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# $OD$-CHARACTERIZATION OF THE AUTOMORPHISM GROUPS OF $O^{\pm}_{10}(2)$[1]

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The prime graph of a finite group was introduced by Gruenberg and Kegel. The degree pattern of a finite group $G$ associated to its prime graph was introduced in [1] and denoted by $D(G)$. The group $G$ is called $k$-fold $OD$-characterizable if there exist exactly $k$ non-isomorphic groups $H$ satisfying

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conditions (1) $|G| = |H|$ and (2) $D(G) = D(H)$. Moreover, a 1-fold $OD$-characterizable group is simply called an *$OD$-characterizable group*. Till now a lot of finite simple groups were shown to be OD-characterizable, and also some finite groups especially the automorphism groups of some finite simple groups were shown not being OD-characterizable but $k$-fold OD-characterizable for some $k > 1$. In the present paper, the authors continue this topic and show that the automorphism groups of orthogonal groups $O_{10}^{+}(2)$ and $O_{10}^{-}(2)$ are $OD$-characterizable.

**Key words** : Prime graph, degree pattern, degree of a vertex, order component.

## 1. INTRODUCTION

Let $G$ be a finite group, $\pi_e(G)$ denotes the set of the element orders of $G$, and $\pi(G)$ the set of all prime divisors of $|G|$. The prime graph of $G$ was defined by Gruenberg and Kegel (ref. to [2]), which was denoted by $\Gamma(G)$ and constructed as follows: The vertex set of this graph is $\pi(G)$, and two distinct vertices $p$ and $q$ are jointed by an edge if and only if $pq \in \pi_e(G)$, in this case, we say vertices $p$ and $q$ are adjacent and denote this fact as $p \sim q$. The number of connected components of $\Gamma(G)$ is denoted by $s(G)$ and the sets of vertices of connected components of $\Gamma(G)$ are denoted as $\pi_i = \pi_i(G)$ $(i = 1, 2, \ldots, s(G))$. If $|G|$ is even, we always assume that $2 \in \pi_1(G)$. Set $T(G) = \{\pi_i(G)|\ i = 1, 2, \cdots, s(G)\}$.

Let $n$ be a positive integer, we use $\pi(n)$ to denote the set of all prime divisors of $n$. If the prime graph of $G$ is known, then $|G|$ can be expressed as a product of $m_1, m_2, \cdots, m_{s(G)}$, where $m_i$ are positive integers such that $\pi(m_i) = \pi_i$. These $m_i s$ were called the *order components* of $G$ by the second author, who proved a lot of finite simple groups can be uniquely determined by its order components. The set of order components of $G$ is denoted as $OC(G) = \{m_1, m_2, \cdots, m_{s(G)}\}$. All further unexplained notations are referred to [3].

*Definition* 1.2 ([1]) — Let $G$ be a finite group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_i$ are primes and $\alpha_i$ are positive integers. For $p \in \pi(G)$, let

$$deg(p) := |\{q \in \pi(G) | p \sim q\}|,$$

which is called the ***degree*** of $p$. Define

$$D(G) := (deg(p_1), deg(p_2), \ldots, deg(p_k)),$$

where $p_1 < p_2 < \cdots < p_k$. $D(G)$ is called the ***degree pattern*** of $G$.

*Definition* 1.3 ([3]) — A group $M$ is called ***$k$-fold OD-characterizable*** if there exist exactly $k$ non-isomorphic groups $G$ such that $|G| = |M|$ and $D(G) = D(M)$. Moreover, a 1-fold OD-characterizable group is simply called an *$OD$-characterizable group*.

*Definition* 1.4 — Let $S$ be a non-abelian simple group. A group $G$ is said to be an almost simple related to $S$ if and only if $S \leq G \leq Aut(S)$.

In a series of articles such as [1, 4-8 and [9], many finite non-abelian simple groups or almost simple groups were shown to be $OD$-characterizable. For convenience, we recall some of them in the following proposition.

*Proposition* 1.4 — A finite group $G$ is $OD$-characterizable if $G$ is one of the following groups:

(1) All sporadic simple groups and their automorphism groups except $Aut(J_2)$ and $Aut(M^cL)$;

(2) The alternating groups $A_p$, $A_{p+1}$, $A_{p+2}$ and the symmetric groups $S_p$ and $S_{p+1}$, where $p$ is a prime;

(3) All finite simple $K_4$-groups except $A_{10}$;

(4) The simple groups of Lie type $L_2(q)$, $L_3(q)$, $U_3(q)$, ${}^2B_2(q)$ and ${}^2G_2(q)$ for certain prime power $q$;

(5) All finite simple $C_{2,2}$-groups;

(6) The alternating groups $A_{p+3}$, where $p+2$ is a composite number, $p+4$ is a prime and $p \in \pi(1000!)$;

(7) The almost simple groups of $Aut(F_4(2))$ and $2 \cdot F_4(2)$;

(8) All finite almost simple $K_3$-groups except $Aut(A_6)$ and $Aut(U_4(2))$.

