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ON LOCALLY DUALLY FLAT (α, β)-METRICS WITH ISOTROPIC S-CURVATURE

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In this paper, we consider locally dually flat (α, β)-metrics with isotropic S-curvature and find some necessary and sufficient conditions under which these metrics reduce to locally Minkowskian metrics.

Key words : (α, β)-metric; locally dually flat metric; S-curvature.

1. INTRODUCTION

For a Finsler metric $F = F(x, y)$, its geodesics curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and given by following

$$G^i = \frac{1}{4} g^{il} \Big\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l} \Big\}, \ \ y \in T_x M.$$

A Finsler metric F is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y \in T_xM$ for any $x \in M$ [15].

In [1], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory [2].

On the other hand, Finsler geometry consists of classical and generalized Finsler geometries. The classical Finsler geometry is the differential geometry of Finsler spaces. More precisely, it is a subject studying manifolds whose tangent spaces carry a norm varying smoothly with the base point. Indeed, Finsler geometry is just Riemannian geometry without the quadratic restriction. Therefore it is natural to extending the construction of locally dually flat metrics for Finsler geometry. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [12]. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structure [6, 7, 17, 19, 20]. A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

where $H = H(x, y)$ is a C^∞ scalar function on $TM_0 = TM \setminus \{0\}$ satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system [6, 7, 16, 19, 20]. In [12], Shen proved that the Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies

$$(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}.$$

In this case, $H = -\frac{1}{6}[F^2]_{x^m} y^m$.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if its geodesic coefficients G^i are in the form

$$G^i = Py^i,$$

where $P : TU = U \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous with degree one, $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$. We call $P(x, y)$ the projective factor of $F(x, y)$.

The S-curvature is a scalar function on TM, which was introduced by the Shen to study volume comparison in Riemann-Finsler geometry [13]. The S-curvature **S** measures the average rate of change of $(T_xM, F_x := F|_{T_xM})$ in the direction $y \in T_xM$. It is known that $\mathbf{S} = 0$ for Berwald metrics. A Finsler metric on an n-dimensional manifold M is said to have isotropic S-curvature if $\mathbf{S} = (n+1)c(x)F$, for some scalar function $c = c(x)$ on M [18].

In this paper, we study locally dually flat (α, β)-metrics with isotropic S-curvature, and find some conditions under which these metrics reduce to locally Minkowskian metrics. First, we prove the following.

***Theorem* 1.1** — *Let $F := \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an non-Randers type $(\alpha, \beta)-$metric on a manifold M of dimension $n \geq 3$. Suppose that $\phi(s)$ is an analytic function or polynomial of s with $\phi'(0) = \phi''(0) = \phi'''(0) = 0$ and $\phi(0) \neq 0$. Then F is locally dually flat with isotropic S-curvature if and only if it is locally Minlowskian.*

For example, see the following.

Example 1 : All polynomial $\phi(s) = c_n s^n + c_{n-1} s^{n-1} + ... + c_4 s^4 + c_0$, $(c_0 \neq 0)$ satisfies in condition of Theorem 1.1.

Then, we prove the following.

***Theorem* 1.2** — *Let $F := \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an non-Randers type $(\alpha, \beta)-$metric on a manifold M of dimension $n \geq 3$. Suppose that $\phi(s)$ is a polynomial of s with $\phi(0) \neq 0$, $\phi'(0) = \phi'''(0) = 0$ and $\frac{\phi''(0)}{\phi(0)} > -1$. Then F is locally dually flat with isotropic S-curvature if and only if it is locally Minlowskian.*

Example 2 : We consider the Ingarden-Tammassy metric $F = \alpha + \frac{\beta^2}{\alpha}$ [9]. Since $\phi(s) = 1 + s^2$, then we have

$$\phi(0) = 1 \neq 0,$$
$$\phi'(0) = 0,$$
$$\frac{\phi''(0)}{\phi(0)} = 2 > -1, \quad \phi''' = 0.$$

Thus Ingarden-Tammassy metric satisfies in condition of Theorem 1.2. It is remarkable that, this metric introduced by Ingarden-Tamassy in [9], when they study physical applications of Finsler metrics in electron optics and thermodynamics.

