n-2$ are the fundamental weights of the group Sp$(n-2)$. With these notations, the restriction of $\wedge^{a,b,c,d}$ to Sp$(n-2)$ is isomorphic to:

$$\wedge^a V(\Gamma_1) \otimes \wedge^b V(\Gamma_1) \otimes \wedge^c V(\Gamma_1) \otimes \wedge^d V(\Gamma_1).$$

Proposition 4 —

1. For $0 \leq r \leq n-2$, we have

$$\wedge^r V(\Gamma_1) \cong V(\Gamma_r) \oplus V(\Gamma_{r-2}) \oplus \cdots \oplus V(\Gamma_1) \quad \text{when } r \text{ is odd}$$

and

$$\wedge^r V(\Gamma_1) \cong V(\Gamma_r) \oplus V(\Gamma_{r-2}) \oplus \cdots \oplus V(\Gamma_0) \quad \text{when } r \text{ is even}$$

with

$$\wedge^r V(\Gamma_1) \cong \wedge^{2n-4-r} V(\Gamma_1).$$

2. For $0 \leq r \leq s \leq n-2$, the Sp$(n-2)$-module $V(\Gamma_r) \otimes V(\Gamma_s)$ can be decomposed into irreducible modules as follows:

$$V(\Gamma_r) \otimes V(\Gamma_s) \cong \sum_{i,j} V(\Gamma_i + \Gamma_j),$$

where the indices of the summation (i, j) are non-negative integers satisfying:

$$\begin{cases} s - r \leq j - i & \leq & 2n - 4 - s - r \\ i + j & \leq & r + s \\ i + j & \equiv & r + s \pmod 2. \end{cases}$$

Conclusion

- The previous proposition allows us to decompose $\wedge^r V(\Gamma_1) \otimes \wedge^s V(\Gamma_1)$ into irreducible $Sp(n-2)$-modules.
- The decomposition of $\wedge^{a,b,c,d}$ is reduced to that of $V(\Gamma_i + \Gamma_j) \otimes V(\Gamma_k + \Gamma_l)$ into irreducible Sp$(n-2)$-modules which can be done using the Steinberg multiplicity formula.
- The decomposition of $\wedge^{a,b,c,d}$ into Sp$(2) \times$ Sp$(n-2)$-modules can be done by gathering the irreducible H-modules in irreducible $Sp(2)$-modules.

3.2. *General case*

We consider now the general case $K = \mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$. We consider a torus T of Sp(1). To decompose the K-module $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ into irreducible K-submodules, we begin by decomposing the restriction of $(\mathfrak{g}/\mathfrak{k})^*$ to $T \times \mathrm{Sp}(q-1) \times \mathrm{Sp}(n-q)$, then the restriction of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ to $T \times \mathrm{Sp}(q-1) \times \mathrm{Sp}(n-q)$ and finally, we come back to K as the case $q = 2$.

As $(\mathfrak{g}/\mathfrak{k})^* \cong V(\lambda_1 + \lambda_{q+1})$, it suffices to study the restriction of the $\mathrm{Sp}(q)$-module $V(\lambda_1)$ to $T \times \mathrm{Sp}(q-1)$. It is easy to show that

$$V(\lambda_1)|_{T \times Sp(q-1)} \cong V(\lambda_1) \oplus V(-\lambda_1) \oplus V(\lambda_2),$$

where $V(\lambda_1)$ and $V(-\lambda_1)$ are trivial and $V(\lambda_2)$ is the standard representation of $\mathrm{Sp}(q-1)$. Then:

$$V(\lambda_1 + \lambda_{q+1})|_{T \times Sp(q-1) \times Sp(n-q)} \cong U_1 \oplus U_2 \oplus U_3,$$

where U_1, U_2, U_3 are the irreducible $T \times \mathrm{Sp}(q-1) \times \mathrm{Sp}(n-q)$-modules of highest weights $\lambda_1 + \lambda_{q+1}$, $-\lambda_1 + \lambda_{q+1}$ and $\lambda_2 + \lambda_{q+1}$ respectively.

The decomposition of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ into irreducible K-submodules can be made recursively as follow:

(i) The first step is given by the previous conclusion.

(ii) The restriction of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ to $T \times \mathrm{Sp}(q-1) \times \mathrm{Sp}(n-q)$ can be decomposed as follow:

- $$\wedge^p(\mathfrak{g}/\mathfrak{k})^* \cong \sum_{i+j+k=p} \wedge^i U_1 \otimes \wedge^j U_2 \otimes \wedge^k U_3.$$
- The decomposition of $\wedge^i U_1 \otimes \wedge^j U_2$ is determined by applying the Proposition 4.
- We decompose $\wedge^k U_3$ recursively.

(iii) To obtain the decomposition of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ as $\mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$-module, we regroup the irreducible $T \times \mathrm{Sp}(q-1)$-modules occurring in the decomposition into irreducible $\mathrm{Sp}(q)$-modules.

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