In this paper, we continue this topic and come to $Aut(O_{10}^{+}(2))$ and $Aut(O_{10}^{-}(2))$ can be OD-characterizable. In order to unify expression of results about two groups, we introduce the following notation. Let $\epsilon \in \{+,-\}$, we use $O_{10}^{\epsilon}(2)$ to denote any one of $O_{10}^{+}(2)$ and $O_{10}^{-}(2)$. Then we can write our main result as following Main Theorem.

*Main Theorem* — If $G$ is a finite group such that $|G| = |Aut(O_{10}^{\epsilon}(2))|$ and $D(G) = D(Aut(O_{10}^{\epsilon}(2)))$, then $G \cong Aut(O_{10}^{\epsilon}(2))$. In particular, $Aut(O_{10}^{\epsilon}(2))$ is OD-characterizable.

## 2. Preliminaries

In this section, we consider some results which will be needed for our further investigations.

*Lemma* 2.1 [10] — Let $S = P_1 \times P_2 \times \cdots \times P_r$, where $P_i$ are isomorphic non-abelian simple groups. Then $Aut(S) = [Aut(P_1) \times Aut(P_2) \times \cdots \times Aut(P_r)] \rtimes \mathbb{S}_r$

*Lemma* 2.2 [3, 11] — Let $S$ be a finite non-abelian simple group. Then the following assertions hold.

(a) If $\pi(S) \subseteq \{2,3,5,7,17,31\}$, then $S$ is isomorphic to one of the following simple groups listed in Table 1.

(b) If $\pi(S) \subseteq \{2,3,5,7,11,17\}$, then $S$ is isomorphic to one of the following simple groups listed in Table 2.

In particular, if $\pi(Out(S)) \neq \varnothing$, then $\pi(Out(S)) \subseteq \{2,3\}$.

**Table** 1. **Finite non-abelian simple groups with** $\pi(G) \subseteq \{2,3,5,7,17,31\}$

| $G$ | $\|G\|$ | $\|Out(G)\|$ | $G$ | $\|G\|$ | $\|Out(G)\|$ |
|---|---|---|---|---|---|
| $A_5$ | $2^2 \cdot 3 \cdot 5$ | 2 | $U_3(5)$ | $2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ | 6 |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | 2 | $S_4(7)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ | 2 |
| $A_6$ | $2^3 \cdot 3^2 \cdot 5$ | 4 | $A_9$ | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 2 |
| $L_2(8)$ | $2^3 \cdot 3^2 \cdot 7$ | 3 | $O_8^+(2)$ | $2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$ | 6 |
| $He$ | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ | 2 | $A_{10}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 2 |
| $O_8^-(2)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$ | 2 | $L_2(31)$ | $2^5 \cdot 3 \cdot 5 \cdot 31$ | 2 |
| $L_4(4)$ | $2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$ | 4 | $U_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7$ | 8 |
| $A_8$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $L_5(2)$ | $2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ | 2 |
| $L_3(4)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 12 | $L_6(2)$ | $2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$ | 2 |
| $U_4(2)$ | $2^6 \cdot 3^4 \cdot 5$ | 2 | $O_{10}^+(2)$ | $2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$ | 2 |
| $S_8(2)$ | $2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$ | 1 | $L_3(5)$ | $2^5 \cdot 3 \cdot 5^3 \cdot 31$ | 2 |
| $L_2(17)$ | $2^4 \cdot 3^2 \cdot 17$ | 2 | $L_2(5^3)$ | $2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$ | 6 |
| $L_2(16)$ | $2^4 \cdot 3 \cdot 5 \cdot 17$ | 4 | $S_6(2)$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7$ | 1 |
| $A_7$ | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $L_2(49)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ | 4 |
| $S_4(4)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 17$ | 4 | $G_2(5)$ | $2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$ | 1 |
| $U_3(3)$ | $2^5 \cdot 3^3 \cdot 7$ | 2 | $J_2$ | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 2 |