Example 3 : Let

$$F = \alpha\Big[1 + (\frac{\beta}{\alpha})^2 + (\frac{\beta}{\alpha})^4 + (\frac{\beta}{\alpha})^5 + \cdots\Big],$$

where $||\beta||_\alpha = b_0 < \frac{1}{2}$ and $|s| < b_0$. Then $\phi = 1 + s^2 + s^4 + s^5 + \cdots$ is a C^∞ function on an open interval $(-b_0, b_0)$ and

$$\phi(0) = 1 \neq 0,$$
$$\frac{\phi''(0)}{\phi(0)} = 2 > -1,$$
$$\phi'(0) = \phi'''(0) = 0.$$

Therefore, F is a Finsler metric satisfies in condition of Theorem 1.2

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM\setminus\{0\}$ the slit tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on TM_0;

(ii) F is positively 1-homogeneous on the fibers of the tangent bundle of M,

(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y+su+tv)\right]|_{s,t=0}, \quad u,v \in T_xM.$$

For a Finsler metric $F = F(x,y)$ on a smooth manifold M, geodesic curves are characterized by the system of second order differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i\big(x, \frac{dx}{dt}\big) = 0,$$

where the local functions $G^i = G^i(x,y)$ are called the spray coefficients, and given by

$$G^i = \frac{1}{4}g^{il}\big\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\big\}.$$

In a standard local coordinates (x^i, y^i) in TM, the vector field $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ is called the spray of F [14].

A Finsler metric F is called a Berwald metric, if G^i are quadratic in $y \in T_xM$ for any $x \in M$. It is proved that on a Berwald manifold (M, F), the parallel translation along any geodesic preserves the Minkowski functionals. Thus, Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathrm{R}^n \;\Big|\; F\Big(y^i\frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}}.$$

In general, the local scalar function $\sigma_F(x)$ can not be expressed in terms of elementary functions, even F is locally expressed by elementary functions.

Let

$$\tau(x,y) := \ln\Big[\frac{\sqrt{\det\Big(g_{ij}(x,y)\Big)}}{\mathrm{Vol}(\mathrm{B}^n(1))} \cdot \mathrm{Vol}\Big\{(y^i) \in \mathrm{R}^n\Big|\; F\Big(y^i\frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}\Big].$$

$\tau = \tau(x,y)$ is a scalar function on TM_0, which is called the *distortion*. For a vector $\mathbf{y} \in T_xM$, let $c(t), -\epsilon < t < \epsilon$, denote the geodesic with $c(0) = x$ and $\dot{c}(0) = \mathbf{y}$. Define

$$\mathbf{S}(\mathbf{y}) := \frac{d}{dt}\Big[\tau\Big(\dot{c}(t)\Big)\Big]|_{t=0}.$$

We call $\mathbf{S}$ the S-curvature. This quantity was introduced in [13] for a volume comparison theorem.

Let $G^i(x,y)$ denote the geodesic coefficients of F in the same local coordinate system. The S-curvature can be defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$. $\mathbf{S}$ said to be *isotropic* if there is a scalar functions $c(x)$ on M such that

$$\mathbf{S}(x,y) = (n+1)c(x)F(x,y).$$

3. Proof of Theorems 1.1 and 1.2

General (α,β)-metrics were first studied by Matsumoto in 1972 as a direct generalization of Randers metrics [10, 11]. This class of Finsler metrics has some applications in Physics [3, 4]. An (α,β)-metric is a Finsler metric on a manifold M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. For an (α,β)-metric, let us define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, β is closed if and only if $s_{ij}=0$. An (α,β)-metric is said to be trivial if $r_{ij}=s_{ij}=0$. Put

$$
\begin{aligned}
&r_{i0}:=r_{ij}y^j,\ \ r_{00}:=r_{ij}y^iy^j,\ \ r_j:=b^ir_{ij},\\
&s_{i0}:=s_{ij}y^j,\ \ \ s_j:=b^is_{ij},\\
&r_0:=r_jy^j,\ \ \ \ s_0:=s_jy^j.
\end{aligned}
$$

In [6], it is proved the following.

Lemma 3.1 ([6]) — Let $F=F(x,y)$ be a Finsler metric on an open subset $U\subset\mathbb{R}^n$. Then F is locally dually flat and projectively flat on U if and only if

$$F_{x^k}=CFF_{y^k},$$

where C is a constant.

Then, Xia proved the following (see Theorem 1.2 in [19]).

***Theorem* 3.2** ([19]) — *Let $F:=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$ be an non-Riemannian (α,β)-metric on a manifold M of dimension $n\geq 3$, where $\beta=b_i(x)y^i\neq 0$. Suppose that $\phi(s)$ is an analytic function with $\phi'(0)=\phi''(0)=0$ or $\phi(s)$ is a polynomial of s with $\phi'(0)=0$. Then F is locally dually flat on M if and only if α and β satisfy*

$$
\begin{aligned}
s_{l0}&=\frac{1}{3}(\beta\theta_l-\theta b_l),\\
r_{00}&=\frac{2}{3}\left[\theta\beta-(\theta_lb^l)\alpha^2\right],\\
G^l_\alpha&=\frac{1}{3}\left[2\theta y^l+\theta^l\alpha^2\right].
\end{aligned}
$$

where $\theta_i(x)y^i$ is a 1-form on M and $\theta^l:=a^{lm}\theta_m$.

