Indian J. Pure Appl. Math., **43**(3): 211-225, June 2012

COMMUTATIVITY OF NEAR-RINGS WITH DERIVATIONS BY USING ALGEBRAIC SUBSTRUCTURES

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(*Received* 12 *October* 2011; *after final revision* 21 *February* 2012;
accepted 15 *March* 2012)

In this paper we use subsets and algebraic substructures of a 3-prime near-ring R admitting a derivation d to study the commutativity of the subsets, the algebraic substructures and the near-ring R under suitable conditions on d, R and the algebraic substructures. The results obtained in this paper generalize several commutativity theorems due to some authors.

Key words : Commutativity of near-rings; derivations; 3-prime near-ring; non-left zero-divisors; semigroup ideal.

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1. Introduction

Throughout this paper, R is a left near-ring and $Z(R)$ is the multiplicative center of R. We say that R is 3-prime if, for all $x, y \in R$ ($xRy = \{0\}$ implies $x = 0$ or $y = 0$). We say that U is a semigroup right (left) ideal of R, if U is a non-empty subset of R satisfies $UR \subseteq U$ ($RU \subseteq U$). We say that U is a semigroup ideal if it is both a semigroup right and left ideal. A map $d : R \to R$ is a derivation on R if d is an additive mapping and $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. If a map $d : R \to R$ satisfies $d(a + b) = d(a) + d(b)$ and $d(ab) = ad(b) + d(a)b$ for all $a, b \in S$, where S is a non-empty subset of R, then we say that d acts as a derivation on S. An element $x \in R$ is called a left (right) zero divisor in R if there exists a non-zero element $y \in R$ such that $xy = 0$ ($yx = 0$). A zero divisor is either a left or a right zero divisor. By a zero-symmetric element $r \in R$, we mean that r satisfies $0r = 0$. If $xr = r$ for all $x \in R$, then r is called a constant element. A near-ring is called zero-symmetric if all of its elements are zero-symmetric elements. For subsets $X, Y \subseteq R$, the symbol $[X, Y]$ will denote the set $\{xy - yx | x \in X, y \in Y\}$. We refer the reader to the books of Meldrum [8] and Pilz [9] for basic definitions and results of near-ring theory.

The study of commutativity of 3-prime near-rings by using derivations was initiated by Bell and Mason in 1987 in [5]. In [6] Hongan generalizes some results of Bell and Mason by using an ideal in a 3-prime near-ring instead of the near-ring itself. Bell generalizes several results of [5] and [6] by using one (two) sided semigroup ideal of the near-ring in his work in [2]. Also, by using semigroup ideals, Bell and Argac generalize other results concerning commutativity on 3-prime near-rings in their works in [1] and [3], especially they generalize Posner's first theorem (see [10]).

In this paper we prove several results concerning commutativity of subsets, algebraic substructures of a near-ring admitting a non-zero derivation and consequently commutativity of the near-ring itself. In Section 2 we give some recent known results and we prove some auxiliary lemmas concerning a suitable algebraic substructure of a near-ring R admitting a non-zero derivation d, where R is either 3-prime or has an element in the algebraic substructure which is not a left zero divisor in R. These results will be very useful in the sequel. As a consequence of Proposition 2.1, any near-ring admitting a derivation is zero-symmetric. In Proposition 2.8 we give the conditions under which the following statements are equivalent for all integer $n \geqslant 2$: (i) $d(nR) = \{0\}$, (ii) $d(nU) = \{0\}$, (iii) $nU = \{0\}$ and (iv) $nR = \{0\}$, where U is a non-zero semigroup ideal of R.

Section 3 is devoted to studying the commutativity of a near-ring admitting a non-zero derivation d such that $d(uv) = d(vu)$ for all $u \in U, v \in V$, where U and V are suitable non-zero algebraic substructures of the near-ring. As a consequence of the results obtained in this section, we generalize Theorem 3 due to Bell and Daif in [4], Theorem 4.1 due to Bell in [2], Theorem 3.4 and Theorem 3.5 due to Bell and Argac in [3] and Theorem 3.9 due to Kamal and Al-Shaalan in [7].

