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FOURIER ANALYSIS ON TRAPEZOIDS WITH CURVED SIDES [1]

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It is well known that smooth periodic functions can be expanded into Fourier series and can be approximated by trigonometric polynomials. The purpose of this paper is to do Fourier analysis for smooth functions on planar domains. A planar domain can often be divided into some trapezoids with curved sides, so first we do the Fourier analysis for smooth functions on trapezoids with curved sides. We will show that any smooth function on a trapezoid with curved sides can be expanded into Fourier sine series with simple polynomial factors, and so it can be well approximated by a combination of sine polynomials and simple polynomials. Then we consider the Fourier analysis on the global domain. Finally, we extend these results to the three-dimensional case.

Key words : Fourier analysis; trapezoid; prism; smooth extension; decomposition; approximation; sine polynomial.

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1. Introduction

It is well known that a smooth periodic function can be well expanded into Fourier series, and so it can be well approximated by trigonometric polynomials [4, 7, 9, 10, 15]. However, for a smooth function f on a general domain D, it is difficult to do Fourier analysis, one often smoothly extends f from D to the global $\mathbb{R}^d$ using the known extension method [1, 6, 8], and then one chooses a box that contains D and approximates it by splines [5]. In this paper, we will study how to do Fourier analysis for a smooth function on a bounded domain with arbitrary shape.

We first consider domain decomposition, this is also a difficult problem. One hundred ago, Schwarz discussed it. Here we do not try to go deep this problem. We only assume that D can be decomposed into some trapezoids with curved sides and we choose a partition such that the number of trapezoids is small as possible. Any trapezoid with a curved side can become a typical domain Ω under an affine transformation:

$$\Omega = \{(x,y); \quad 0 \le x \le 1, \quad 0 \le y \le g(x)\},$$

where

$$0 < g(x) < 1 \ \ (0 \le x \le 1).$$

Suppose that f is a smooth function on Ω. We do Fourier analysis as follows:

Step 1 : We extend f to a smooth function F on a $[0,1]^2$ and $F(x,1) = 0$ $(0 \le x \le 1)$. Here we present a new extension method to satisfy the above condition and the extension function F possesses very simple representative on $[0,1]^2 \setminus \Omega$.

Step 2 : We decompose F into $F = \phi_1 + \phi_2 + \phi_3$ on $[0,1]^2$, where ϕ_1 is a bivariate polynomial of degree 2, ϕ_2 is a sum of the form $\sum_1^3 \psi_i \, h_i$, in which each ψ_i is a univariate smooth periodic odd function and each h_i is a univariate polynomial of degree 1, and ϕ_3 is a bivariate smooth periodic odd function.

Step 3 : We expand each ψ_i and ϕ_3 into Fourier sine series. Finally, we get the Fourier sine expansion of f on Ω, with simple polynomial factors.

From Step 1 and Step 2, we see that a smooth function f on a trapezoid with a curved side can be expressed into a combination of periodic odd functions and polynomials of degree ≤ 2. Since a periodic odd function can be well approximated by sine polynomials, a smooth function f on a trapezoid with a curved side can be well approximated by a combination of polynomials of degree ≤ 2 and sine polynomials. Moreover, we give the precise approximation order. For trapezoids with two curved sides, we have also similar results.

For a smooth function f on a bounded domain D, we assume that

$$D = \bigcup_{j=1}^{M} D_j \qquad (\text{a disjoint union}),$$

where each D_j is a trapezoid with curved sides. Denote the restriction of f on D_j by f_j. Then the approximation of f on D is reduced to the approximation of f_j on D_j by a combination of polynomials of degree 2 and sine polynomials of degree $2N$ $(j = 1, ..., M)$. In our approximation process, the number M of subdomains $\{D_j\}$ is determined by the shape of the domain D and is independent of N.

In contrast to our approximation, for spline approximation, the approximation error tends to zero only if the number of subdomain tends infinite.

This paper is organized as follows. For a smooth function on a trapezoid with a curved side, in Sections 3 and 4, we give smooth extension theorem and decomposition theorem. In Sections 5, 6, and 7, we discuss Fourier expansion and the corresponding approximation theorem on trapezoids with curved sides. In Section 8, we discuss the global approximation of functions on a general domain. Finally, in Section 9, we extend these results to the three-dimensional case.

2. PRELIMINARIES

Let D be a simply connected domain of $\mathbb{R}^d$. We say $f \in H^\alpha(D)$ $(0 < \alpha \leq 1)$ if there exists a constant $0 < M < \infty$ such that for any $t,\ t' \in D$,

$$|f(t) - f(t')| \leq M \parallel t - t' \parallel^\alpha,$$

where $\| \cdot \|$ is the norm of the space $\mathbb{R}^d$. We say $f \in W^r H^\alpha(\Omega)$ if its derivatives

$$\frac{\partial^{i_1+\cdots+i_d} f}{\partial t_1^{i_1} \cdots \partial t_d^{i_d}} \in H^\alpha(D) \quad \text{for} \quad 0 \le t_1 + \cdots + t_d \le r.$$

We say f is a $\lambda-$periodic function on $\mathbb{R}^d$ if $f(t+n\lambda) = f(t)$ for $t \in \mathbb{R}^d$, $n \in \mathbb{Z}$. We say f is an odd function on $\mathbb{R}^d$ if $f(\varepsilon_1 t_1, \varepsilon_2 t_2, \cdots, \varepsilon_d t_d) = (-1)^{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_d} f(t_1, t_2, \cdots, t_d)$, where each $\varepsilon_k = 1$ or -1.

By T_N^d denote the set of all $d-$variate sine polynomials of the form

$$\sum_{n_1,\ldots,n_d=-N}^{N} c_{n_1,n_2,\ldots,n_d} \sin(\pi n_1 t_1) \sin(\pi n_2 t_2) \cdots \sin(\pi n_d t_d),$$

where each coefficient $c_{n_1,\ldots,n_d}$ is a constant.

Proposition 2.1 [10] — Let f be a $2-$periodic (odd) function, and for some $r \in Z_+$ and $0 < \alpha \le 1$, $\frac{\partial^r f}{\partial x_i^r} \in H^\alpha(\mathbb{R}^d)$ $(i = 1, ..., d)$. Then the best approximation of f in T_N^d is estimated as follows. For $1 \le p \le \infty$,

$$E_N(f)_p := \min_{q \in T_N^d} \| f - q \|_p = O\left(\frac{1}{N^{r+\alpha}}\right),$$

where $\| h \|_p = \left(\int_{[-1,1]^d} |h(t)|^p \, dt \right)^{\frac{1}{p}}$.

Let f be a $2-$periodic odd function on $\mathbb{R}^d$ and $f \in L_2([-1,1]^d)$. Then f can be expanded into the Fourier sine series [9]:

$$f(t_1, \cdots, t_d) = \sum_{n_1,\cdots,n_d=1}^{\infty} \alpha_{n_1,\cdots,n_d} \sin(\pi n_1 t_1) \cdots \sin(\pi n_d t_d) \qquad (L_2),$$

where the coefficient $\alpha_{n_1,\cdots,n_d} = \int_{[-1,1]^d} f(t_1, \cdots, t_d) \sin(\pi n_1 t_1) \cdots \sin(\pi n_d t_d) \, dt_1 \cdots dt_d$.

