Indian J. Pure Appl. Math., **43**(3): 227-249, June 2012

SOME PROPERTIES FOR A CLASS OF SYMMETRIC FUNCTIONS WITH APPLICATIONS

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(*Received* 28 *June* 2010; *after final revision* 11 *December* 2010; *accepted* 20 *March* 2012)

For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, the symmetric function $\psi_n(x, r)$ is defined by

$$\psi_n(x,r) = \psi_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \le i_1 < i_2 \cdots < i_r \le n} \prod_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}},$$

where $r = 1, 2, \cdots, n$ and $i_1, i_2, \cdots, i_n$ are positive integers. In this article, the Schur convexity, Schur multiplicative convexity and Schur harmonic

[1]This research was supported by the Natural Science Foundation of China under Grants 11071069 and 11171307, the Natural Science Foundation of Hunan Province under Grant 09JJ6003, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

convexity of $\psi_n(x,r)$ are discussed. As applications, some inequalities are established by use of the theory of majorization.

Key words : Symmetric function, Schur convex, Schur multiplicatively convex, Schur harmonic convex.

1. INTRODUCTION

In this paper, we adopt the notation and terminology as follows: $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space $(n \geq 2)$, $\mathbb{R}^n_+ = \{(x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{N} = \{1, 2, \cdots, n, \cdots\}$. For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we denote by

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n), \\ xy &= (x_1 y_1, x_2 y_2, \cdots, x_n y_n), \\ \alpha x &= (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), \\ e^x &= (e^{x_1}, e^{x_2}, \cdots, e^{x_n}), \\ \alpha + x &= (\alpha + x_1, \alpha + x_2, \cdots, \alpha + x_n) \end{aligned}$$

and

$$\alpha - x = (\alpha - x_1, \alpha - x_2, \cdots, \alpha - x_n).$$

Moreover, we denote by

$$x^\alpha = (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha),$$

$$\log x = (\log x_1, \log x_2, \cdots, \log x_n)$$

and

$$\frac{1}{x} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right)$$

for $x \in \mathbb{R}^n_+$.

For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, $r \in \mathbb{N}$ and $r \leq n$, the Hamy symmetric function $H_n(x, r)$ is defined by Hara, Uchiyama and Takahasi [19] as follows:

$$H_n(x,r) = H_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \left(\prod_{j=1}^{r} x_{i_j} \right)^{\frac{1}{r}},$$

where $i_1, i_2, \cdots, i_n \in \mathbb{N}$.

Corresponding to this is the r-th order Hamy mean

$$\sigma_n(x,r) = \sigma_n(x_1, x_2, \cdots, x_n; r) = \frac{1}{C_n^r} H_n(x,r),$$

where $C_n^r = \frac{n!}{(n-r)!r!}$. Hara, Uchiyama and Takahasi [19] established the following refinement of the classical arithmetic and geometric means inequalities:

$$G_n(x) = \sigma_n(x,n) \leq \sigma_n(x, n-1) \leq \cdots \leq \sigma_n(x,2) \leq \sigma_n(x,1) = A_n(x).$$

Here, $A_n(x) = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $G_n(x) = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ denote the classical arithmetic and geometric means of x, respectively. We also denote the harmonic mean of x by $H_n(x) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$.

The paper [2] by Ku, Ku and Zhang contains some interesting inequalities including the fact that $(\sigma_n(x,r))^{\frac{1}{r}}$ is log-concave, the more results can be found in the book [1] by Bullen.

Recently, the Schur convexity of the Hamy symmetric function $H_n(x, r)$ was discussed and some analytic inequalities were established by Guan [17].

In this article, we define the following new symmetric function

$$\psi_n(x,r) = \psi_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \prod_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}}, \qquad (1.1)$$

for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, $r \in \mathbb{N}$ and $r \leq n$. Here, $i_1, i_2, \cdots, i_n \in \mathbb{N}$.

The main purpose of this paper is to discuss the Schur convexity, Schur harmonic convexity and Schur multiplicative convexity for the symmetric function $\psi_n(x, r)$. As applications, some inequalities are established by use of the theory of majorization.

Schur convex and Schur multiplicatively convex functions are defined as follows.

Definition 1.1 — Let $E \subseteq \mathbb{R}^n$ be a set, a real-valued function F is said to be Schur convex on E if

$$F(x_1, x_1, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n)$$

for each pair of n-tuples $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in E, such that $x \prec y$, i.e.

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, k = 1, 2, \cdots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component of x. F is said to be Schur concave if $-F$ is Schur convex.

Definition 1.2 — Let $E \subseteq \mathbb{R}_+^n$ be a set, a real-valued function $F : E \to \mathbb{R}_+$ is said to be Schur multiplicatively convex on E if $F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n)$ for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\log x \prec \log y$. F is said to be Schur multiplicatively concave if $\frac{1}{F}$ is Schur multiplicatively convex.