Section 4 is devoted to studying commutativity under the condition $d(uv) = -d(vu)$ for all u, v in an algebraic substructure of a near-ring R admitting a non-zero derivation d to obtain that R is a commutative ring of characteristic 2. Section 5 is devoted to studying commutativity of a near-ring admitting two non-zero derivations d and g such that $d(v)g(u) = g(u)d(v)$ for all $v \in V$ and $u \in U$, where V is a non-zero semigroup ideal of R and U is a non-zero one sided semigroup ideal of R. As a consequence of the results obtained in this section we generalize Theorem 3.1 in [2] and Theorem 4.13 in [7].

2. Preliminaries and Some Results

Throughout this section we give some well-known basic lemmas and prove some auxiliary results which will be used in the next sections of the paper. We begin first by the following result.

Proposition 2.1 — Let U be a non-empty subset of a near-ring R. Suppose d is an additive map on R which acts as a derivation on U. Then

(i) U contains no non-zero constant elements.

(ii) If $0u \in U$ for all $u \in U$, then U consists of zero-symmetric elements. In particular, if U is a semigroup left ideal of R, then U consists of zero-symmetric elements.

PROOF : (i) Suppose there exists a constant element $z \in U - \{0\}$. Thus, $d(z) = d(zz) = zd(z) + d(z)z = zd(z) + z$. Since $xzd(z) = zd(z)$ for all $x \in R$, we have that $zd(z)$ is a constant element and then

$$xd(z) = x(zd(z) + z) = xzd(z) + xz = zd(z) + z = d(z)$$

for all $x \in R$ which means that $d(z)$ is also a constant element. It follows that $zd(z) = d(z)$ and $d(z) = zd(z) + z = d(z) + z$. Hence, $z = 0$, a contradiction. Therefore, U contains no non-zero constant elements.

(ii) For all $u \in U$, we have that $0u$ is a constant element lies in U. From (i), we get that $0u = 0$. Therefore, U consists of zero-symmetric elements.

From Proposition 2.1, any near-ring admitting a derivation is zero-symmetric.

Lemma 2.2 — Let R be a near-ring.

(i) [2, Lemma 1.2(iii)] If $x \in Z(R)$ is not a zero divisor in R and either yx or xy in $Z(R)$, then $y \in Z(R)$.

(ii) [5, Lemma 3(ii)] If $x \in Z(R)$ is not a zero divisor in R and $x + x \in Z(R)$, then $(R, +)$ is abelian.

(iii) [5, Lemma 3(i)] If R is 3-prime and $x \in Z(R) - \{0\}$, then x is not a zero divisor in R.

Moreover, if R admits a derivation d, then we have the following results

(iv) [11, Lemma 2] $x \in Z(R)$ implies $d(x) \in Z(R)$.

(v) [11, Proposition 1] $d(xy) = xd(y) + d(x)y = d(x)y + xd(y)$ for all $x, y \in R$.

(vi) [5, Lemma 1] R satisfies the partial distributive law: $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$ for all $x, y, z \in R$.

Lemma 2.3 [2, Lemma 1.3(i)] — Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U. If $Ux = \{0\}$ ($xU = \{0\}$) for some $x \in R$, then $x = 0$.

Lemma 2.4 [2, Lemma 1.3(iii)] — Let R be a 3-prime near-ring with a non-zero semigroup right ideal U. If an element x of R centralizes U, then $x \in Z(R)$.

Lemma 2.5 [2, Lemma 1.5] — Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U. If $U \subseteq Z(R)$, then R is a commutative ring.

Lemma 2.6 [2, Lemma 1.4] — Let R be a 3-prime near-ring with a non-zero semigroup ideal U.

(i) If $x, y \in R$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.

(ii) Let d be a non-zero derivation on R. If $x \in R$ and $d(U)x = \{0\}$ ($xd(U) = \{0\}$), then $x = 0$.

Lemma 2.7 [2, Lemma 1.3(ii)] — Let R be a 3-prime near-ring with a non-zero derivation d and a non-zero semigroup right (left) ideal U. Then $d(U) \neq \{0\}$.

Proposition 2.8 — Let R be a 3-prime near-ring with a non-zero semigroup ideal U and a non-zero derivation d. Then for every integer $n \geqslant 2$, the following statements are equivalent:

(i) $d(nR) = \{0\}$.

(ii) $d(nU) = \{0\}$.