3. Smooth Extension of Functions on Trapezoids with a Curved Side

Without loss of generality, we assume that a trapezoid with a curved side is

$$\Omega = \{(x, y) : \quad 0 \leq x \leq 1, \;\; 0 \leq y \leq g(x)\}, \tag{3.1}$$

where $g \in W^1 H^\beta([0,1])$ and $0 < g(x) < 1$ $(0 \leq x \leq 1)$. Clearly, $\Omega \subset [0,1]^2$.

Let $f \in W^2 H^\alpha(\Omega)$ and Γ be the curved side of Ω. Now we smoothly extend f from Ω to $[0,1]^2$. Denote $\Omega^1 = \{(x,y) : \; 0 \leq x \leq 1, \;\; g(x) \leq y \leq 1\}$. Then

$$\Omega \bigcup \Omega^1 = [0,1]^2 \quad \text{and} \quad \Omega \bigcap \Omega^1 = \{(x,y) : \; y = g(x), \, 0 \leq x \leq 1\} = \Gamma. \tag{3.2}$$

Denote

$$a(x) = \frac{f(x, g(x))}{1 - g(x)}, \qquad b(x) = \frac{a(x) + \frac{\partial f}{\partial y}(x, g(x))}{(1 - g(x))^2}, \qquad 0 \leq x \leq 1. \tag{3.3}$$

For $0 \leq x \leq 1, \; y \in \mathbb{R}$, we define a function

$$P(x,y) = a(x)(1-y) + b(x)(1-y)^2(y - g(x)). \tag{3.4}$$

Lemma 3.1 — $P(x,y)$ has the following properties:

(i) $P \in W^1 H^\gamma([0,1] \times \mathbb{R})$, where $\gamma = \min\{\alpha, \beta\}$, and

(ii) for $0 \leq x \leq 1$, $\qquad P(x, g(x)) = f(x, g(x)), \qquad P(x,1) = 0,$

$$\frac{\partial P}{\partial x}(x, g(x)) = \frac{\partial f}{\partial x}(x, g(x)), \qquad \frac{\partial P}{\partial y}(x, g(x)) = \frac{\partial f}{\partial y}(x, g(x)).$$

Proof : We first prove that

$$a(x) \in W^1 H^\beta([0,1]), \qquad b(x) \in W^1 H^\gamma([0,1]). \tag{3.5}$$

Note that

$$\frac{d}{dx}(f(x, g(x))) = f_x(x, g(x)) + f_y(x, g(x)) \, g'(x), \tag{3.6}$$

where $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Since $f \in W^2\, H^\alpha(\Omega)$ and $g \in W^1\, H^\beta([0,1])$, the functions $f_x(x, g(x))$ and $f_y(x, g(x))$ are continuous on $[0,1]$ and

$$f_x(x, g(x)), \quad f_y(x, g(x)) \in H^1([0,1]). \tag{3.7}$$

Again, since $g' \in H^\beta([0,1])$, we have

$$\begin{aligned}
&|f_y(x_1, g(x_1))\, g'(x_1) - f_y(x_2, g(x_2))\, g'(x_2)| \\
&\le |f_y(x_1, g(x_1))|\, |g'(x_1) - g'(x_2)| + |g'(x_2)|\, |f_y(x_1, g(x_1)) - f_y(x_2, g(x_2))| \\
&\le K_1|x_1 - x_2|^\beta + K_2|x_1 - x_2| \le K_3|x_1 - x_2|^\beta, \qquad 0 \le x_1,\ x_2 \le 1.
\end{aligned}$$

Hereafter, K_i's are different constants. From this and (3.6)-(3.7), we get $\frac{d}{dx}(f(x, g(x))) \in H^\beta([0,1])$. Again, by the first formula in (3.3) and $0 < g(x) < 1$, we have $a'(x) \in H^\beta([0,1])$, so we get $a(x) \in W^1\, H^\beta([0,1])$.

Note that

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y}(x, g(x))\right) = f_{xy}(x, g(x)) + f_{y^2}(x, g(x))\, g'(x), \tag{3.8}$$

where $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$ and $f_{y^2} = \frac{\partial^2 f}{\partial y^2}$. Since $f \in W^2\, H^\alpha(\Omega)$ and $g \in W^1\, H^\beta(\Omega)$, we get

$$f_{xy}, \quad f_{y^2} \in H^\alpha(\Omega) \qquad \text{and} \qquad g \in H^1([0,1]), \quad g' \in H^\beta([0,1]).$$

So we have

$$\begin{aligned}
|f_{xy}(x_1, g(x_1)) - f_{xy}(x_2, g(x_2))| &\le K_1(|x_1 - x_2| + |g(x_1) - g(x_2)|)^\alpha \\
&\le K_2|x_1 - x_2|^\alpha \qquad (0 \le x_1,\ x_2 \le 1),
\end{aligned}$$

i.e.,

$$f_{xy}(x, g(x)) \in H^\alpha([0,1]). \tag{3.9}$$

Similarly, we have $f_{y^2}(x, g(x)) \in H^\alpha([0,1])$. So we have

$$\begin{aligned}
|f_{y^2}(x_1, g(x_1))\, g'(x_1) &- f_{y^2}(x_2, g(x_2))\, g'(x_2)| \\
&\le |f_{y^2}(x_1, g(x_1))|\, |g'(x_1) - g'(x_2)| \\
&\quad + |g'(x_2)|\, |f_{y^2}(x_1, g(x_1)) - f_{y^2}(x_2, g(x_2))| \\
&\le K_3|x_1 - x_2|^\gamma, x_1,\ x_2 \le 1,
\end{aligned}$$

where $\gamma = \min\{\alpha, \beta\}$. So $f_{y^2}(x, g(x))\, g'(x) \in H^\gamma([0, 1])$. From this and (3.8)-(3.9), we get

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y}(x, g(x))\right) \in H^\gamma([0, 1]).$$

So $\frac{\partial f}{\partial y}(x, g(x)) \in W^1\, H^\gamma([0, 1])$. Again, since $a(x),\ g \in W^1\, H^\beta([0, 1])$ and $0 < g(x) < 1$, by (3.3), we get $b(x) \in W^1\, H^\gamma([0, 1])$.

From (3.5) and $g \in W^1\, H^\beta([0, 1])$, and $\gamma = \min\{\alpha, \beta\}$, it follows by (3.4) that (i) holds.

We easily check that four equalities in (ii) hold, for example,

$$\begin{aligned} \frac{\partial P}{\partial x}(x,\ g(x)) &= a'(x)(1 - g(x)) - b(x)(1 - g(x))^2\, g'(x) \\ &= \frac{\partial f}{\partial x}(x,\ g(x))(0 \leq x \leq 1). \end{aligned}$$

Lemma 3.1 is proved. □

Remark 3.2 : In $P \in W^1\, H^\gamma([0, 1] \times \mathbb{R})$, the hidden constant depends on the equation $g(x)$ of the curved side of Ω.

Define

$$F(x, y) = \begin{cases} f(x, y), & (x, y) \in \Omega, \\ P(x, y), & (x, y) \in \Omega^1, \end{cases} \tag{3.10}$$

where $P(x, y)$ is defined in (3.4). Then we have the following extension theorem.