Now, let $\phi=\phi(s)$ be a positive C^∞ function on $(-b_0,b_0)$. For a number $b\in[0,b_0)$, let

$$\Phi:=-(Q-sQ')\{n\Delta+1+sQ\}-(b^2-s^2)(1+sQ)Q'' \tag{1}$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q' \tag{2}$$

$$Q = \frac{\phi'}{\phi - s\phi'}. \tag{3}$$

$$Q' = \frac{\phi\phi''}{(\phi - s\phi')^2} \tag{4}$$

$$Q'' = \frac{\phi'\phi'' + \phi\phi'''}{(\phi - s\phi')^2} + \frac{2s\phi\phi''^2}{(\phi - s\phi')^3}. \tag{5}$$

Then we have

$$\Phi = -(Q - sQ')(n+1)\Delta + (b^2 - s^2)\{(Q - sQ')Q' - (1 + sQ)Q''\}. \tag{6}$$

In [5], Cheng-Shen study the class of (α, β)-metrics of non-Randers type $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ with isotropic S-curvature and obtain the following.

***Theorem* 3.3** ([19]) — *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an non-Riemannian (α, β)-metric on a manifold and $b := \|\beta_x\|_\alpha$. Suppose that $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constant $k_1 > 0$, k_2 and k_3. Then F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

(a) β satisfies

$$r_{ij} = \varepsilon\{b^2 a_{ij} - b_i b_j\}, \quad s_j = 0, \tag{7}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2} \tag{8}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\varepsilon$.

(b) β satisfies

$$r_{ij} = 0, \quad s_j = 0. \tag{9}$$

In this case, $\mathbf{S} = 0$, regardless of choices of a particular ϕ.

By a direct computation, we can obtain a formula for mean Cartan torsion of an (α,β)-metric as follows

$$I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i).$$

It is remarkable that, Cheng-Wang-Wang prove that the condition $\Phi = 0$ characterizes the Riemannian metrics among (α,β)-metrics [8]. Hence, in the continue, we suppose that $\Phi \neq 0$.

By using the Theorem 3.3, we are going to consider locally dually flat (α,β)-metric with isotropic S-curvature.

Lemma 3.4 — Let $F := \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an non-Randers type $(\alpha,\beta)-$metric on a manifold M of dimension $n \geq 3$. Suppose that $\phi(s)$ is an analytic function or polynomial of s with $\phi'(0) = \phi''(0) = \phi'''(0) = 0$ and $\phi(0) \neq 0$. Then F is of isotropic S-curvature if and only if β satisfies $r_{ij} = 0$ and $s_j = 0$.

PROOF : In (1),(2) ,(3), (4) and (5), putting $s = 0$ imply that

$$Q(0) = Q'(0) = Q''(0) = 0,$$
$$\Delta(0) = 1, \quad \Phi(0) = 0.$$

Thus in $s = 0$, we have

$$[(b^2 - s^2)\Phi]|_0 = b^2\Phi(0) = 0.$$

On the other hand, by assumptions we get

$$-2(n+1)k\phi(0)\Delta^2(0) = -2k(n+1)k\phi(0) \neq 0.$$

Then in $s = 0$, we have

$$(b^2 - s^2)\Phi \neq -2(n+1)k\phi\Delta^2.$$

This implies that

$$(b^2 - s^2)\Phi \neq -2(n+1)k\phi\Delta^2.$$

Therefore the part (a) of Theorem 3.3 is not hold. □

Lemma 3.5 — Let $F := \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an non-Randers type $(\alpha, \beta)-$metric on a manifold M of dimension $n \geq 3$. Suppose that $\phi(s)$ is a polynomial of s with $\phi(0) \neq 0$, $\phi'(0) = \phi'''(0) = 0$ and $\frac{\phi''(0)}{\phi(0)} > -1$. Then F is of isotropic S-curvature if and only if β satisfies $r_{ij} = 0$ and $s_j = 0$.