Next, we define the Schur harmonic convexity.

Definition 1.3 — Let $E \subseteq \mathbb{R}_+^n$ be a set. A real-valued function F is said to be Schur harmonic convex on E if

$$F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n) \tag{1.2}$$

for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\frac{1}{x} \prec \frac{1}{y}$. F is said to be Schur harmonic concave if inequality (1.2) is reversed.

The Schur convexity was introduced by Schur [24] in 1923, Hardy, Littlewood and Pólya were also interested in some inequalities that are related to the Schur convexity [20]. It has many important applications in extended mean values [11], theory of statistical experiments [29], graphs and matrices [13], combinatorial optimization [21], reliability [22], gamma functions [26], information-theoretic topics [14], stochastic orderings [28], analytic inequalities [4, 6, 9, 42, 43] and other related fields. Recently, the Schur multiplicative and harmonic convexities were introduced and investigated in [2, 3, 5, 7, 8, 10, 15, 16, 25, 30, 33-41].

2. Lemmas

In this section, we introduce and establish some Lemmas, which are used in the proof of our main results.

Lemma 2.1 [24] — Let $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a continuous symmetric function. If f is differentiable in $\mathbb{R}^n_+$, then f is Schur convex in $\mathbb{R}^n_+$ if and only if

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \tag{2.1}$$

for all $i, j = 1, 2, \cdots, n$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$. And f is Schur concave in $\mathbb{R}^n_+$ if and only if inequality (2.1) is reversed for all $i, j = 1, 2, \cdots, n$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$. Here f is a symmetric function in $\mathbb{R}^n_+$ which means that $f(Px) = f(x)$ for any $x \in \mathbb{R}^n_+$ and any $n \times n$ permutation matrix P.

Remark 2.1 : Since f is symmetric, the Schur's condition in Lemma 2.1, i.e. (2.1) can be reduced as

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geq 0.$$

Lemma 2.2 [12, 18] — Let $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a continuous symmtric function. If f is differentiable in $\mathbb{R}^n_+$, then f is Schur multiplicatively convex in $\mathbb{R}^n_+$ if and only if

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}\right) \geq 0 \tag{2.2}$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$. And f is Schur multiplicatively concave in $\mathbb{R}^n_+$ if and only if inequality (2.2) is reversed.

Lemma 2.3 — Let $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a continuous symmetric function. If f is differentiable in $\mathbb{R}^n_+$, then f is Schur harmonic convex in $\mathbb{R}^n_+$ if and only if

$$(x_1 - x_2)\left(x_1^2\frac{\partial f}{\partial x_1} - x_2^2\frac{\partial f}{\partial x_2}\right) \geq 0 \tag{2.3}$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$. And f is Schur harmonic concave in $\mathbb{R}^n_+$ if and only if inequality (2.3) is reversed.

PROOF : From Definitions 1.1 and 1.3, we clearly see the fact that

$f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is Schur harmonic convex if and only if $F(x) = \dfrac{1}{f(\frac{1}{x})} : \mathbb{R}^n_+ \to \mathbb{R}_+$ is Schur concave.

This fact, Lemma 2.1 and Remark 2.1 together with elementary calculation imply that Lemma 2.3 is true.

Lemma 2.4 [16-18] — Suppose that $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$ and $\sum_{i=1}^n x_i = s$. If $c \geq s$, then

$$\frac{c-x}{\frac{nc}{s}-1} = \left(\frac{c-x_1}{\frac{nc}{s}-1}, \frac{c-x_2}{\frac{nc}{s}-1}, \cdots, \frac{c-x_n}{\frac{nc}{s}-1}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

Lemma 2.5 [16] — Suppose that $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$ and $\sum_{i=1}^n x_i = s$. If $c \geq 0$, then

$$\frac{c+x}{\frac{nc}{s}+1}=\left(\frac{c+x_1}{\frac{nc}{s}+1},\frac{c+x_2}{\frac{nc}{s}+1},\cdots,\frac{c+x_n}{\frac{nc}{s}+1}\right)\prec(x_1,x_2,\cdots,x_n)=x.$$

Lemma 2.6 [32] — Suppose that $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$ and $\sum_{i=1}^n x_i = s$. If $0\le\lambda\le1$, then

$$\frac{s-\lambda x}{n-\lambda}=\left(\frac{s-\lambda x_1}{n-\lambda},\frac{s-\lambda x_2}{n-\lambda},\cdots,\frac{s-\lambda x_n}{n-\lambda}\right)\prec(x_1,x_2,\cdots,x_n)=x.$$

3. MAIN RESULTS

***Theorem* 3.1** — *For $r\in\{1,2,\cdots,n\}$, the symmetric function $\psi_n(x,r)$ is Schur convex in $\mathbb{R}^n_+$.*

PROOF : By Lemma 2.1 and Remark 2.1 we only need to prove that

$$(x_1-x_2)\left(\frac{\partial\psi_n(x,r)}{\partial x_1}-\frac{\partial\psi_n(x,r)}{\partial x_2}\right)\ge0. \tag{3.1}$$

for all $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$ and $r=1,2,\cdots,n$.