***Theorem* 3.3** — *Suppose that $f \in W^2\, H^\alpha(\Omega)$, where Ω is a trapezoid with a curved side which is stated in (3.1) and $g \in W^1\, H^\beta([0, 1])$ and $0 < g(x) < 1\ (0 \leq x \leq 1)$. If the extension F is stated as in (3.10), then F satisfies the condition $F \in W^1\, H^\gamma([0, 1]^2)$ and $F(x, 1) = 0\ (0 \leq x \leq 1)$, where $\gamma = \min\{\alpha, \beta\}$.*

PROOF : Note that $f \in W^2\, H^\alpha(\Omega)$, $P \in W^1\, H^\gamma(\Omega^1)$, and $\Omega \bigcap \Omega^1 = \Gamma = \{(x, y) :\ y = g(x),\ 0 \leq x \leq 1\}$. By Lemma 3.1 (ii), we have

$$P(x,y) = f(x,y), \qquad \frac{\partial P}{\partial x}(x,y) = \frac{\partial f}{\partial x}(x,y), \qquad \frac{\partial P}{\partial y}(x,y) = \frac{\partial f}{\partial y}(x,y), \quad (x,y) \in \Gamma.$$

From this, by (3.10), $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are continuous on $[0,1]^2$. Take two points z_1 and z_2 on $[0,1]^2$. When $z_1,\ z_2 \in \Omega$, by $\frac{\partial f}{\partial x} \in H^1(\Omega)$, we have

$$\left|\frac{\partial F}{\partial x}(z_1) - \frac{\partial F}{\partial x}(z_2)\right| = \left|\frac{\partial f}{\partial x}(z_1) - \frac{\partial f}{\partial x}(z_2)\right| \le K_1 \parallel z_1 - z_2 \parallel .$$

When $z_1,\ z_2 \in \Omega^1$, we have

$$\left|\frac{\partial F}{\partial x}(z_1) - \frac{\partial F}{\partial x}(z_2)\right| = \left|\frac{\partial P}{\partial x}(z_1) - \frac{\partial P}{\partial x}(z_2)\right|.$$

By Lemma 3.1 (i), $\frac{\partial P}{\partial x} \in H^\gamma(\Omega^1)$, so we get

$$\left|\frac{\partial F}{\partial x}(z_1) - \frac{\partial F}{\partial x}(z_2)\right| \le K_2 \parallel z_1 - z_2 \parallel^\gamma .$$

Let $z_1 \in \Omega$ and $z_2 \in \Omega^1$. Denote the intersection of the curved side Γ with straight line through z_1 and z_2 by z^*. By Lemma 3.1 (ii), $\frac{\partial P}{\partial x}(z^*) = \frac{\partial f}{\partial x}(z^*)$. From this and (3.10), we get

$$\begin{aligned} |\tfrac{\partial F}{\partial x}(z_1) - \tfrac{\partial F}{\partial x}(z_2)| &= |\tfrac{\partial f}{\partial x}(z_1) - \tfrac{\partial P}{\partial x}(z_2)| \\ &\le |\tfrac{\partial f}{\partial x}(z_1) - \tfrac{\partial f}{\partial x}(z^*)| + |\tfrac{\partial P}{\partial x}(z^*) - \tfrac{\partial P}{\partial x}(z_2)| \\ &\le K_1 \parallel z_1 - z_2 \parallel^\gamma . \end{aligned}$$

So $\frac{\partial F}{\partial x} \in H^\gamma([0,1]^2)$. Similarly, we have $\frac{\partial F}{\partial y} \in H^\gamma([0,1]^2)$. This implies $F \in W^1 H^\gamma([0,1]^2)$. Finally, by Lemma 3.1 (ii) and (3.10), we get $F(x,1) = P(x,1) = 0$ $(0 \le x \le 1)$. Theorem 3.3 is proved. □

4. Decomposition of Smooth Functions on $[0,1]^2$

In the preceding section, we have smoothly extended $f \in W^2 H^\alpha(\Omega)$ from the trapezoid Ω with a curved side to the smooth function F on the square $[0,1]^2$, which is stated in (3.10). By Theorem 3.2, we know that the extension function F satisfies that $F \in W^1 H^\gamma([0,1]^2)$ $(\gamma = \min\{\alpha, \beta\})$, and $F(x,1) = 0$ $(0 \le x \le 1)$.

In this section, we will express the smooth extension F as a combination of smooth periodic functions and polynomials of degree ≤ 2.

First, we give the following decomposition of F on $[0,1]^2$

$$F(x,y) = \phi_1(x,y) + \phi_2(x,y) + \phi_3(x,y) \qquad ((x,y) \in [0,1]^2), \tag{4.1}$$

where

$$\phi_1(x,y) = F(0,0)(1-x)(1-y) + F(1,0)x(1-y), \tag{4.2}$$

$$\phi_2(x,y) = (1-x)\phi_{21}(y) + x\phi_{22}(y) + (1-y)\phi_{23}(x), \tag{4.3}$$

here $\phi_{21}(t) = F(0,t) - \phi_1(0,t)$, $\phi_{22}(t) = F(1,t) - \phi_1(1,t)$, $\phi_{23}(t) = F(t,0) - \phi_1(t,0)$, and

$$\phi_3(x,y) = F(x,y) - \phi_1(x,y) - \phi_2(x,y). \tag{4.4}$$

Lemma 4.1 — Let F be decomposed as in (4.1)-(4.4). Then

(i) $\phi_1(x,y) = F(x,y)$ on the four vertexes of $[0,1]^2$, and

(ii) $\phi_3(x,y) = 0$ on the boundary $\partial([0,1]^2)$.

Proof : By (4.2), we have $\phi_1(0,0) = F(0,0)$, $\phi_1(1,0) = F(1,0)$, and $\phi_1(0,1) = \phi_1(1,1) = 0$. Again , by Theorem 3.3, we get $F(0,1) = F(1,1) = 0$, and so $F(0,1) = \phi_1(0,1)$ and $F(1,1) = \phi_1(1,1)$. (i) follows.

By (4.3) and (i), we get

$$\begin{aligned} \phi_2(x,0) &= (1-x)(F(0,0) - \phi_1(0,0)) + x(F(1,0) \\ &- \phi_1(1,0)) + (F(x,0) - \phi_1(x,0)) = F(x,0) - \phi_1(x,0) \quad (0 \le x \le 1). \end{aligned}$$

Again, by (4.4), $\phi_3(x,0)=0\ (0\le x\le 1)$. By (4.3) and (i), we get

$$\phi_2(x,1)=(1-x)(F(0,1)-\phi_1(0,1))+x(F(1,1)-\phi_1(1,1))=0 \qquad (0\le x\le 1).$$

By Theorem 3.2 and (4.2), we get $F(x,1)=\phi_1(x,1)=0\ (0\le x\le 1)$ and by (4.4), $\phi_3(x,1)=0\ (0\le x\le 1)$.

Similarly, we get $\phi_3(0,y)=\phi_3(1,y)=0\ (0\le y\le 1)$. So we get (ii). Lemma 4.1 is proved. □

First, we extend each ϕ_{2i} in (4.3) to a 2-periodic odd function ϕ_{2i}^* as follows:

$$\widetilde{\phi}_{2i}(t) = \begin{cases} \phi_{2i}(t), & 0\le t\le 1,\\ -\phi_{2i}(-t), & -1\le t\le 0,\end{cases}$$

$$\phi_{2i}^*(t+2n\pi)=\widetilde{\phi}_{2i}(t),\ -1\le t\le 1,\quad n\in\mathbb{Z}\ \ (i=1,2,3).$$

By Lemma 4.1 (i), we have $\phi_{2i}(0)=\phi_{2i}(1)=0$. Therefore, the above ϕ_{2i}^* is well-defined.