(iii) $nU = \{0\}$.

(iv) $nR = \{0\}$.

PROOF : (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) Since for all $u \in U, r \in R$, we have $0 = d(nru) = d(r(nu)) = rd(nu) + d(r)(nu) = d(r)(nu)$. By Lemma 2.6(ii), we get $nU = \{0\}$.

(iii) $\Rightarrow$ (iv) Since for all $u \in U, r \in R$, we get $0 = nur = u(nr)$. Thus, $nR = \{0\}$ by Lemma 2.3.

(iv) $\Rightarrow$ (i) is trivial.

Proposition 2.9. — Let R be a 3-prime near-ring with a non-zero semigroup left ideal U such that its elements are distributive in R. Then R is a ring.

PROOF : For all $x, y, r \in R, u \in U$, we have $[(x + y)r - yr - xr]u = (x + y)ru - yru - xru = 0$. Since R is 3-prime, we have $(x + y)r = xr + yr$ and R is a distributive near-ring. By using Theorem 9.28 in [8], the elements of R^2 commute with each other under the addition operation. So $r(a+b-a-b) = ra+rb+r(-a)+r(-b) = 0$ for all $r, a, b \in R$. Thus, we have $R(a+b-a-b) = \{0\}$ for all $a, b \in R$. The primeness of R implies that $a + b = b + a$ for all $a, b \in R$ and $(R, +)$ is abelian. Therefore, R is a ring.

Lemma 2.10. — Let R be a near-ring with a non-zero derivation d such that $d(vu) = d(uv)$ (resp. $d(vu) = -d(uv)$) for all $v \in V, u \in U$, where U is a non-empty subset of R and V is a non-empty subsemigroup of $(R, \cdot)$ such that $UV \subseteq V$. Then $d(u)v[wu - uw] = 0$ (resp. $d(u)v[wu - (-u)(-w)] = 0$) for all $v, w \in V, u \in U$. Moreover, if there exists $a \in U$ such that $d(a)$ is not a left zero-divisor in R, then $uv = vu$ (resp. $uv = -vu$) for all $u \in U, v \in V$.

PROOF : For all $u \in U, v \in V$, we have $d((uv)u) = d(u(uv))$. So $0 = d(u(uv-vu)) = ud(vu-uv)+d(u)(vu-uv) = d(u)(vu-uv)$. Then $d(u)vu = d(u)uv$. Replacing v by vw, where $w \in V$, we get $d(u)vwu = d(u)uvw = d(u)vuw$ and hence $d(u)v[wu - uw] = 0$ for all $v, w \in V, u \in U$.

If there exists $a \in U$ such that $d(a)$ is not a left zero-divisor in R, then from $d(u)(vu - uv) = 0$, we have $va = av$ for all $v \in V$. Using $UV \subseteq V$, we get $d(auv) = d(uva) = d(u(av)) = d(avu)$ which means $d(a(uv - vu)) = d(a)[uv - vu] = 0$ and then $uv = vu$ for all $u \in U, v \in V$. The proof for the second case is similar.

3. THE CONDITION $d(uv) = d(vu)$

In this section, we study the commutativity of addition and multiplication operations of some kinds of near-rings, each admitting a non-zero derivation d satisfying the condition $d(uv) = d(vu)$ on some suitable algebraic substructures of the near-ring. As a consequence of results obtained in this section, we generalized some results due to Bell, Daif and Argac.

***Theorem* 3.1** — *Let R be a 3-prime near-ring with a non-empty subset U and a non-zero semigroup ideal V. If R admits a non-zero derivation d such that $d(vu) = d(uv)$ for all $v \in V, u \in U$ and $d(U) \neq \{0\}$, then $U \subseteq Z(R)$ and $(R, +)$ is abelian. In particular, if $U = \{r\} \neq \{0\}$, then either ($r \in Z(R)$ and $(R, +)$ is abelian) or $d(r) = 0$.*