The proof is divided into seven cases.

Case 1 : If $r=1$, then (1.1) leads to

$$\psi_n(x,1)=\psi_n(x_1,x_2,\cdots,x_n;1)=\sum_{i=1}^n\frac{1+x_i}{x_i}. \tag{3.2}$$

From (3.2) we have

$$\begin{aligned}(x_1-x_2)\left(\frac{\partial\psi_n(x,1)}{\partial x_1}-\frac{\partial\psi_n(x,1)}{\partial x_2}\right) &= \frac{(x_1-x_2)^2(x_1+x_2)}{x_1^2x_2^2}\\ &\ge 0.\end{aligned}$$

Case 2 : If $n=2$ and $r=2$, then from (1.1) we clearly see that

$$\psi_2(x,2)=\psi_2(x_1,x_2;2)=\frac{(1+x_1)(1+x_2)}{x_1x_2}. \tag{3.3}$$

Equation (3.3) leads to

$$\begin{aligned}(x_1-x_2)\left(\frac{\partial\psi_2(x,2)}{\partial x_1}-\frac{\partial\psi_2(x,2)}{\partial x_2}\right) &= \frac{(x_1-x_2)^2(1+x_1+x_2)}{x_1^2x_2^2}\\ &\geq 0.\end{aligned}$$

Case 3 : If $n=3$ and $r=2$, then by (1.1) we have

$$\begin{aligned}\psi_3(x,2) &= \psi_3(x_1,x_2,x_3;2)\\ &= \frac{1+x_1}{x_1}\left(\frac{1+x_2}{x_2}+\frac{1+x_3}{x_3}\right)+\frac{1+x_2}{x_2}\frac{1+x_3}{x_3}. \qquad (3.4)\end{aligned}$$

Equation (3.4) gives

$$\begin{aligned}&(x_1-x_2)\left(\frac{\partial\psi_3(x,2)}{\partial x_1}-\frac{\partial\psi_3(x,2)}{\partial x_2}\right)\\ &= \frac{(x_1-x_2)^2}{x_1^2x_2^2}\left[(1+x_1+x_2)+\frac{(1+x_3)(x_1+x_2)}{x_3}\right]\\ &\geq 0.\end{aligned}$$

Case 4 : If $n\geq 4$ and $r=2$, then (1.1) leads to

$$\begin{aligned}\psi_n(x,2) &= \psi_n(x_1,x_2,\cdots,x_n;2)\\ &= \frac{1+x_1}{x_1}\frac{1+x_2}{x_2}+\left(\frac{1+x_1}{x_1}+\frac{1+x_2}{x_2}\right)\sum_{i=3}^{n}\frac{1+x_i}{x_i}\\ &\quad+\sum_{3\leq i<j\leq n}\frac{(1+x_i)(1+x_j)}{x_ix_j}. \qquad (3.5)\end{aligned}$$

Elementary computation and (3.5) yield

$$\begin{aligned}&(x_1-x_2)\left(\frac{\partial\psi_n(x,2)}{\partial x_1}-\frac{\partial\psi_n(x,2)}{\partial x_2}\right)\\ &= \frac{(x_1-x_2)^2}{x_1^2x_2^2}\left[(1+x_1+x_2)+(x_1+x_2)\sum_{i=3}^{n}\frac{1+x_i}{x_i}\right]\\ &\geq 0.\end{aligned}$$

Case 5 : If $n \geq 3$ and $r = n$, then

$$\psi_n(x,n) = \psi_n(x_1, x_2, \cdots, x_n; n) = \prod_{i=1}^{n} \frac{1+x_i}{x_i} \tag{3.6}$$

and

$$\begin{aligned}
&(x_1 - x_2)\left(\frac{\partial \psi_n(x,n)}{\partial x_1} - \frac{\partial \psi_n(x,n)}{\partial x_2}\right) \\
&= \frac{(x_1-x_2)^2(1+x_1+x_2)}{x_1x_2(1+x_1)(1+x_2)}\psi_n(x,n) \\
&\geq 0.
\end{aligned}$$