PROOF : By (1), (2), (3), (4) and (5), if $s = 0$ then we have

$$Q(0) = Q''(0) = 0, \;\; Q'(0) > -1,$$
$$\Delta(0) = 1 + b^2 Q'(0) > 1 - b^2 > 0,$$
$$\Phi(0) = 0.$$

Thus in $s = 0$, we have

$$[(b^2 - s^2)\Phi]|_0 = b^2\Phi(0) = 0.$$

Since $\phi(0) \neq 0$ and $\Delta(0) > 0$, then

$$-2(n+1)k\phi(0)\Delta^2(0) \neq 0.$$

Thus in the point $s = 0$, we have

$$(b^2 - s^2)\Phi(0) \neq -2(n+1)k\phi(0)\Delta^2(0).$$

This implies that

$$(b^2 - s^2)\Phi \neq -2(n+1)k\phi\Delta^2.$$

It follows that the part (a) of Theorem 3.3 is not hold. □

***Theorem* 3.6 —** *Let $F := \alpha\phi(s),\ s = \frac{\beta}{\alpha}$ be a locally dually flat non-Randers type $(\alpha, \beta)-$metric on a manifold M of dimension $n \geq 3$, $\phi(s)$ is an analytic function or polynomial of s with $\phi'(0) = \phi''(0) = \phi'''(0) = 0$ and $\phi(0) \neq 0$. Suppose that F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on M. Then F is locally projectively flat in adapted coordinate systems.*

PROOF : Let $G^i = G^i(x, y)$ and $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x, y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we have

$$G^i = \bar{G}^i_\alpha + Py^i + Q^i, \tag{10}$$

where

$$P = \alpha^{-1}\Theta\Big[-2Q\alpha s_0 + r_{00}\Big] \tag{11}$$

$$Q^i = \alpha Q {s^i}_0 + \Psi\Big[-2Q\alpha s_0 + r_{00}\Big]b^i \tag{12}$$

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big[(\phi - s\phi') + (b^2 - s^2)\phi''\Big]} \tag{13}$$

$$\Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \tag{14}$$

By the Lemma 3.4, F is isotropic S-curvature if and only if the following hold

$$r_{00} = 0, \tag{15}$$

$$s_j = 0. \tag{16}$$

By Theorem 3.2, we have

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l), \tag{17}$$

$$G^m_\alpha = \frac{1}{3}\left[2\theta y^l + \theta^l\alpha^2\right], \tag{18}$$

$$r_{00} = \frac{2}{3}\left[\theta\beta - (\theta_l b^l)\alpha^2\right]. \tag{19}$$

By (15) and (19), we obtain

$$(\theta_l b^l)\alpha^2 = (\theta)\beta \tag{20}$$

Since α^2 is a irreducible polynomial, then (20) reduces to following

$$\theta = 0, \tag{21}$$

$$\theta_l b^l = 0. \tag{22}$$

Then (17), (18) and (19) reduce to following

$$s_{l0} = 0, \tag{23}$$

$$G^m_\alpha = 0, \tag{24}$$

$$r_{00} = 0. \tag{25}$$

By (16), we get

$$s_0 = 0$$

and by (23) we have

$$s^l_0 = 0.$$

By (10), we have $G^i = 0$. □

***Theorem* 3.7** — *Let $F := \alpha\phi(s),\ s = \frac{\beta}{\alpha}$ be a locally dually flat non-Randers type $(\alpha, \beta)-$metric on M^n $(n \geq 3)$, where $\phi(s)$ is a polynomial of s with $\phi(0) \neq 0$, $\phi'(0) = \phi'''(0) = 0$ and $\frac{\phi''(0)}{\phi(0)} > -1$. Suppose that F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on M. Then F is locally projectively flat in adapted coordinate systems.*

PROOF : By the Lemma 3.5, F is isotropic S-curvature, $\mathbf{S} = (n+1)cF$, if and only if $r_{00} = 0$ and $s_j = 0$. Then by using the same method used in Theorem 3.6, we get the proof. □

Proof of Theorem 1.1 : Under the assumption we conclude that F is dually flat and projectively flat in any adapted coordinate system. By Lemma 3.1, we have

$$F_{x^k} = CFF_{y^k}.$$

Thus the spray coefficients $G^i = Py^i$ are given by

$$P = \frac{1}{2}CF.$$

Since $G^i = 0$, then $P = 0$ and thus $C = 0$. It implies that $F_{x^k} = 0$ and then F is a locally Minkowskian metric in the adapted coordinated system. This completes the proof. □

Proof of Theorem 1.2 : By the same method used in Theorem 1.1, we get $G^i = 0$ which implies that $F_{x^k} = 0$. Thus F is a locally Minkowskian metric in the adapted coordinated system. □

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