**Table** 2. **Finite non-abelian simple groups with** $\pi(S) \subseteq \{2, 3, 5, 7, 11, 17\}$

| $S$ | $\|S\|$ | $\|Out(S)\|$ | $S$ | $\|S\|$ | $\|Out(S)\|$ |
|---|---|---|---|---|---|
| $A_5$ | $2^2 \cdot 3 \cdot 5$ | 2 | $U_3(5)$ | $2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ | 6 |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | 2 | $S_4(7)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ | 2 |
| $A_6$ | $2^3 \cdot 3^2 \cdot 5$ | 4 | $A_9$ | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 2 |
| $L_2(8)$ | $2^3 \cdot 3^2 \cdot 7$ | 3 | $O_8^+(2)$ | $2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$ | 6 |
| $L_2(17)$ | $2^4 \cdot 3^2 \cdot 17$ | 2 | $L_2(11)$ | $2^2 \cdot 3 \cdot 5 \cdot 7$ | 3 |
| $L_2(16)$ | $2^4 \cdot 3 \cdot 5 \cdot 17$ | 4 | $U_5(2)$ | $2^{10} \cdot 3^5 \cdot 5 \cdot 11$ | 2 |
| $A_7$ | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $U_6(2)$ | $2^{15} \cdot 3^6 \cdot 5 \cdot 7$ | 6 |
| $S_4(4)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 17$ | 4 | $A_{11}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$ | 2 |
| $U_3(3)$ | $2^5 \cdot 3^3 \cdot 7$ | 2 | $J_2$ | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 2 |
| $He$ | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ | 2 | $A_{10}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 2 |
| $O_8^-(2)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$ | 2 | $A_{12}$ | $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$ | 2 |
| $L_4(4)$ | $2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$ | 4 | $U_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7$ | 8 |
| $A_8$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $M_{11}$ | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | 1 |
| $L_3(4)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 12 | $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 2 |
| $U_4(2)$ | $2^6 \cdot 3^4 \cdot 5$ | 2 | $M_{22}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 2 |
| $S_8(2)$ | $2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$ | 1 | $HS$ | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ | 2 |
| $L_2(49)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ | 4 | $M^cL$ | $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ | 2 |
| $O_{10}^-(2)$ | $2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$ | 2 | $S_6(2)$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7$ | 1 |

*Lemma* 2.3 [12] — Let $G$ be a Frobenius group with kernel $F$ and complement $C$. Then the following assertions hold.

(a) $F$ is a nilpotent group.

(b) $|F| \equiv 1(mod\ |C|)$.

*Lemma* 2.4 [13] — Let $G$ be a Frobenius group of even order with $H$ and $K$ its Frobenius kernel and Frobenius complement, respectively. Then $s(G) = 2$ and T(G)= $\{\pi(K), \pi(H)\}$.

*Lemma* 2.5 ([2] Theorem A) — Let $G$ be a finite group with $s(G) \geq 2$, then G is one of the following groups:

(a) $G$ is a Frobenius or 2-Frobenius group;

(b) $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $H$ and $G/K$ are $\pi_1$-groups and $K/H$ is a non-abelian simple, where $\pi_1$ is the prime graph component containing $2$, $H$ is a nilpotent group, and $|G/H| \Big| \mid \text{Aut } (K/H)|$. Moreover, any odd order component of $G$ is also an odd order component of $K/H$.

*Remark* 1 : A group $G$ is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$ respectively.

*Lemma* 2.6 [14] — Let $G$ be a finite group such that $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following assertions hold.

(a) There is a finite non-abelian simple group $S$ such that $S \leq \overline{G} = G/K \leq Aut(S)$ for the maximal normal solvable subgroup $K$ of $G$. Furthermore, $t(S) \geq t(G) - 1$.

(b) One of the following statements holds:

(1) $S \cong A_7$ or $L_2(q)$ for some odd prime power $q$ and $t(S) = t(2, S) = 3$;

(2) For every prime $r \in \pi(G)$ non-adjacent to 2 in $\Gamma(G)$, a Sylow $r$-subgroup of $G$ is isomorphic to a Sylow $r$-subgroup of $S$. In particular, $t(2, S) \geq t(2, G)$.

## 3. Proof of The Main Theorem

Now we start to prove the Main Theorem. At first we prove the Main Theorem holds for $Aut(O^+_{10}(2))$.