Case 6 : If $n \geq 4$ and $r = n-1$, then

$$\begin{aligned}
\psi_n(x,n-1) &= \psi_n(x_1, x_2, \cdots, x_n; n-1) \\
&= \frac{1+x_1}{x_1}\frac{1+x_2}{x_2}\psi_{n-2}(x_3, x_4, \cdots, x_n; n-3) \\
&\quad + \left(\frac{1+x_1}{x_1} + \frac{1+x_2}{x_2}\right)\prod_{i=3}^{n}\frac{1+x_i}{x_i}
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&(x_1 - x_2)\left(\frac{\partial \psi_n(x,n-1)}{\partial x_1} - \frac{\partial \psi_n(x,n-1)}{\partial x_2}\right) \\
= \ & \frac{(x_1-x_2)^2}{x_1^2x_2^2}\Bigg[(1+x_1+x_2)\psi_{n-2}(x_3, x_4, \cdots, x_n; n-3) \\
&+(x_1+x_2)\prod_{i=3}^{n}\frac{1+x_i}{x_i}\Bigg] \\
\geq \ & 0.
\end{aligned}$$

Case 7 : If $n \geq 5$ and $3 \leq r \leq n-2$, then

$$\begin{aligned}
\psi_n(x,r) &= \psi_n(x_1, x_2, \cdots, x_n; r) \\
&= \frac{1+x_1}{x_1}\frac{1+x_2}{x_2}\psi_{n-2}(x_3, x_4, \cdots, x_n; r-2) \\
&\quad + \left(\frac{1+x_1}{x_1} + \frac{1+x_2}{x_2}\right)\psi_{n-2}(x_3, x_4, \cdots, x_n; r-1) \\
&\quad +\psi_{n-2}(x_3, x_4, \cdots, x_n; r)
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&(x_1-x_2)\left(\frac{\partial\psi_n(x,r)}{\partial x_1}-\frac{\partial\psi_n(x,r)}{\partial x_2}\right)\\
= &\frac{(x_1-x_2)^2}{x_1^2x_2^2}\Big[(1+x_1+x_2)\psi_{n-2}(x_3,x_4,\cdots,x_n;r-2)\\
&+(x_1+x_2)\psi_{n-2}(x_3,x_4,\cdots,x_n;r-1)\Big]\\
\geq &\ 0.
\end{aligned}$$

Therefore, (3.1) follows from Cases 1-7 and the proof of Theorem 3.1 is completed.

***Theorem* 3.2** — *For* $r\in\{1,2,\cdots,n\}$, *the symmetric function* $\psi_n(x,r)$ *is Schur multiplicatively convex in* $\mathbb{R}^n_+$.

PROOF : According to Lemma 2.2 we only need to prove that

$$(\log x_1-\log x_2)\left(x_1\frac{\partial\psi_n(x,r)}{\partial x_1}-x_2\frac{\partial\psi_n(x,r)}{\partial x_2}\right)\geq 0 \tag{3.9}$$

for all $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$ and $r=1,2,\cdots,n$.

The proof is divided into seven cases.

Case 1 : If $r=1$, then (3.2) leads to

$$\begin{aligned}
&(\log x_1-\log x_2)\left(x_1\frac{\partial\psi_n(x,1)}{\partial x_1}-x_2\frac{\partial\psi_n(x,1)}{\partial x_2}\right)\\
= &\frac{(\log x_1-\log x_2)(x_1-x_2)}{x_1x_2}\\
\geq &\ 0.
\end{aligned}$$

Case 2 : If $n=2$ and $r=2$, then (3.3) yields

$$\begin{aligned}
&(\log x_1-\log x_2)\left(x_1\frac{\partial\psi_2(x,2)}{\partial x_1}-x_2\frac{\partial\psi_2(x,2)}{\partial x_2}\right)\\
= &\frac{(\log x_1-\log x_2)(x_1-x_2)}{x_1x_2}\\
\geq &\ 0.
\end{aligned}$$

Case 3 : If $n = 3$ and $r = 2$, then from (3.4) we have

$$\begin{aligned}
&(\log x_1 - \log x_2)\left(x_1 \frac{\partial \psi_3(x,2)}{\partial x_1} - x_2 \frac{\partial \psi_3(x,2)}{\partial x_2}\right)\\
= \; & \frac{(\log x_1 - \log x_2)(x_1 - x_2)}{x_1 x_2}\left(1 + \frac{1+x_3}{x_3}\right)\\
\geq \; & 0.
\end{aligned}$$

Case 4 : If $n \geq 4$ and $r = 2$, then (3.5) implies

$$\begin{aligned}
&(\log x_1 - \log x_2)\left(x_1 \frac{\partial \psi_n(x,2)}{\partial x_1} - x_2 \frac{\partial \psi_n(x,2)}{\partial x_2}\right)\\
= \; & \frac{(\log x_1 - \log x_2)(x_1 - x_2)}{x_1 x_2}\left(1 + \sum_{i=3}^{n} \frac{1+x_i}{x_i}\right)\\
\geq \; & 0.
\end{aligned}$$