PROOF : From Lemma 2.10, we deduce that $d(u)v[wu - uw] = 0$ for all $v, w \in V, u \in U$. By Lemma 2.6(i), we have for all $u \in U$ either $d(u) = 0$ or u centralizes V. Taking $u = a$ such that $0 \neq d(a) \in d(U) \neq \{0\}$, we get $a \in Z(R) - \{0\}$ by Lemma 2.4. Therefore, a is not a left zero-divisor in R by Lemma 2.2(iii). Since $a \in Z(R)$, we have $d((av)u) = d(u(av)) = d(a(uv))$ and then $ad(vu - uv) + d(a)(vu - uv) = 0$ for all $v \in V, u \in U$. Therefore, $d(a)(vu - uv) = 0$ for all $v \in V, u \in U$. Using Lemma 2.2(iv), we obtain $d(a) \in Z(R) - \{0\}$, so it is not a left zero-divisor in R. Thus, $vu = uv$ for all $v \in V, u \in U$. Hence, U centralizes V and $U \subseteq Z(R)$ by Lemma 2.4. Now $a^2 \in Z(R)$, whence $d(a^2) \in Z(R)$ by Lemma 2.2(iv). Then $ad(a) + d(a)a = 2d(a)a = d(a)(2a) \in Z(R)$. Using Lemma 2.2(iii), $d(a) \neq 0$ is not a left zero-divisor in R. It follows that $2a \in Z(R)$ by Lemma 2.2(i). Hence, $(R, +)$ is abelian by Lemma 2.2(ii).

The following corollary generalizes Theorem 4.1 of [2], Theorem 3.4 of [3], Theorem 3 of [4] and Theorem 3.9 of [7]

Corollary 3.2 — Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U and a non-zero semigroup ideal V. If R admits a non-zero derivation d such that $d(vu) = d(uv)$ for all $v \in V, u \in U$, then R is a commutative ring.

PROOF : From Theorem 3.1 and Lemma 2.7, we have $U \subseteq Z(R)$. Hence, R is a commutative ring by Lemma 2.5.

The following example shows that the condition "V is a non-zero semigroup ideal" in Corollary 3.2 is not redundant even for rings.

Example 3.1 : Let R be the non-commutative prime ring $M_2(F)$, where F is any field, U_1 the non-zero semigroup right ideal $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} | a, b \in F \right\}$ and U_2 the non-zero semigroup left ideal $R \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \mid a, c \in F \right\}$. Notice that neither U_1 nor U_2 is a semigroup ideal of R. Define $d : R \to R$ to be the non-zero inner derivation induced by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and define $D : R \to R$ to be the non-zero inner derivation induced by $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Thus, $d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -c & a-d \\ 0 & c \end{bmatrix}$ and $D\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} b & 0 \\ d-a & -b \end{bmatrix}$, where $a, b, c, d \in F$. Now for all $X = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \in U_1$, we have

$$\begin{aligned} d(XY) &= d\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}\right) = d\left(\begin{bmatrix} ac & ad \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & ac \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & ca \\ 0 & 0 \end{bmatrix} = d\left(\begin{bmatrix} ca & cb \\ 0 & 0 \end{bmatrix}\right) = d\left(\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = d(YX). \end{aligned}$$

Also, for all $X = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, Y = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \in U_2$, we have

$$\begin{aligned} D(XY) &= D\left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}\right) = D\left(\begin{bmatrix} ac & 0 \\ bc & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ -ac & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ -ca & 0 \end{bmatrix} = D\left(\begin{bmatrix} ca & 0 \\ da & 0 \end{bmatrix}\right) = D\left(\begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\right) = D(YX). \end{aligned}$$

If we take either $U = V = U_1$ or $U = V = U_2$, then both $d(XY) = d(YX)$ and $D(XY) = D(YX)$ for all $X, Y \in U = V$. Therefore, "V is a non-zero semigroup ideal" in Corollary 3.2 is not redundant.

The following result generalizes Theorem 3.5 of [3].

***Theorem* 3.3** — *Let R be a near-ring with a derivation d, a semigroup right ideal U and $a, b \in U$ such that b and $d(a)$ are not left zero-divisors in R. If $d(uv) = d(vu)$ for all $u, v \in U$, then R is a commutative ring.*

PROOF : From Lemma 2.10, we have $uv = vu$ for all $u, v \in U$. Now, $b(bx)y = (bx)by = (by)bx = bbyx$ for all $x, y \in R$. Hence, $xy = yx$ and R is a commutative ring.