*Lemma* 3.1 — If $G$ is a finite group such that $|G| = |Aut(O_{10}^+(2))|$ and $D(G) = D(Aut(O_{10}^+(2)))$, then $G \cong Aut(O_{10}^+(2))$. In particular, $Aut(O_{10}^+(2))$ is $OD$-characterizable.

PROOF : By the hypothesis we have that $|G| = |Aut(O_{10}^+(2))| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$ and $D(G) = D(Aut(O_{10}^+(2))) = (4, 4, 2, 2, 2, 0)$. Hence $\{2, 3, 5, 6, 7, 10, 14, 17, 31, 34\} \subseteq \pi_e(G)$ and $\{35, 85, 119, 31p\} \cap \pi_e(G) = \varnothing$ for any $p \in \pi(G)$. Evidently, $s(G) = 2$, $\pi_1(G) = \{2, 3, 5, 7, 17\}$ and $\pi_2(G) = \{31\}$.

We assert that $G$ is neither Frobenius nor 2-Frobenius group. Otherwise, if $G$ is a Frobenius group, then $G = NH$, where $N$ is the Frobenius kernel and $H$ the Frobenius complement, and $T(G) = \{\pi(N), \pi(H)\} = \{\{2, 3, 5, 7, 17\}, \{31\}\}$. Since $|H|$ divides $|N| - 1$ by Lemma 2.3, it follows that $|N| = 2^{31} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$ and $|H| = 31$. But at this moment, $H$ cannot act fixed-point-freely on the Sylow 17-subgroup $P_{17}$ as $31 \nmid (17 - 1)$, a contradiction.

If $G$ is an 2-Frobenius group as described in Remark 1. Then $T(G) = \{\pi(H) \bigcup \pi(G/K), \pi(K/H)\}$, it follows that $|K/H| = 31$. On the other hand, $G/K \leq Aut(K/H) \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, and so $|G/K||30$, which implies $\{7, 17, 31\} \subseteq \pi(K)$. Consequently, it follows that $7,\ 17 \in \pi(H)$. Since $H$ is nilpotent, the 7- Sylow subgroup of $H$ is normal in $G$, which makes the element of order 31 can act on $7-$Sylow subgroup. But it easy to see the action is trivial, which is a contradiction to $K$ is a Frobenius group.

Now, by Lemma 2.5, $G$ has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that $N$ and $G/G_1$ are $\pi_1$-groups and $G_1/N$ is a non-abelian simple group, $N$ is a nilpotent group. Note that the prime graph of $G_1/N$ must have one odd component as the same as the odd component $\{31\}$, it follows by Lemma 2.2 and Table 1 that $G_1/N$ is isomorphic to one of the following simple groups: $L_2(5^3)$, $G_2(5)$, $L_5(2)$, $L_6(2)$ and $O_{10}^+(2)$.

Comparing the orders of above simple groups with $|Aut(O_{10}^+(2))|$, we see that

$G_1/N$ cannot be one of $L_2(5^3)$, $G_2(5)$ and $L_6(2)$. That is to say, $G_1/N \cong L_5(2)$ or $O_{10}^+(2)$.

If $G_1/N \cong L_5(2)$, then $5 \in \pi(N)$ by Table 1, hence the element of order 31 must fixed-point-freely on a subgroup of order 5 in $N$, which implies that 31 and 5 are jointed in the prime graph of $G$, a contradiction.

If $G_1/N \cong O_{10}^+(2)$, then $O_{10}^+(2) \leq G/N \leq Aut(O_{10}^+(2))$ since $G/N \leq Aut(G_1/N)$.

If $G/N \cong Aut(O_{10}^+(2))$, then $N = 1$ and $G \cong Aut(O_{10}^+(2))$ by $|G| = |Aut(O_{10}^+(2))|$.

If $G/N \cong O_{10}^+(2)$, then $|N| = 2$ and so $N \leq Z(G)$. Therefore $G$ has an element of order $2 \cdot 31$, a contradiction to that 31 is an isolated point of the prime graph of $G$. This completes the proof of Lemma 3.1.

Now we prove the Main Theorem holds for $Aut(O_{10}^-(2))$.

*Lemma* 3.2 — If $G$ is a finite group such that $|G| = |Aut(O_{10}^-(2))|$ and $D(G) = D(Aut(O_{10}^-(2)))$, then $G \cong Aut(O_{10}^-(2))$. In particular, $Aut(O_{10}^-(2))$ is $OD$-characterizable.