Case 5 : If $n \geq 3$ and $r = n$, then (3.6) leads to

$$\begin{aligned}
&(\log x_1 - \log x_2)\left(x_1 \frac{\partial \psi_n(x,n)}{\partial x_1} - x_2 \frac{\partial \psi_n(x,n)}{\partial x_2}\right)\\
= \; & \frac{(\log x_1 - \log x_2)(x_1 - x_2)}{(1+x_1)(1+x_2)}\psi_n(x,n)\\
\geq \; & 0.
\end{aligned}$$

Case 6 : If $n \geq 4$ and $r = n-1$, then (3.7) implies

$$\begin{aligned}
&(\log x_1 - \log x_2)\left(x_1 \frac{\partial \psi_n(x,n-1)}{\partial x_1} - x_2 \frac{\partial \psi_n(x,n-1)}{\partial x_2}\right)\\
= \; & \frac{(\log x_1 - \log x_2)(x_1 - x_2)}{x_1 x_2}\Bigg[\psi_{n-2}(x_3, x_4, \cdots, x_n; n-3)\\
& + \prod_{i=3}^{n} \frac{1+x_i}{x_i}\Bigg]\\
\geq \; & 0.
\end{aligned}$$

Case 7 : If $n \geq 5$ and $3 \leq r \leq n-2$, then (3.8) gives

$$
\begin{aligned}
& (\log x_1 - \log x_2)\left(x_1 \frac{\partial \psi_n(x,r)}{\partial x_1} - x_2 \frac{\partial \psi_n(x,r)}{\partial x_2}\right) \\
= \quad & \frac{(\log x_1 - \log x_2)(x_1 - x_2)}{x_1 x_2}\Big[\psi_{n-2}(x_3, x_4, \cdots, x_n; r-2) \\
& + \psi_{n-2}(x_3, x_4, \cdots, x_n; r-1)\Big] \\
\geq \quad & 0.
\end{aligned}
$$

Therefore, (3.9) follows from Cases 1-7 and the proof of Theorem 3.2 is completed.

***Theorem* 3.3** — *For $r \in \{1, 2, \cdots, n\}$, the symmetric function $\psi_n(x,r)$ is Schur harmonic concave in $\mathbb{R}^n_+$.*

PROOF : By Lemma 2.3 we only need to prove that

$$
(x_1 - x_2)\left(x_1^2 \frac{\partial \psi_n(x,r)}{\partial x_1} - x_2^2 \frac{\partial \psi_n(x,r)}{\partial x_2}\right) \leq 0 \tag{3.10}
$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$ and $r = 1, 2, \cdots, n$.

The proof is divided into four cases.

Case 1 : If $r = 1$, then (3.2) leads to

$$
(x_1 - x_2)\left(x_1^2 \frac{\partial \psi_n(x,1)}{\partial x_1} - x_2^2 \frac{\partial \psi_n(x,1)}{\partial x_2}\right) = 0.
$$

Case 2 : If $n \geq 2$ and $r = 2$, then (3.3)-(3.5) yield

$$
(x_1 - x_2)\left(x_1^2 \frac{\partial \psi_n(x,2)}{\partial x_1} - x_2^2 \frac{\partial \psi_n(x,2)}{\partial x_2}\right) = -\frac{(x_1 - x_2)^2}{x_1 x_2} \leq 0.
$$

Case 3 : If $n \geq 3$ and $r = n$, then from (3.6) we have

$$\begin{aligned}
&(x_1 - x_2)\left(x_1^2 \frac{\partial \psi_n(x,n)}{\partial x_1} - x_2^2 \frac{\partial \psi_n(x,n)}{\partial x_2}\right) \\
= \quad & -\frac{(x_1 - x_2)^2}{(1 + x_1)(1 + x_2)} \psi_n(x,n) \\
\leq \quad & 0.
\end{aligned}$$

Case 4 : If $n \geq 4$ and $3 \leq r \leq n - 1$, then (3.7) and (3.8) lead to

$$\begin{aligned}
&(x_1 - x_2)\left(x_1^2 \frac{\partial \psi_n(x,r)}{\partial x_1} - x_2^2 \frac{\partial \psi_n(x,r)}{\partial x_2}\right) \\
= \quad & -\frac{(x_1 - x_2)^2}{x_1 x_2} \psi_{n-2}(x_3, x_4, \cdots, x_n; r - 2) \\
\leq \quad & 0.
\end{aligned}$$

Therefore, (3.10) follows from Cases 1-4 and the proof of Theorem 3.3 is completed.