Remark 3.1 : Let R be a near-ring with a non-zero derivation d. Consider the following two statements:

(*) R is 3-prime and $d(vu) = d(uv)$ for all $u, v \in U$, where U is a non-zero semigroup ideal of R.

(**) $d(vu) = d(uv)$ for all $u, v \in U$, where U is a non-zero semigroup ideal of R and there exist $a, b \in U$ such that $d(a)$ and b are not left zero divisors in R.

(i) The statement (*) implies the statement (**). Indeed, statement (*) implies that R is commutative by Corollary 3.2. Thus, R is without non-zero divisors of zero using Lemma 2.2(iii).

(ii) The converse need not be true. Let R be the polynomial ring $\mathbb{Z}_4[x]$ and d the usual derivative. Then R is commutative and $d(xy) = d(yx)$ for all $x, y \in R$. Moreover, $d(x^5) = 5(x^4) = x^4$ is not a zero divisor in R. But R is not prime since $2xR2x = R(2x)(2x) = R(4x^2) = \{0\}$ and $2x \neq 0$. Observe that $U = R$ and $d(a) = b$ in this example. So the statement (**) is weaker than the the statement (*).

In the following result we give a simple proof of Theorem 2.1 of [2].

***Theorem* 3.4** — *Let R be a 3-prime near-ring with a non-zero derivation d and a non-zero semigroup left ideal (resp. semigroup right ideal) U. If $d(U) \subseteq Z(R)$, then R is a commutative ring.*

PROOF : Since $d(U) \neq \{0\}$ by Lemma 2.7, there exists $a \in U$ such that $d(a) \in Z(R) - \{0\}$ and then it is not a zero-divisor in R by Lemma 2.2(iii). Since $vd(uv) = d(uv)v$ for all $u, v \in U$, we have $vd(u)v + vud(v) = d(u)vv + ud(v)v$ by Lemma 2.2(vi). As $d(u), d(v) \in Z(R)$, we get $d(v)[vu - uv] = 0$ for all $u, v \in U$. Taking $v = a$, we deduce that a centralizes U. Now $vd(ua) = d(ua)v$ for all $u, v \in U$ implies $vd(u)a + vud(a) = d(u)av + ud(a)v$. Thus, $d(a)[vu - uv] = 0$ and then $uv = vu$ for all $u, v \in U$. For all $x \in R, v \in U$, we have $vd(xa) = d(xa)v$ (resp. $vd(ax) = d(ax)v$) and hence $vd(x)a + vxd(a) = d(x)av + xd(a)v$ (resp. $vad(x) + vd(a)x = ad(x)v + d(a)xv$). Since $vd(x)a = d(x)av$ (resp. $vad(x) = ad(x)v$), we get $d(a)[vx - xv] = 0$ for all $x \in R, v \in U$ which means $U \subseteq Z(R)$. Therefore, R is a commutative ring by Lemma 2.5.

4. THE CONDITION $d(uv) = -d(vu)$

In this section, we prove some lemmas which are useful to obtain some results similar to the results of Section 3 by using the condition $d(uv) = -d(vu)$ instead of the condition $d(uv) = d(vu)$. Moreover, with the condition $d(uv) = -d(vu)$ we prove that the near-ring is a commutative ring of characteristic two. We begin with the following lemma.

Lemma 4.1 — Let R be a 3-prime near-ring with a non-zero semigroup right ideal U.

(i) If $ua = (-a)(-u)$ for all $u \in U, a \in A$, where A is a subset of R, then $-A \subseteq Z(R)$.

(ii) If $ua = (-a)(-u)$ for all $u \in U, a \in A$, where A is a subgroup of $(R, +)$, then $A \subseteq Z(R)$.

(iii) If $uj = (-j)(-u)$ for all $u \in U, j \in J$, where J is a non-zero right R-subgroup of R or a non-zero semigroup left ideal of R, then R is a commutative ring.

PROOF : (i) From $ua = (-a)(-u)$ for all $u \in U, a \in A$, we have $-ua = -(-a)(-u)$ and then $u(-a) = (-a)u$. Using Lemma 2.4, we get $-a \in Z(R)$ for all $a \in A$.

(ii) It is direct from (i) since $-A = A$.