Next, we extend ϕ_3 to a 2-periodic odd function on $[-1,1]^2$ by

$$\widetilde{\phi}_3(x,y)=\begin{cases} \phi_3(x,y), & (x,y)\in[0,1]^2,\\ -\phi_3(-x,y), & (x,y)\in[-1,0]\times[0,1],\\ \phi_3(-x,-y), & (x,y)\in[-1,0]^2,\\ -\phi_3(x,-y), & (x,y)\in[0,1]\times[-1,0]\end{cases} \tag{4.5}$$

and

$$\phi_3^*(x+2n,\ y+2m)=\widetilde{\phi}_3(x,\ y) \qquad ((x,y)\in[-1,1]^2,\quad n,\ m\in\mathbb{Z}). \tag{4.6}$$

By Lemma 4.1 (ii), we have $\phi_3(x,y)=0\ ((x,y)\in\partial([0,1]^2))$. Therefore, (4.5) and (4.6) are well-defined and

$$\phi_3^*(x,y)=0 \qquad \text{for} \quad x\in\mathbb{Z} \quad \text{or} \quad y\in\mathbb{Z}. \tag{4.7}$$

Lemma 4.2 — Let ϕ_{21}^*, ϕ_{22}^*, ϕ_{23}^*, and ϕ_3^* be stated as above. Then

(i) ϕ_{2i}^* is a 2-periodic odd function and $\phi_{2i}^* \in W^1\, H^\gamma(\mathbb{R})$ $(i = 1, 2, 3)$;

(ii) ϕ_3 is a 2-periodic odd function and $\phi_3^* \in W^1\, H^\gamma(\mathbb{R}^2)$, where $\gamma = \min\{\alpha, \beta\}$.

PROOF : Since the proof of (i) is easier than that of (ii), we only prove (ii).

Consider two neighboring squares $D = [0,1]^2$ and $D_{-1} = [-1,0] \times [0,1]$. Since $F \in W^1\, H^\gamma(D)$ and ϕ_1 is a polynomial, by (4.3) and (4.4), we see that $\phi_3 \in W^1\, H^\gamma(D)$. By (4.5) and (4.6), we deduce that

$$\phi_3^* \in W^1\, H^\gamma(D), \qquad \phi_3^* \in W^1\, H^\gamma(D_{-1}). \tag{4.8}$$

Let $J = D \bigcap D_{-1}$. Then $J = \{(x,y) : x = 0,\ 0 \le y \le 1\}$. By (4.7), $\phi_3^*(x,y) = 0$ on J. By (4.8), $\phi_3^* \in C(D \bigcup D_{-1})$. Since ϕ_3^* is odd, we have $\phi_3^*(-x,y) = -\phi_3^*(x,y)$ and $\frac{\partial \phi_3^*}{\partial x}(-x,y) = \frac{\partial \phi_3^*}{\partial x}(x,y)$. This implies that

$$\lim_{x\to 0, x<0} \frac{\partial \phi_3^*}{\partial x}(x,y) = \lim_{x\to 0, x>0} \frac{\partial \phi_3^*}{\partial x}(x,y).$$

From this and (4.8), we obtain that for $(x_0, y_0) \in J$,

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in D}} \frac{\partial \phi_3^*}{\partial x}(x,y) = \lim_{\substack{(x,y)\in(x_0,y_0)\\(x,y)\in D_{-1}}} \frac{\partial \phi_3^*}{\partial x}(x,y).$$

On the other hand, since $\phi_3^*(x,y) = 0$ on J, we obtain that for $(x_0, y_0) \in J$,

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in D}} \frac{\partial \phi_3^*}{\partial y}(x,y) = \frac{\partial \phi_3}{\partial y}(0, y_0) = 0,$$

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in D_{-1}}} \frac{\partial \phi_3^*}{\partial y}(x,y) = \frac{\partial \phi_3^*}{\partial y}(0, y_0) = 0.$$

Again, by (4.8), we have

$$\frac{\partial \phi^*}{\partial x},\ \frac{\partial \phi^*}{\partial y} \in C(D_0 \bigcup D_{-1}). \tag{4.9}$$

Since $\frac{\partial \phi^*}{\partial x},\ \frac{\partial \phi^*}{\partial y} \in H^\gamma(D_0)$ and $\frac{\partial \phi^*}{\partial x},\ \frac{\partial \phi^*}{\partial y} \in H^\gamma(D_{-1})$, by (4.9), we have

$$\phi_3^* \in W^1\, H^\gamma([-1,1] \times [0,1]). \tag{4.10}$$

Since ϕ_3^* is a 2-periodic function, by (4.10), we have

$$\phi_3^* \in W^1 H^\gamma([0,2] \times [0,1]). \tag{4.11}$$

Combining (4.10) with (4.11), we get $\phi_3^* \in W^1 H^\gamma([-1,2] \times [0,1])$. Since ϕ_3^* is an odd function, we have $\phi_3^* \in W^1 H^\gamma([-1,2] \times [-1,0])$. An argument similar to (4.10) deduces that $\phi_3^* \in W^1 H^\gamma([-1,2] \times [-1,1])$. By periodicity, $\phi_3^* \in W^1 H^\gamma([-1,2]^2)$. Continuing this procedure, by periodicity, we have $\phi_3^* \in W^1 H^\gamma(\mathbb{R}^2)$. Lemma 4.2 is proved. □

From this, we obtain the following decomposition theorem.

***Theorem* 4.3** — *Let F be stated as above: $F \in W^1 H^\gamma([0,1]^2)$ and $F(x,1) = 0$ $(0 \le x \le 1)$. Then*

$$F(x,y) = \phi_1^*(x,y) + \phi_2^*(x,y) + \phi_3^*(x,y) \qquad ((x,y) \in [0,1]^2),$$

where (i) ϕ_1^* is a polynomial of degree 2 and

$$\phi_1^*(x,y) = F(0,0)(1-x)(1-y) + F(1,0)x(1-y);$$

(ii) $\phi_2^*(x,y) = (1-x)\phi_{21}^*(y) + x\phi_{22}^*(y) + (1-y)\phi_{23}^*(x)$,

here each ϕ_{2i}^* are 2-periodic odd function and $\phi_{2i}^* \in W^1 H^\gamma(\mathbb{R})$, and $\phi_{21}^*(y) = F(0,y) - F(0,0)(1-y)$, $\phi_{22}^*(y) = F(1,y) - F(1,0)(1-y)$ $(0 \le y \le 1)$, and $\phi_{23}^*(x) = F(x,0) - F(0,0)(1-x) - F(1,0)x$ $(0 \le x \le 1)$;

(iii) $\phi_3^*(x,y)$ is a 2-periodic odd function and $\phi_3^* \in W^1 H^\gamma(\mathbb{R}^2)$ and

$$\phi_3^*(x,y) = F(x,y) - \phi_1^*(x,y) - \phi_2^*(x,y) \qquad ((x,y) \in [0,1]^2).$$

5. Fourier Expansion with Simple Polynomial Factors

Based on the smooth extension in Section 3 and the decomposition in Section 4, for the smooth function f on the trapezoid Ω with a curved side, we will give its Fourier sine expansion with simple polynomial factors.