4. Applications

In this section, we establish some inequalities by use of Theorems 3.1-3.3 and the theory of majorization.

***Theorem* 4.1** : *Let* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, $s = \sum_{i=1}^n x_i$ *and* $r \in \{1, 2, \cdots, n\}$, *then*

(1) $\psi_n(x,r) \geq \psi_n(\frac{c-x}{\frac{nc}{s}-1}, r)$ for $c \geq s$;

(2) $\psi_n(x,r) \leq \psi_n\left(\frac{cH_n(x)-1}{cx-1} x, r\right)$ for $c \geq \sum_{i=1}^n \frac{1}{x_i}$;

(3) $\psi_n(x,r) \geq \psi_n(\frac{c+x}{\frac{nc}{s}+1}, r)$ for $c \geq 0$;

(4) $\psi_n(x,r) \leq \psi_n\left(\frac{cH_n(x)+1}{cx+1}x, r\right)$ for $c \geq 0$;

(5) $\psi_n(x,r) \geq \psi_n(\frac{s-\lambda x}{n-\lambda}, r)$ for $0 \leq \lambda \leq 1$;

(6) $\psi_n(x,r) \leq \psi_n\left(\frac{n-\lambda}{\sum_{i=1}^{n}\frac{1}{x_i}-\frac{\lambda}{x}}, r\right)$ for $0 \leq \lambda \leq 1$.

PROOF : Theorem 4.1(1) and (2) follow from Lemma 2.4 together with Theorems 3.1 and 3.3.

Theorem 4.1(3) and (4) follow from Lemma 2.5 together with Theorems 3.1 and 3.3.

Theorem 4.1(5) and (6) follow from Lemma 2.6 together with Theorems 3.1 and 3.3.

***Theorem* 4.2** — *Let* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$ *and* $r \in \{1, 2, \cdots, n\}$, *then*

$$(i) \qquad \sum_{1\leq i_1<i_2<\cdots<i_r\leq n}\prod_{j=1}^{r}\frac{1+x_{i_j}}{x_{i_j}} \geq C_n^r\left[\frac{A_n(1+x)}{A_n(x)}\right]^r;$$

$$(ii) \qquad \sum_{1\leq i_1<i_2<\cdots<i_r\leq n}\prod_{j=1}^{r}(1+x_{i_j}) \leq C_n^r\left[A_n(1+x)\right]^r.$$

PROOF : We clearly see that

$$(A_n(x), A_n(x), \cdots, A_n(x)) \prec (x_1, x_2, \cdots, x_n) = x. \tag{4.1}$$

Therefore, Theorem 4.2(i) follows from (4.1) and Theorem 3.1 together with (1.1), and Theorem 4.2(ii) follows from (4.1) and Theorem 3.3 together with (1.1).

Corollary 4.1 — If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$, then

(i) $\quad \dfrac{G_n(x)}{G_n(1+x)} \leq \dfrac{A_n(x)}{A_n(1+x)}$;

(ii) $\quad G_n(1+x) \leq A_n(1+x)$.

PROOF : Corollary 4.1 follows from Theorem 4.2 with $r = n$.

Remark 4.1 : Inequality in Corollary 4.1(ii) is the well-known unweighted arithmetic-geometric means inequality, and inequality in Corollary 4.1(i) was proved by V. Govedarica and M. V. Jovanović in paper [15].

***Theorem* 4.3** — *Let* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ *and* $r \in \{1, 2, \cdots, n\}$, *then*

$$(i) \quad \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}} \le C_n^r \left[1 + \frac{1}{H_n(x)}\right]^r ;$$

$$(ii) \quad \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} (1 + x_{i_j}) \ge C_n^r \left[1 + H_n(x)\right]^r .$$

PROOF : It is easy to see that

$$\left(\frac{1}{H_n(x)}, \frac{1}{H_n(x)}, \cdots, \frac{1}{H_n(x)}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) = \frac{1}{x}. \tag{4.2}$$

Therefore, Theorem 4.3(i) follows from (4.2) and Theorem 3.3 together with (1.1), and Theorem 4.3(ii) follows from (4.2), and Theorem 3.1 together with (1.1).

If we take $r = 1$ and $r = n$ in Theorem 4.3, respectively, then we get the following Corollaries 4.2 and 4.3.

Corollary 4.2 — If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, then

(*i*) $A_n\left(\frac{1+x}{x}\right) \le 1 + \frac{1}{H_n(x)}$;

(*ii*) $A_n(1 + x) \ge 1 + H_n(x)$.