PROOF : By the hypothesis we have that $|G| = |Aut(O_{10}^-(2))| = 2^{21} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$ and $D(G) = D(Aut(O_{10}^+(2))) = (4,4,3,3,1,1)$. Hence $\{2,3,5,6,7,10,11,14,15,17,21,35\} \subseteq \pi_e(G)$ and $\{55,77,85,119,187\} \cap \pi_e(G) = \varnothing$. By $deg(17) = 1$, there exists unique prime $p_2 \in \pi(G)$ such that $17 \sim p_2$ in $\Gamma(G)$. Because $|\pi(G)| = 6$, there are another 4 primes except 17 and $p_2$, which are not adjacent to 17. If these four vertices, say $p_3$, $p_4$, $p_5$ and $p_6$, are pairwise adjacent, then $deg(p_3) = deg(p_4) = deg(p_5) = deg(p_6) \geq 3$, and so $deg(p_2) \geq 2$, which contradicts our assumption $D(G) = (4,4,3,3,1,1)$. Hence there exist at least two vertices, for example, $p_3$ and $p_4$, being non-adjacent to each other. For convenience, denote $p_1 = 17$. Let $\triangle$ be an independent set of $\Gamma(G)$

and $\triangle' = \{p_1, p_3, p_4\}$. It follows that $|\triangle'| \leq |\triangle|$. Since $\triangle' = \{p_1, p_3, p_4\}$, it follows that $t(G) \geq 3$. Furthermore, $t(2, G) \geq 2$ since $deg(2) = 4$ and $|\pi(G)| = 6$. By Lemma 2.6, there is a finite non-abelian simple group $S$ such that $S \leq \overline{G} = G/K \leq Aut(S)$, where $K$ is a maximal normal solvable subgroup of $G$. Since $S \leq \overline{G} = G/K$ and $\pi(G) = \{2,\ 3,\ 5,\ 7,\ 11,\ 17\}$, it follows that $\pi(S) \subseteq \{2,\ 3,\ 5,\ 7,\ 11,\ 17\}$, that is, $S$ is isomorphic to one of the finite simple groups in Table 2. In the following, we continue the proof in two cases.

*Case* 1 : Suppose $17 \notin \pi(S)$.

If $17 \notin \pi(S)$, then $17 \notin \pi(Aut(S))$ by Lemma 2.2. It follows that $17 \in \pi(K)$ since $S \leq \overline{G} = G/K \leq Aut(S)$. Let $T = \{5, 7, 11\}$. We assert that $17 \sim r$ for any $r \in T$.

If $r \in \pi(K)$, Then one can take a 17-Sylow subgroup $P_{17}$ and a $r$-Sylow subgroup $P_r$ of $K$ such that $P_{17}P_r$ is a Hall $\{17, r\}$-subgroup of $K$ since K is solvable. It follows that $|P_{17}P_r| = 17 \cdot r$ or $17 \cdot 5^2$ since $|G|_{17} = 17, |G|_{11} = 11$ and $|G|_5 = 5^2$. Calculating the number of Sylow of subgroups of $P_{17}P_r$, one has that $P_{17}P_r$ is nilpotent and so $17 \sim r$ in $\Gamma(G)$.

If $r \notin \pi(K)$, then $G = KN_G(P_{17})$ by Frattini argument. Therefore the normalizer $N_G(P_{17})$ contains an element of order $r$, say $x$. Thus $< x > P_{17}$ is a nilpotent subgroup of $G$ of order $17 \cdot r$ or $17 \cdot 5^2$. Hence $17 \sim r$ in $\Gamma(G)$. Now the assertion holds.

By checking groups in Table 2 one-by-one, we have either $r \in \pi(K)$ or $r \notin \pi(K)$ for all $r \in T$. That is to say, the assertion holds for all groups in Table 2. Hence $deg(17) \geq 3$, a contradiction.

*Case* 2 : Suppose $17 \in \pi(S)$.

If $17 \in \pi(S)$, then $17 \notin \pi(K)$ since $|G|_{17} = 17$ and $S \leq \overline{G} = G/K \leq Aut(S)$. According to Table 2, $S$ is isomorphic to one of the following simple groups: $L_2(2^4)$, $L_2(13^2)$, $L_3(2^4)$, $L_4(2^2)$, $L_2(17)$, $U_4(2^2)$, $U_3(17)$, $S_4(2^2)$,

$S_4(13)$, $S_6(2^2)$, $S_8(2)$, $O_7(2^2)$, $O_9(2)$, $O_8^+(2)$, $O_8^-(2)$, $O_{10}^-(2)$, $F_4(2)$, $A_{17}$, $A_{18}$, $A_{19}$, $A_{20}$, $A_{21}$, $A_{22}$, $He$, $J_3$ and ${}^2E_6(2)$.