Let $f \in W^2\, H^\alpha(\Omega)$ and Ω be stated as in (3.1). Again, let

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in \Omega, \\ \\ P(x,y), & (x,y) \in \Omega^1 \end{cases} \tag{5.1}$$

as in (3.10), where $P(x,y)$ is stated in (3.4).

Step 1 : We expand the smooth extension F on $[0,1]^2$ into the Fourier sine series with polynomial factors. We decompose F as in Theorem 4.3:

$$F(x,y) = \phi_1^*(x,y) + \phi_2^*(x,y) + \phi_3^*(x,y) \quad ((x,y) \in [0,1]^2), \tag{5.2}$$

where $\phi_1^*,\ \phi_2^*$, and ϕ_3^* are stated as in Theorem 4.3 (i)-(iii).

First we expand $\phi_2^*(x,y)$ into the Fourier sine series with polynomial factors of degree 1. In Theorem 4.3 (ii), each ϕ_{2i}^* is a 2-periodic odd function and $\phi_{2i}^* \in W^1\, H^\gamma(\mathbb{R})\ (i = 1,2,3)$, we can expand them into the Fourier sine series

$$\phi_{2i}^*(t) = \sum_{n=1}^{\infty} \alpha_{ni}\, \sin(\pi n t) \quad (i = 1,2,3), \tag{5.3}$$

where $\alpha_{n,i} = 2 \int\limits_0^1 \phi_{2i}^*(t)\, \sin(\pi n t)\, dt$.

Since $\phi_3^* \in W^1\, H^\gamma(\mathbb{R}^2)$ is a 2-periodic odd function, by Theorem 4.3 (iii), we can expand it into Fourier sine series

$$\phi_3^*(x,y) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} c_{n_1 n_2}\, \sin(\pi n_1 x)\, \sin(\pi n_2 y), \tag{5.5}$$

where the Fourier coefficients

$$c_{n_1 n_2} = 4 \int\limits_{[0,1]^2} (F(x,y) - \phi_1^*(x,y) - \phi_2^*(x,y))\, \sin(\pi n_1 x)\, \sin(\pi n_2 y)\, dx\, dy, \tag{5.6}$$

From this and Theorem 4.3, we obtain that for $(x,y)\in[0,1]^2$,

$$\begin{aligned} F(x,y) &= F(0,0)(1-x)(1-y)+F(1,0)x(1-y)+(1-y) \\ \sum_{n=1}^{\infty}\alpha_{n3}\,\sin(\pi nx) &+ (1-x)\sum_{n=1}^{\infty}\alpha_{n1}\,\sin(\pi ny)+x\sum_{n=1}^{\infty}\alpha_{n2}\,\sin(\pi ny) \\ &+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}c_{n_1n_2}\,\sin(\pi n_1x)\,\sin(\pi n_2y), \end{aligned} \tag{5.7}$$

where the Fourier coefficients $\alpha_{n1},\ \alpha_{n2},\ \alpha_{n3}$ which are stated in (5.4) and $c_{n_1n_2}$ is stated in (5.6).

Step 2 : We expand f on the trapezoid Ω with a curved side into the Fourier sine series with simple polynomial factors. By (5.1), we have $F(x,y)=f(x,y)\ ((x,y)\in\Omega)$ and

$$F(0,t)=\begin{cases} f(0,t), & 0\le t\le g(0),\\ P(0,t), & g(0)\le t\le 1,\end{cases} \qquad F(1,t)=\begin{cases} f(1,t), & 0\le t\le g(1),\\ P(1,t), & g(1)\le t\le 1.\end{cases}$$

From this, we can rewrite representations of the Fourier coefficients α_{n1}, α_{n2}, α_{n3}, and $c_{n_1n_2}$ in (5.7). Considering the restriction on Ω of Formula (5.7), we obtain the Fourier sine expansion of the function $f(x,y)$ on a trapezoid Ω with a curved side.

***Theorem* 5.1** — *Let* $f\in W^2\,H^{\alpha}(\Omega)$ *and* $\Omega=\{(x,y)\,:\,0\le x\le 1,\,0\le y\le g(x)\}$, *where* $g\in W^1\,H^{\beta}([0,1])$ *and* $0<g(x)<1\,(0\le x\le 1)$. *Then the Fourier sine expansion of* f *with simple polynomial factors is*

$$\begin{aligned} f(x,y) &= f(0,0)(1-x)(1-y)+f(1,0)x(1-y) \\ &+ (1-y)\sum_{n=1}^{\infty}\alpha_{n3}\,\sin(\pi nx)+(1-x)\sum_{n=1}^{\infty}\alpha_{n1}\,\sin(\pi ny) \\ &+ x\sum_{n=1}^{\infty}\alpha_{n2}\,\sin(\pi ny)+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}c_{n_1n_2}\,\sin(\pi n_1x)\,\sin(\pi n_2y), \\ & \quad (x,y)\in\Omega, \end{aligned}$$

where

$$\alpha_{n1} = 2\int_0^{g(0)} f(0,t)\,\sin(\pi nt)\,dt + 2\int_{g(0)}^{1} P(0,t)\,\sin(\pi nt)\,dt - \frac{2}{n\pi}\,f(0,0),$$

$$\alpha_{n2} = 2\int_0^{g(1)} f(1,t)\,\sin(\pi nt)\,dt + 2\int_{g(1)}^{1} P(1,t)\,\sin(\pi nt)\,dt - \frac{2}{n\pi}\,f(1,0),$$

$$\alpha_{n3} = 2\int_0^{1} f(t,0)\,\sin(\pi nt)\,dt - \frac{2}{n\pi}\,f(0,0) + \frac{2(-1)^n}{n\pi}\,f(1,0),$$

$$\begin{aligned} c_{n_1 n_2} &= 4\int_\Omega f(x,y)\,\sin(\pi n_1 x)\,\sin(\pi n_2 y)\,dx\,dy + 4\int_{\Omega^1} P(x,y) \\ &\quad \sin(\pi n_1 x)\,\sin(\pi n_2 y)\,dx\,dy \\ &+ \frac{4}{\pi^2}\frac{(-1)^{n_1} f(1,0) - f(0,0)}{n_1 n_2} - \frac{2}{\pi}\left(\frac{\alpha_{n_1 1}}{n_1} - \frac{(-1)^{n_1}\alpha_{n_2 2}}{n_1} + \frac{\alpha_{n_1 3}}{n_2}\right), \end{aligned}$$

here $P(x,y)$ *is stated in (3.4) and* $\Omega^1 = \{(x,y):\ 0 \le x \le 1,\ g(x) \le y \le 1\}$.

Remark 5.2 : By (3.3) and (3.4), $P(0,t)$ and $P(1,t)$ are both polynomials of degree 3. Therefore, the integrals

$$\int_{g(0)}^{1} P(0,t)\,\sin(\pi nt)\,dt \quad \text{and} \quad \int_{g(1)}^{1} P(1,t)\,\sin(\pi nt)\,dt$$

are computed easily. By (3.3)-(3.4), the integral

$$\int_{\Omega^1} P(x,y)\,\sin(\pi n_1 x)\,\sin(\pi n_2 y)\,dx\,dy$$

can be computed by the values of f and $\frac{\partial f}{\partial y}$ on the curved side of Ω and the values of $g(x)$.