Corollary 4.3 — If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, then

$$(i) \quad \frac{G_n(x)}{G_n(1 + x)} \ge \frac{H_n(x)}{1 + H_n(x)};$$

$$(ii) \quad G_n(1 + x) \ge 1 + H_n(x).$$

***Theorem* 4.4** — *Let* $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$ *and* $r\in\{1,2,\cdots,n\}$, *then*

$$\sum_{1\le i_1<i_2<\cdots<i_r\le n}\prod_{j=1}^{r}\frac{1+x_{i_j}}{x_{i_j}}\ge C_n^r\left[\frac{1+G_n(x)}{G_n(x)}\right]^r.$$

PROOF : We clearly see that

$$\log\left(G_n(x),G_n(x),\cdots,G_n(x)\right)\prec\log(x_1,x_2,\cdots,x_n). \tag{4.3}$$

Therefore, Theorem 4.4 follows from (4.3) and Theorem 3.2 together with (1.1).

If we take $r=1$ and $r=n$ in Theorem 4.4, respectively, then we get

Corollary 4.4 — If $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$, then

$$A_n\left(\frac{1+x}{x}\right)\ge\frac{1+G_n(x)}{G_n(x)}$$

and

$$G_n(1+x)\ge 1+G_n(x).$$

***Theorem* 4.5** — *If* $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$, *then*

(i) $$\sum_{1\le i_1<i_2\cdots<i_r\le n}\prod_{j=1}^{r}\frac{2+x_{i_j}}{1+x_{i_j}}\le 2^{r-1}\frac{2+\sum_{i=1}^n x_i}{1+\sum_{i=1}^n x_i}C_{n-1}^{r-1}+2^rC_{n-1}^r$$
for $1\le r\le n-1$;

(ii) $$\prod_{i=1}^{n}\frac{2+x_i}{1+x_i}\le 2^{n-1}\frac{2+\sum_{i=1}^n x_i}{1+\sum_{i=1}^n x_i};$$

(iii) $$\sum_{1\le i_1<i_2\cdots<i_r\le n}\prod_{j=1}^{r}(2+x_{i_j})\ge 2^{r-1}(2+\sum_{i=1}^{n}x_i)C_{n-1}^{r-1}+2^rC_{n-1}^r$$
for $1\le r\le n-1$;

(iv) $$\prod_{i=1}^{n}(2+x_i)\ge 2^{n-1}(2+\sum_{i=1}^{n}x_i).$$

PROOF : Theorem 4.5 follows from Theorem 3.1 and Theorem 3.3 together with (1.1) and the fact that

$$(1+x_1, 1+x_2, \cdots, 1+x_n) \prec (1+\sum_{i=1}^{n} x_i, 1, 1, \cdots, 1).$$

***Theorem* 4.6** — *Let $\mathcal{A} = A_1A_2\cdots A_{n+1}$ be a $n-$dimensional simplex in R^n and P be an arbitrary point in the interior of $\mathcal{A}$. If B_i is the intersection point of straight line A_iP and hyperplane $\sum_i = A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}, i = 1, 2, \cdots, n+1$. Then for $r \in \{1, 2, \cdots, n+1\}$ we have*

$$(i) \quad \sum_{1\le i_1<i_2\cdots<i_r\le n+1} \prod_{j=1}^{r} \frac{A_{i_j}B_{i_j} + PB_{i_j}}{PB_{i_j}} \ge C_{n+1}^{r}(n+2)^r;$$

$$(ii) \quad \sum_{1\le i_1<i_2\cdots<i_r\le n+1} \prod_{j=1}^{r} \frac{A_{i_j}B_{i_j} + PB_{i_j}}{A_{i_j}B_{i_j}} \le C_{n+1}^{r}\left(\frac{n+2}{n+1}\right)^r;$$

$$(iii) \quad \sum_{1\le i_1<i_2\cdots<i_r\le n+1} \prod_{j=1}^{r} \frac{A_{i_j}B_{i_j} + PA_{i_j}}{PA_{i_j}} \ge C_{n+1}^{r}\left(\frac{2n+1}{n}\right)^r;$$

$$(iv) \quad \sum_{1\le i_1<i_2\cdots<i_r\le n+1} \prod_{j=1}^{r} \frac{A_{i_j}B_{i_j} + PA_{i_j}}{A_{i_j}B_{i_j}} \le C_{n+1}^{r}\left(\frac{2n+1}{n+1}\right)^r.$$

PROOF : It is easy to see that $\sum_{i=1}^{n+1} \frac{PB_i}{A_iB_i} = 1$ and $\sum_{i=1}^{n+1} \frac{PA_i}{A_iB_i} = n$, these identities imply that

$$\left(\frac{1}{n+1}, \frac{1}{n+1}, \cdots, \frac{1}{n+1}\right) \prec \left(\frac{PB_1}{A_1B_1}, \frac{PB_2}{A_2B_2}, \cdots, \frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right) \tag{4.4}$$

and

$$\left(\frac{n}{n+1}, \frac{n}{n+1}, \cdots, \frac{n}{n+1}\right) \prec \left(\frac{PA_1}{A_1B_1}, \frac{PA_2}{A_2B_2}, \cdots, \frac{PA_{n+1}}{A_{n+1}B_{n+1}}\right). \tag{4.5}$$

Therefore, Theorem 4.6 follows from (4.4), (4.5), Theorem 3.1, Theorem 3.3 and (1.1).