Comparing the orders of $G$ and one of the following simple groups $L_2(13^2), L_3(2^4)$, $U_4(2^2)$, $U_3(17)$, $S_4(13)$, $S_6(2^2)$, $O_7(2^2)$, $O_8^+(2)$, $F_4(2)$, $A_{17}$, $A_{18}$, $A_{19}$, $A_{20}$, $A_{21}$, $A_{22}$, $He$, $J_3$ and ${}^2E_6(2)$, we get that $S$ cannot be any one of these groups.

If $S \cong L_2(2^4)$, then $K$ is a $\{3,5,7,11\}$-group since $|\pi(Out(L_2(2^4)))| = 2$. Because $K$ is solvable, we can consider a Hall $\{5,11\}$-subgroup $P_5P_{11}$ and a Hall $\{7,11\}$-subgroup $P_7P_{11}$. It is easy to get that $|P_5P_{11}| = 5^2{\cdot}11$ and $|P_7P_{11}| = 7{\cdot}11$ by the order of $G$. Consequently $P_5P_{11}$ and $P_7P_{11}$ are abelian groups. Hence $5 \sim 11$ and $7 \sim 11$, which implies that $deg(11) \geq 2$, a contradiction.

If $S \cong L_4(4)$, then $K$ is an $\{2,3,11\}$-group. Let $T = \{5,7,17\}$ and $P_{11} \in Syl_{11}(K)$, then $G = KN_G(P_{11})$ by Frattini argument. Thus $T \subseteq N_G(P_{11})$. Now using N-C Theorem the factor group $N_G(P_{11})/C_G(P_{11})$ is isomorphic to a subgroup of $Aut(P_{11}) \cong \mathbb{Z}_{10}$. Hence $|N_G(P_{11})/C_G(P_{11})| | 10$, which implies that 7 and 17 divides $|C_G(P_{11})|$. Thus $7 \sim 11$, $17 \sim 11$, and $deg(11) \geq 2$, a contradiction.

By the same reason as above we can prove that $S$ cannot be isomorphic to one of the following simple groups: $L_2(17)$, $S_4(2^2)$, $S_8(2)$, $O_9(2)$ and $O_8^-(2)$.

Therefore $S$ can only be isomorphic to the simple group $O_{10}^-(2)$. Since $S \leq \overline{G} = G/K \leq Aut(S)$, we obtain that $O_{10}^-(2) \leq G/K \leq Aut(O_{10}^-(2))$.

If $G/K \cong Aut(O_{10}^-(2))$, then $K = 1$ and $G \cong Aut(O_{10}^-(2))$ since $|G| = |Aut(O_{10}^-(2))|$.

If $G/K \cong O_{10}^-(2)$, then $|K| = 2$ and so $K \leq Z(G)$. Therefore, $G$ is a central extension of $\mathbb{Z}_2$ by $O_{10}^-(2)$. By [3], it is easy to see that $G$ can only be a split extension of $\mathbb{Z}_2$ by $O_{10}^-(2)$. Hence, $G \cong \mathbb{Z}_2 \times O_{10}^-(2)$. In the case that $deg(11) = 2$, which leads to a contradiction. This completes the proof of the

Lemma 3.2.

*Proof of Main Theorem* : Main Theorem follows from Lemma 3.1 and Lemma 3.2.

In 1989, W. J. Shi in [15] put forward the following conjecture:

*Conjecture* : Let $G$ be a group and $M$ a finite simple group. Then $G \cong M$ if and only if (1) $|G| = |M|$ and (2) $\pi_e(G) = \pi_e(M)$.

In fact, this conjecture is valid for $Aut(O_{10}^{\epsilon}(2))$ since $|G| = |Aut(O_{10}^{\epsilon}(2))|$ and $\pi_e(G) = \pi_e(Aut(O_{10}^{\epsilon}(2)))$ implies that $|G| = |Aut(O_{10}^{\epsilon}(2))|$ and $D(G) = D(Aut(O_{10}^{\epsilon}(2)))$. Hence we have the following corollary.

*Corollary* 1 — If $G$ is a finite group such that $|G| = |Aut(O_{10}^{\epsilon}(2))|$ and $\pi_e(G) = \pi_e(Aut(O_{10}^{\epsilon}(2)))$. Then $G \cong Aut((O_{10}^{\epsilon}(2)))$.

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