6. Approximation of Functions on Trapezoids with a Curved Side

Let the trapezoid Ω with a curved side be stated as (3.1). Again, let $f \in W^2\, H^\alpha(\Omega)$, and $F(x,y)$ be the smooth extension of f from Ω to $[0,1]^2$ which is stated as in (5.1). By Theorem 4.3 and (5.1), we know that

$$f(x,y) = f(0,0)(1-x)(1-y) + f(1,0)x(1-y)$$

$$+(1-x)\phi_{21}^*(y) + x\phi_{22}^*(y) + (1-y)\phi_{23}^*(x) + \phi_3^*(x,y), \qquad (x,y) \in \Omega, \quad (6.1)$$

where each ϕ_{2i}^* is a 2-periodic odd function, and $\phi_{2i}^* \in W^1\, H^\gamma(\mathbb{R})$ $(\gamma = \min\{\alpha,\ \beta\})$, and $\phi_3^*(x,y)$ is a bivariate 2-periodic odd function and $\phi_3^*(x,y) \in W^1\, H^\gamma(\mathbb{R}^2)$.

For a $N \in \mathbb{Z}_+$, let $\tau_N(\phi_{2i}^*)$ be the best approximation sine polynomial of ϕ_{2i}^* in the space T_N^1, and $\tau_N(\phi_3^*)$ be the best approximation sine polynomial of ϕ_3^* in the space T_N^2. Define a combination of polynomials of degree ≤ 2 and sine polynomials of degree $\leq 2N$

$$\begin{aligned} q_N(x,y) &= f(0,0)(1-x)(1-y) + f(1,0)x(1-y) \\ &+ (1-x)\tau_N(\phi_{21}^*)(y) + x\tau_N(\phi_{22}^*)(y) \qquad (6.2) \\ &+ (1-y)\tau_N(\phi_{23}^*)(x) + \tau_N(\phi_3^*)(x,y). \end{aligned}$$

From this, we have

***Theorem* 6.1** — *Let $f \in W^2\, H^\alpha(\Omega)$ and $\Omega = \{(x,y):\ 0 \leq x \leq 1,\ 0 \leq y \leq g(x)\}$, where $g \in W^1\, H^\beta([0,1])$ and $0 < g(x) < 1$ $(0 \leq x \leq 1)$. Then*

$$\| f - q_N \|_{L_p(\Omega)} = O\left(\frac{1}{N^{1+\gamma}}\right),$$

where q_N is stated in (6.2) and $\gamma = \min\{\alpha,\beta\}$.

PROOF : By (6.1) and (6.2), we obtain the deviation

$$\begin{aligned} f(x,y) - q_N(x,y) &= (1-x)(\phi_{21}^*(y) - \tau_N(\phi_{21}^*)) + x(\phi_{22}^*(y) \\ &\quad -\tau_N(\phi_{22}^*)(y)) \\ &+ (1-y)(\phi_{23}^*(x) - \tau_N(\phi_{23}^*)(x)) + \phi_3^*(x,y) \\ &\quad -\tau_N(\phi_3^*)(x,y). \end{aligned}$$

This implies that for any $1 \le p < \infty$,

$$\begin{aligned} \| f - q_N \|_{L_p([0,1]^2)} &\le \sum_{i=1}^{3} \| \phi_{2i}^* - \tau_N(\phi_{2i}^*) \|_{L_p([0,1])} \\ &\quad + \| \phi_3^* - \tau_N(\phi_3^*) \|_{L_p([0,1]^2)} . \end{aligned} \tag{6.3}$$

Since each ϕ_{2i}^* is a 2-periodic odd function and $\phi_{2i}^* \in W^1\, H^\gamma(\mathbb{R})$, and ϕ_3^* is a 2-periodic odd function and $\phi_3^* \in W^1\, H^\gamma(\mathbb{R}^2)$, by Proposition 2.1, we have

$$\| \phi_{2i}^* - \tau_N(\phi_{2i}^*) \|_{L_p([0,1])} = O\left(\frac{1}{N^{1+\gamma}}\right) \quad (i = 1,2,3),$$

$$\| \phi_3^* - \tau_N(\phi_3^*) \|_{L_p([0,1]^2)} = O\left(\frac{1}{N^{1+\gamma}}\right).$$

Finally, by (6.3), we get the desired result. □

By Remark 3.2, we know that in the $O\left(\frac{1}{N^{1+\alpha}}\right)$, the hidden constant depends on $g(x)$.

7. TRAPEZOIDS WITH TWO CURVED SIDES

In this section we discuss Fourier analysis of smooth functions on trapezoids with two curved sides. We only give the corresponding results. Their arguments are similar to the case of trapezoids with a curved side. We omit the detail here.

Let Ω be a trapezoid with two curved sides $\Omega := \{(x, y) : 0 \le x \le 1, \tau(x) \le y \le g(x)\}$, where $0 < \tau(x) < g(x) < 1$ $(0 \le x \le 1)$ and $g \in W^1 H^{\beta}([0,1]),\ \tau \in W^1 H^{\beta^*}([0,1])$.

Consider a smooth function $f \in W^2 H^{\alpha}(\Omega)$ on Ω. Denote

$$\Omega^1 = \{(x,y): \quad 0 \le x \le 1, \quad g(x) \le y \le 1\},$$

$$\Omega^2 = \{(x,y): \quad 0 \le x \le 1, \quad 0 \le y \le \tau(x)\}.$$

Then $\Omega^1 \bigcup \Omega \bigcup \Omega^2 = [0,1]^2$.

For $0 \le x \le 1,\ y \in \mathbb{R}$, we define $P(x,y)$ as in (3.4). We again define

$$P^*(x,y) = \frac{f(x,\tau(x))\, y}{\tau(x)} + \left(\frac{\partial f}{\partial y}(x,\tau(x)) - \frac{f(x,\tau(x))}{\tau(x)} \right) \frac{y^2(y-\tau(x))}{\tau^2(x)}.$$

Then we have the following extension theorem of f from Ω to $[0,1]^2$.

***Theorem* 7.1** — *Let*

$$F(x,y) = \begin{cases} P(x,y), & (x,y) \in \Omega^1, \\ f(x,y), & (x,y) \in \Omega, \\ P^*(x,y), & (x,y) \in \Omega^2. \end{cases}$$

Then $F \in W^1 H^{\gamma}([0,1])$ $(\gamma = \min\{\alpha, \beta, \beta^*\})$ *and* $F(x,0) = F(x,1) = 0$ $(0 \le x \le 1)$.

Furthermore, we have the following decomposition theorem of F on $[0,1]^2$.

***Theorem* 7.2** — *For* $(x,y) \in [0,1]^2$*, we have*

$$F(x,y) = (1-x)F(0,y) + xF(1,y) + \phi(x,y), \tag{7.1}$$

where $F(0,y)$ *and* $F(1,y)$ *can be extended to periodic odd univariate functions* $F^*(0,y)$ *and* $F^*(1,y)$*, which belong to* $W^1 H^{\gamma}(\mathbb{R})$ *and* $\phi(x,y)$ *can be extended* *to a bivariate periodic odd function $\phi^*(x,y)$ which belongs to $W^1\, H^\gamma(\mathbb{R}^2)$, where $\gamma = \min\{\alpha, \beta, \beta^*\}$.*

We expand $F(0,y)$, $F(1,y)$, and $\phi^*(x,y)$ into the Fourier sine series. We get the Fourier sine expansion with simple polynomial factor of F on $[0,1]^2$. Considering the restriction of this expansion theorem on Ω, we will get the Fourier sine series of f with simple polynomial factors on Ω.