Remark 4.2 : Mitrinović, Pečairć and Volenec [27, p.473-479] established a series of inequalities for $\frac{PA_i}{A_iB_i}$ and $\frac{PB_i}{A_iB_i}$, $i = 1, 2, \cdots, n+1$. Obvious, our inequalities in Theorem 4.6 are different from theirs.

***Theorem* 4.7** — *Suppose that $A \in M_n(C)(n \geq 2)$ is a complex matrix, $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\sigma_1, \sigma_2, \cdots, \sigma_n$ are the eigenvalues and singular values of A, respectively. If A is a positive definite Hermitian matrix and $r \in \{1, 2, \cdots, n\}$, then*

$$(i) \quad \sum_{1\leq i_1<i_2\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{1+\lambda_{i_j}}{\lambda_{i_j}} \geq C_n^r \left(\frac{n+trA}{trA}\right)^r;$$

$$(ii) \quad \sum_{1\leq i_1<i_2\cdots<i_r\leq n} \prod_{j=1}^{r} (1+\lambda_{i_j}) \leq C_n^r \left(\frac{n+trA}{n}\right)^r;$$

$$(iii) \quad \sum_{1\leq i_1<i_2\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{2+\lambda_{i_j}}{1+\lambda_{i_j}} \geq C_n^r \left(\frac{1+\sqrt[n]{det(I+A)}}{\sqrt[n]{det(I+A)}}\right)^r;$$

$$(iv) \quad \sum_{1\leq i_1<i_2\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{trA+\lambda_{i_j}}{\lambda_{i_j}} \geq C_n^r \left(\frac{trA+\sqrt[n]{detA}}{\sqrt[n]{detA}}\right)^r;$$

$$(v) \quad \sum_{1\leq i_1<i_2\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{1+\lambda_{i_j}}{\lambda_{i_j}} \leq \sum_{1\leq i_1<i_2\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{1+\sigma_{i_j}}{\sigma_{i_j}}.$$

PROOF : (i)–(ii) We clearly see that $\lambda_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \lambda_i = trA$, these lead to

$$\left(\frac{trA}{n}, \frac{trA}{n}, \cdots, \frac{trA}{n}\right) \prec (\lambda_1, \lambda_2, \cdots, \lambda_n). \tag{4.6}$$

Therefore, Theorem 4.7(i) and (ii) follows from (4.6), Theorem 3.1, Theorem 3.3 and (1.1).

(iii) It is easy to see that $1+\lambda_1, 1+\lambda_2, \cdots, 1+\lambda_n$ are the eigenvalues of matrix $I + A$ and $\prod_{i=1}^{n}(1+\lambda_i) = det(I + A)$, these yield that

$$\log\left(\sqrt[n]{det(I+A)},\ \sqrt[n]{det(I+A)},\cdots,\ \sqrt[n]{det(I+A)}\right) \prec \log(1+\lambda_1, 1+\lambda_2,\cdots,1+\lambda_n). \tag{4.7}$$

Therefore, Theorem 4.7(iii) follows from (4.7), Theorem 3.2 and (1.1).

(iv) Theorem 4.7(iv) follows from (1.1), Theorem 3.2 and the fact that

$$\log\left(\frac{\sqrt[n]{detA}}{trA},\frac{\sqrt[n]{detA}}{trA},\cdots,\frac{\sqrt[n]{detA}}{trA}\right) \prec \log\left(\frac{\lambda_1}{trA},\frac{\lambda_2}{trA},\cdots,\frac{\lambda_n}{trA}\right).$$

(v) A result due to H. Weyl [31] (see also [24, p.231]) gives

$$\log(\lambda_1,\lambda_2,\cdots,\lambda_n) \prec \log(\sigma_1,\sigma_2,\cdots,\sigma_n). \tag{4.8}$$

Therefore, Theorem 4.7(v) follows from (4.8), Theorem 3.2 and (1.1).

Remark 4.3 : The theorems give these results but it is not an approach that would be used for this purpose.

ACKNOWLEDGEMENT

The authors wish to thank the anonymous referees for their careful reading of the manuscript and their fruitful comments and suggestions.

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