(iii) If J is a non-zero right R-subgroup of R, then $J \subseteq Z(R)$ from (ii). If J is a non-zero semigroup left ideal of R, then $-J \subseteq Z(R)$ by (i) and observe that $-J$ is a non-zero semigroup left ideal of R. In both cases, R is a commutative ring by Lemma 2.5.

Proposition 4.2 — Let R be a 3-prime near-ring such that $uv = -vu$ for all $u \in U, v \in V$, where U and V are non-zero semigroup ideals of R. Then R is a commutative ring of characteristic 2.

PROOF : For all $u, w \in U, v \in V$, we get $uwv = -vuw = vu(-w) = (-uv)(-w) = u(-v)(-w)$ which means $u[wv - (-v)(-w)] = 0$. Lemma 2.3 implies $wv = (-v)(-w)$ for all $v \in V, w \in U$. Using Lemma 4.1(iii), R is a commutative ring. Now $uvw = -vwu = wuv = -uvw = uv(-w)$ for all $u, w \in U, v \in V$ and then $0 = uv(w + w) = uv(2w)$. Lemma 2.3 implies $v(2w) = 0$ for all $w \in U, v \in V$. Again, Lemma 2.3 implies $2U = \{0\}$. Thus, $2R = \{0\}$ by Proposition 2.8.

***Theorem* 4.3** — *Let R be a 3-prime near-ring with a non-zero derivation d such that $d(uy) = -d(yu)$ for all $u \in U, y \in A$, where U is a non-zero semigroup ideal of R and A is a non-empty subset of R. Then $-A \subseteq Z(R)$ and $(R, +)$ is abelian or $d(A) = \{0\}$. In particular, if $A = \{r\}$ and $d(r) \neq 0$, then $-r \in Z(R)$ and $(R, +)$ is abelian.*

PROOF : From Lemma 2.10, we deduce that $d(y)u[vy - (-y)(-v)] = 0$. Lemma 2.6 implies that for all $y \in A$ either $d(y) = 0$ or $vy = (-y)(-v)$. If $d(A) \neq \{0\}$, then there exists $a \in A$ such that $d(a) \neq 0$. It follows that $va = (-a)(-v)$ for all $v \in U$. Using Lemma 4.1(i), $-a \in Z(R)$ and then $d(-a) \in Z(R) - \{0\}$ by Lemma 2.2(iv). From $d(uy) = -d(yu)$ for all $u \in U, y \in A$, we have $d(((-a)u)y) = -d(y((-a)u)) = -d((y(-a))u) = -d((-a)yu)$ and then

$d((-a)(uy+yu))=0$. So we deduce that

$$d(-a)[uy+yu]=0 \text{ for all } u\in U, y\in A. \tag{4.1}$$

Replacing u by uv, where $v\in U$, we have

$$d(-a)uvy=-d(-a)yuv=d(-a)yu(-v)=(-d(-a)uy)(-v)=d(-a)u(-y)(-v)$$

and so $d(-a)u[vy-(-y)(-v)]=0$. Using $d(-a)\neq 0$ and Lemma 2.6(i), we have $vy=(-y)(-v)$ for all $v\in U, y\in A$. Lemma 4.1(i) implies that $-A\subseteq Z(R)$. Now $(-a)^2\in Z(R)$, whence $d((-a)^2)\in Z(R)$ by Lemma 2.2(iv). Then $(-a)d(-a)+d(-a)(-a)=2(d(-a))(-a)=d(-a)(2(-a))\in Z(R)$. Using Lemma 2.2(iii), $d(-a)$ is not a left zero-divisor in R. It follows that $2(-a)\in Z(R)$ by Lemma 2.2(i). Hence, $(R,+)$ is abelian by Lemma 2.2(ii).

Corollary 4.4 — Let R be a 3-prime near-ring with a non-zero derivation d and a non-zero semigroup ideal U.

(i) If $d(uy)=-d(yu)$ for all $u\in U, y\in A$, where A is a subgroup of R, then $A\subseteq Z(R)$ and R is of characteristic 2 or $d(A)=\{0\}$.

(ii) If $d(vu)=-d(uv)$ for all $v\in V, u\in U$, where V is a non-zero semigroup left ideal of R, then R is a commutative ring.

(iii) If $d(vu)=-d(uv)$ for all $v\in V, u\in U$, where V is a non-zero right R-subgroup of R, then R is a commutative ring of characteristic 2.