Let $\tau_N(F^*(0,y))$ and $\tau_N(F^*(1,y))$ be the best approximation sine polynomials of $F^*(0,y)$ and $F^*(1,y)$ in the space T_N^1, respectively, and $\tau_N(\phi^*)$ be the best approximation sine polynomial of ϕ^* in the space T_N^2. In (7.1), replacing $F(0,y)$, $F(1,y)$, and $\phi(x,y)$ by $\tau_N(F^*(0,y))$, $\tau_N(F^*(1,y))$, and $\tau_N(\phi^*)$, respectively, we get a combination of polynomial of degree 1 and sine polynomials of degree $2N$:

$$\widetilde{q}_N(x,y) = (1-x)\tau_N(F^*(0,y)) + x\tau_N(F^*(1,y)) + \tau_N(\phi^*). \tag{7.2}$$

Finally, we get the following approximation theorem:

***Theorem* 7.3** — *Let $\widetilde{q}_N(x,y)$ be stated in (7.2). Then*

$$\| f - \widetilde{q}_N \|_{L_p(\Omega)} = O\left(\frac{1}{N^{1+\gamma}}\right),$$

where $\gamma = \min\{\alpha, \beta_1, \beta_2\}$.

8. Approximation of Functions on General Planar Domains

Suppose that D_2 is a bounded planar domain whose boundary possesses the smoothness $W^1\, H^\alpha$ and which can be decomposed as follows:

$$D_2 = \bigcup_{j=1}^{M} \Omega_j \qquad (\text{a disjoint union}),$$

where each Ω_j is a trapezoid with curved sides and each Ω_j can become

$$\widetilde{\Omega}_j = \{(x,y), \quad 0 \le x \le 1, \;\; \tau_j(x) \le y \le g_j(x)\} \qquad (0 < \tau_j(x) < g_j(x) < 1)$$

under an affine transformation L_j, i.e., $L_j\Omega_j = \widetilde{\Omega}_j$. Since the boundary curves belong to $W^1\, H^\alpha$, we have $\tau_j \in W^1\, H^\alpha$ and $g_j \in W^1\, H^\alpha$.

Let $f \in W^2\, H^\alpha(D_2)$ and f_j is its restriction on Ω_j. Denote $\widetilde{f}_j = f_j \circ L_j^{-1}$. Then $\widetilde{f}_j \in W^2\, H^\alpha(\widetilde{\Omega}_j)$. By Sections 6 and 7, we know that for any $N \in \mathbb{Z}_+$, there exists a combination $\widetilde{q}_N^{(j)}(x, y)$ similar to (7.2) such that

$$\| \widetilde{f}_j - \widetilde{q}_N^{(j)} \|_{L_p(\Omega_j)} = O\left(\frac{1}{N^{1+\alpha}}\right).$$

Setting $q_N^{(j)} = \widetilde{q}_N^{(j)} \circ L_j$, we have

$$\| f_j - q_N^{(j)} \|_{L_p(\Omega_j)} = O\left(\frac{1}{N^{1+\alpha}}\right) \qquad (j = 1, ..., M),$$

where $q_N^{(j)}(x, y)$ is a combination of polynomials of degree ≤ 2 and sine polynomials of degree $2N$. From this and $D = \sum\limits_{j=1}^{M} \Omega_j$ (where M is independent of N), we deduce that

$$\| f - \sum_{j=1}^{M} q_N^{(j)} \chi_{\Omega_j} \|_{L_p(D)} = O\left(\frac{1}{N^{1+\alpha}}\right).$$

Here χ_{Ω_j} is the characteristic function of Ω_j. This is a global approximation with M elements. In this approximation process, the original domain D_2 is divided M trapezoids with curved sides, where the number M depends only on the shape of the domain D_2 and is independent of N.

In spline approximation, for a smooth function on a bounded domain, based on known smooth extension theorems [1,6,8] and partition $\Delta = \{C\}$ of the domain [5, Section 6.2], one constructs piecewise polynomial as its approximation tool. In approximation error, the fundamental infinitesimal is the partition diameter

$$\text{diam}\, \Delta = \max_{C \in \Delta} \text{diam}\, C.$$

From this, we see that the approximation error tends zero only if the cardinality of the collection $\Delta := \{C\}$ tends infinite. Therefore, in this point, the approximation proposed by this paper is quite different from spline approximation.

9. FOURIER ANALYSIS OF FUNCTIONS ON THE THREE-DIMENSIONAL DOMAINS

Now we extend the results given in Sections 3-8 from the two-dimensional case to the three-dimensional case. Since the methods of arguments are similar, here we omit their proofs.

It is well-known that a three-dimensional domain with arbitrary shape can be divided into some prisms with curved surfaces.

9.1. *Prisms with a curved top*

Let Ω be a prism with a curved top and

$$\Omega = \{(x, y, z) : \quad 0 \le x,\ y \le 1,\ \ 0 \le z \le q(x, y)\}, \tag{9.1}$$

where $q(x, y) \in W^1\, H^\beta([0, 1]^2)$ and $0 < q(x, y) < 1$ $((x, y) \in [0, 1]^2)$.

Let $f \in W^2\, H^\alpha(\Omega)$. Denote

$$Q = \{(x, y, z) :\ 0 \le x, y \le 1,\ q(x, y) \le z \le 1\}. \tag{9.2}$$

We first smoothly extend f from Ω to $[0, 1]^3$. For $(x, y, z) \in Q$, define

$$\begin{aligned} P(x, y, z) \quad = \quad & \frac{f(x, y, q(x, y))(1 - z)}{1 - q(x, y)} + \left(\frac{f(x, y, q(x, y))}{1 - q(x, y)} + \frac{\partial f}{\partial z}(x, y, q(x, y)) \right) \\ & \frac{(z - q(x, y))(1 - z)^2}{(1 - q(x, y))^2} \end{aligned} \tag{9.3}$$

and

$$\Phi(x, y, z) = \begin{cases} f(x, y, z), & (x, y, z) \in \Omega, \\ \\ P(x, y, z), & (x, y, z) \in Q. \end{cases}$$

Then $\Phi(x, y, z) \in W^1\, H^\gamma([0, 1]^3)$ $(\gamma = \min\{\alpha, \beta\})$ and $\Phi(x, y, 1) = 0$ $((x, y) \in [0, 1]^2)$, where $\gamma = \min\{\alpha, \beta\}$.

We decompose Φ as $\Phi(x,y,z) = V_1(x,y,z) + V_2(x,y,z) + V_3(x,y,z) + V_4(x,y,z), \ (x,y,z) \in [0,1]^3$, where

$$\begin{aligned} V_1(x,y,z) &= \Phi(0,0,0)(1-x)(1-y)(1-z) + \Phi(0,1,0)(1-x)y(1-z) \\ &+ \Phi(1,0,0)x(1-y)(1-z) + \Phi(1,1,0)xy(1-z) \end{aligned}$$

is a polynomial of degree 3; The second term is

$$\begin{aligned} V_2(x,y,z) &= \Phi_1(x,0,0)(1-y)(1-z) + \Phi_1(x,1,0)y(1-z) \\ &+ \Phi_1(0,y,0)(1-x)(1-z) + \Phi_1(1,y,0)x(1-z) \\ &+ \Phi_1(0,0,z)(1-x)(1-y) + \Phi_1(0,1,z)(1-x)y \\ &+ \Phi_1(1,0,z)x(1-y) + \Phi_1(1,1,z)xy \qquad (\Phi_1 = \Phi - V_1), \end{aligned}$$

here the first factor of each term can be extended to a 2-periodic odd function in $W^1 H^\gamma(\mathbb{R})$. Therefore, it can be expanded into univariate Fourier sine series. Furthermore, it can be L_p-approximated with approximation order $O(\frac{1}{N^{1+\gamma}})$ by univariate sine polynomials of degree $\leq N$. From this, we deduce that $V_2(x,y,z)$ can be L_p-approximated with approximation order $O(\frac{1}{N^{1+\gamma}})$ by a combination of bivariate polynomials of degree ≤ 2 and univariate sine polynomials of degree $\leq N$; The third term is

$$\begin{aligned} V_3(x,y,z) &= \Phi_2(0,y,z)(1-x) + \Phi_2(1,y,z)x + \Phi_2(x,0,z)(1-y) \\ &+ \Phi_2(x,1,z)y + \Phi_2(x,y,0)(1-z) \quad (\Phi_2 = \Phi_1 - V_2), \end{aligned}$$

here the first factor of each term can be extended to 2-periodic odd function in $W^1 H^\gamma(\mathbb{R}^2)$ and can be expanded into bivariate Fourier sine series. Furthermore, it can be L_p-approximated with approximation order $O(\frac{1}{N^{1+\gamma}})$ by bivariate sine polynomials of degree $\leq 2N$. From this, we deduce that $V_3(x,y,z)$ can be L_p-approximated with approximation order $O(\frac{1}{N^{1+\gamma}})$ by a combination of univariate polynomials of degree ≤ 1 and bivariate sine polynomials of degree $\leq 3N$ Finally, the fourth term is

$$V_4(x,y,z) = \Phi_2(x,y,z) - V_3(x,y,z),$$

here $V_4(x,y,z)$ can be extended to a 2-periodic odd function in the space $W^1\, H^\gamma(\mathbb{R}^3)$ and can be expanded into three-variate Fourier sine series, and can be approximated by three-variate sine polynomials with approximation order $O(\frac{1}{N^{1+\gamma}})$.

From this, for the smooth function f on Ω, we can obtain its Fourier sine expansion with simple polynomial factors. Moreover, we can obtain the L_p-approximation of f with approximation order $O(\frac{1}{N^{1+\gamma}})$ by a combination of three-variate polynomials of degree 3 and three-variate sine polynomials of degree $3N$.

Prisms with two curved surfaces

Let Ω be a prism with two curved surfaces

$$\Omega = \{(x,y,z): \qquad 0 \le x,y \le 1, \quad q^*(x,y) \le z \le q(x,y)\},$$

where $q(x,y) \in W^1\, H^\beta([0,1]^2)$ and $q^*(x,y) \in W^1\, H^{\beta^*}([0,1]^2), \quad 0 < q(x,y) < q^*(x,y) < 1, \ (x,y) \in [0,1]^2$.

Let $f \in W^2\, H^\alpha(\Omega)$. Now we smoothly extend f from Ω to $[0,1]^3$. Define Q as in (9.2). Define Q^* as follows:

$$Q^* = \{(x,y): \qquad 0 \le x,y \le 1, \quad 0 \le z \le q^*(x,y)\}.$$

Then $Q \bigcup \Omega \bigcup Q^* = [0,1]^3$.

For $0 \le x,y \le 1, \ z \in \mathbb{R}$, we define $P(x,y,z)$ as in (9.3). We again define

$$P^*(x,y,z) = \frac{f(x,y,q^*(x,y))\, z}{q^*(x,y)} + \left(\frac{\partial f}{\partial z}(x,y,q^*(x,y)) - \frac{f(x,y,q^*(x,y))}{q^*(x,y)} \right) \frac{z^2(z-q^*(x,y))}{(q^*(x,y))^2}.$$

Let

$$\Phi(x,y,z) = \begin{cases} P(x,y,z), & (x,y,z) \in Q, \\ f(x,y,z), & (x,y,z) \in \Omega, \\ P^*(x,y,z), & (x,y,z) \in Q^*. \end{cases}$$

Then $\Phi(x,y,z) \in W^1 H^\gamma([0,1]^3)$, where $\gamma = \min\{\alpha, \beta, \beta^*\}$.

We decompose Φ on $[0,1]^3$ as

$$\Phi(x,y,z) = W_1(x,y,z) + W_2(x,y,z) + W_3(x,y,z), \qquad (x,y,z) \in [0,1]^3,$$

where the first term is

$$\begin{aligned} W_1(x,y,z) &= \Phi(0,0,z)(1-x)(1-y) + \Phi(0,1,z)(1-x)\,y + \Phi(1,0,z) \\ &\quad x\,(1-y) + \Phi(1,1,z)\,x\,y, \end{aligned} \tag{9.4}$$

here the first factor of each term can be extended to a 2-periodic odd function in $W^1 H^\gamma(\mathbb{R})$. The second term is

$$W_2(x,y,z) = \Psi(0,y,z)(1-x) + \Psi(1,y,z)\,x + \Psi(x,0,z)\,(1-y) + \Psi(x,1,z)\,y, \tag{9.5}$$

where $\Psi = \Phi - W_1$, here the first factor of each term can be extended to 2-periodic odd function in $W^1 H^\gamma(\mathbb{R}^2)$. The third term is

$$W_3(x,y,z) = \Psi(x,y,z) - W_2(x,y,z), \tag{9.6}$$

here $W_3(x,y,z)$ can be extended to 2-periodic odd function in $W^1 H^\gamma(\mathbb{R}^3)$. Expanding the periodic functions in (9.4)-(9.6) into Fourier sine series, we get the Fourier sine expansion of f with simple polynomial factors. Replacing each periodic function by the corresponding best approximation sine polynomial, we get the corresponding approximation theorem.

Approximation of functions on three-dimensional domains

Suppose that D_3 is a bounded three-dimensional domain whose boundary surfaces belong to $W^1 H^\alpha$. We decompose D_3 as follows:

$$D_3 = \bigcup_{j=1}^{M} \Omega_j^{(3)} \qquad \text{(a disjoint union)},$$

where each $\Omega_j^{(3)}$ is a prism with curved surfaces.

Let $f \in W^2 H^\alpha(D_3)$ and its restriction on Ω_j by f_j. Similar to the argument in Section 8, based on subsections 9.1 and 9.2, we easily find the combination $Q_N^{(j)}(x, y, z)$ of polynomials of degree ≤ 3 and sine polynomials of degree $\le 3N$ such that

$$\| f - \sum_{j=1}^{M} Q_N^{(j)} \chi_{\Omega_j^{(3)}} \|_{L^p(D_3)} = O\left(\frac{1}{N^{1+\alpha}}\right),$$

where $\chi_{\Omega_j^{(3)}}$ is the characteristic function of $\Omega_j^{(3)}$ and M is independent of N.

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