PROOF : (i) Theorem 4.3 implies that $d(A)=\{0\}$ or $-A\subseteq Z(R)$. But $A=-A$, so if $d(A)\neq\{0\}$, then $A\neq\{0\}$ and $A\subseteq Z(R)$. Suppose $d(A)\neq\{0\}$. From equation (4.1) in the proof of Theorem 4.3 and Lemma 2.2(iii), we deduce that $yu=-uy=-yu=y(-u)$ for all $u\in U, y\in A$ and hence $y[u+u]=0$. Choose $y\in A-\{0\}$. Then $y\in Z(R)-\{0\}$ and is not a zero-divisor in R by Lemma 2.2(iii). So $2U=\{0\}$ and Proposition 2.8 implies that $2R=\{0\}$.

(ii) From Lemma 2.7 and Theorem 4.3, we have $-V\subseteq Z(R)$. But $-V$ is a non-zero semigroup left ideal of R. So R is a commutative ring by Lemma 2.5.

(iii) Directly from (i), Lemma 2.7 and Lemma 2.5.

5. THE CONDITION $d(v)g(u) = g(u)d(v)$

In this section, we study the commutativity of near-rings, each admitting two non-zero derivations d and g satisfying the condition $d(v)g(u) = g(u)d(v)$ ($d(v)g(u) = -g(u)d(v)$) on some suitable algebraic substructures of the near-ring. We start with the following lemmas.

Lemma 5.1 [1, Theorem 2.3] — Let U be a non-zero semigroup ideal of a 3-prime near-ring R with $2R \neq \{0\}$. If d_1 and d_2 are derivations on R such that $d_1(x)d_2(y) = -d_2(y)d_1(x)$ for all $x, y \in U$, then $d_1 = 0$ or $d_2 = 0$.

Lemma 5.2 [2, Lemma 3.2] — Let R be a 3-prime near-ring with a non-zero derivation d and a non-zero semigroup ideal U of R such that $d^2(U) \neq \{0\}$. If $a \in R$ and $[a, d(U)] = \{0\}$, then $a \in Z(R)$.

The following result generalizes Theorem 3.1 of [2] and Theorem 4.13 of [7].

***Theorem* 5.3** — *Let R be a 3-prime near-ring with non-zero derivations d and g such that $d^2(V) \neq \{0\}$, where V is a non-zero semigroup ideal of R. If $d(v)g(u) = g(u)d(v)$ for all $v \in V, u \in U$, where U is a non-zero semigroup right (left) ideal of R, then R is a commutative ring.*

PROOF : $d(v)g(u) = g(u)d(v)$ for all $v \in V, u \in U$ implies that $[g(u), d(V)] = \{0\}$ for all $u \in U$. By Lemma 5.2, we have $g(U) \subseteq Z(R)$ and then R is a commutative ring by Theorem 3.4.

Corollary 5.4 — Let R be a 3-prime near-ring with non-zero derivations d and g such that $2R \neq \{0\}$. If $d(v)g(u) = g(u)d(v)$ for all $v \in V, u \in U$, where V is a non-zero semigroup ideal of R and U is a non-zero semigroup right (left) ideal of R, then R is a commutative ring.

PROOF : From Theorem 5.3, R is a commutative ring or $d^2(V) = \{0\}$. If $d^2(V) = \{0\}$, then $0 = d^2(vw) = 2d(v)d(w) = d(v)[2d(w)]$ for all $v, w \in V$. Using Lemma 2.6(ii), we have $2d(V) = \{0\}$. Thus, $2R = \{0\}$ by Proposition 2.8, a contradiction. Therefore, R is a commutative ring.

Corollary 5.5 — Let R be a 3-prime near-ring with non-zero derivations d and g such that $d^2(U) \neq \{0\}$, where U is a non-zero semigroup ideal of R. If $d(u)g(v) = -g(v)d(u)$ for all $u, v \in U$, then R is a commutative ring of characteristic 2.

PROOF : From Lemma 5.1, we get $2R = \{0\}$. So $d(u)g(v) = g(v)d(u)$ for all $u, v \in U$ and R is commutative by Theorem 5.3. Therefore, R is a commutative ring of characteristic 2.

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