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A CHARACTERIZATION OF FINSLER METRICS OF CONSTANT FLAG CURVATURE

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Using the notion of C-projectively flatness, we give a new characterization of Finsler metrics of constant flag curvature.

Key words : Finsler metrics; flag curvature; Weyl curvature; C-projective transformations.

1. INTRODUCTION

One of the important problems in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature [10]. On the other hand, there are some well-known projective invariants of Finsler metrics namely, Weyl curvature, generalized Douglas -Weyl curvature [5] and another C-projective invariant **H**-curvature [7]. Weyl introduces a projective invariant for Riemannian metrics. Then Douglas extendes Weyl's projective invariant to Finsler metrics. Finsler metrics with vanishing projective Weyl curvature are called *Weyl metrics*. Z. Szabó proves that Weyl metrics are exactly Finsler metrics of scalar flag curvature.

Recently, the non-Riemannian quantity $\mathbf{H}$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics, has been studied extensively (for more details see [6, 7]). Akbar-Zadeh proves that for a Weyl manifold of dimension $n \geq 3$, the flag curvature is constant if and only if $\mathbf{H} = 0$. Then, it is natural to find other projectively invariant quantity which characterizes Finsler metrics of constant flag curvature [9].

In this paper, we define a projective invariant so called $\overline{W}$-curvature. We show that the $\overline{W}$-curvature is another candidate for characterizing Finsler metrics of constant flag curvature. More precisely, we prove the following.

***Theorem* 1.1** — *Let (M, F) be a connected Finsler manifold with dimension $n \geq 3$. Then F is of constant flag curvature if and only if $\overline{W} = 0$.*

We set the Berwald connection on Finsler manifolds. The vertical and horizontal covariant derivatives of Berwald connection are denoted by ',' and '|', respectively.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on TM_0 (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite,

$$g_y(u, v) := \frac{1}{2}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where $G^i := \frac{1}{4} g^{il}\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\}y^j y^k$ is called the associated *spray* to (M, F). The projection of an integral curve of $\mathbf{G}$ is called a *geodesic* in M. A diffeomorphism $f : (M, F) \to (M, \bar{F})$ between two Finsler manifolds is called a projective transformation, if f maps every geodesic of F to a geodesic of $\bar{F}$ as a point set.

The Riemannian curvature tensor of Berwald connection are given by ${K^i}_{hjk} = d_jG^i_{hk} + G^m_{hk}G^i_{mj} - d_kG^i_{hj} - G^m_{hj}G^i_{mk}$, where $d_k = \partial_k - G^m_k\dot{\partial}_m$, $\partial_k = \frac{\partial}{\partial x^k}$, $\dot{\partial}_k = \frac{\partial}{\partial y^k}$, $G^i_k = \dot{\partial}_kG^i$ and $G^i_{jk} = \dot{\partial}_jG^i_k$ (for more details, see [7]).

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald. For a flag $P = \mathrm{span}\{y, u\} \subset T_xM$ with flagpole y, $\mathbf{K} = \mathbf{K}(P, y)$ stands for *flag curvature*. When F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P$, which is just the sectional curvature of P in Riemannian geometry. We say that a Finsler metric F is *of scalar curvature* if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent bundle TM_0. If $\mathbf{K} = K(x)$, then F is said to be of *isotropic flag curvature*. If $\mathbf{K} = constant$, then F is said to be of *constant flag curvature*.

3. Special Projective Weyl Curvature

Let $\phi : F^n \to \bar{F}^n$ be a projective transformation. Then, there exists a positive homogeneous scalar function $P(x, y)$ of degree one satisfying

$$\bar{G}^i = G^i + Py^i.$$

In this case, P is called the projective factor. Under a projective transformation with projective factor P, the Riemannian curvature tensor of Berwald connection changes as follows

$$\bar{K}^i{}_{hjk} = K^i{}_{hjk} + y^i\dot{\partial}_hQ_{jk} + \delta^i_hQ_{jk} + \delta^i_j\dot{\partial}_hQ_k - \delta^i_k\dot{\partial}_hQ_j, \tag{1}$$

where $Q_i = d_iP - PP_i$ and $Q_{ij} = \dot{\partial}_iQ_j - \dot{\partial}_jQ_i$. A projective transformation with projective factor P is said to be *C-projective* if $Q_{ij} = 0$.

Lemma 3.1 — Every C-projective mapping $\phi : F^n \to \bar{F}^n$ with projective factor P satisfies $P_{ij|s}y^s = 0$.

PROOF : First, we prove that

$$P_{ik|s}y^s = y^s\dot{\partial}_i Q_{ks}. \tag{2}$$

We know that $\dot{\partial}_k d_i = d_i\dot{\partial}_k - G^r_{ik}\dot{\partial}_r$. Hence, we get

$$\begin{aligned} y^s\dot{\partial}_i Q_{ks} &= y^s(\dot{\partial}_i d_s P_k - \dot{\partial}_i d_k P_s) \\ &= y^s(d_s P_{ik} - G^r_{si}P_{kr} - d_k P_{is} + G^r_{ik}P_{sr}) \\ &= y^s d_s P_{ik} - G^s_k P_{is} + G^s_i P_{sk} \\ &= P_{ik|s}y^s \end{aligned} \tag{3}$$

By assumption $Q_{ij} = 0$. Then (2) implies the desired result. □

Let X be a projective vector field on a Finsler manifold (M, F). Suppose that the vector field X in a local coordinate (x^i) on M is written in the form $X = X^i(x)\partial_i$. Then the complete lift of X is denoted by $\hat{X}$ and locally defined by $\hat{X}=X^i\partial_i + y^j\partial_j X^i\dot{\partial}_i$. Suppose that $\pounds_{\hat{X}}$ stands for Lie derivative with respect to the complete lift of X. Then we have

$$\pounds_{\hat{X}} G^i = P y^i, \tag{4}$$

$$\pounds_{\hat{X}} G^i_k = \delta^i_k P + y^i P_k, \tag{5}$$

$$\pounds_{\hat{X}} G^i_{jk} = \delta^i_j P_k + \delta^i_k P_j + y^i P_{jk}, \tag{6}$$

$$\pounds_{\hat{X}} G^i_{jkl} = \delta^i_j P_{kl} + \delta^i_k P_{jl} + \delta^i_l P_{kj} + y^i P_{jkl}, \tag{7}$$

$$\pounds_{\hat{X}} K^i_{jkl} = \delta^i_j(P_{l|k} - P_{k|l}) + \delta^i_l P_{j|k} - \delta^i_k P_{j|l} + y^i\dot{\partial}_j(P_{l|k} - P_{k|l}). \tag{8}$$

Since $Q_{ij} = P_{i|j} - P_{j|i}$, we have

$$\pounds_{\hat{X}} K^i_{jkl} = \delta^i_j Q_{lk} + \delta^i_l P_{j|k} - \delta^i_k P_{j|l} + y^i\dot{\partial}_j Q_{lk}. \tag{9}$$

We have

$$\dot{\partial}_j P_{k|l} = P_{jk|l} - P_r G^r_{jkl}. \tag{10}$$

Contracting i and k in (9) and using (2), we get

$$\pounds_{\hat{X}} K_{jl} = P_{l|j} - nP_{j|l} + P_{jl|s} y^s, \tag{11}$$

where $K_{jl} := K^r_{jrl}$. Now, suppose that $Q_{ij} = 0$. Then by Lemma 3.1, we have $P_{jl|s} y^s = 0$. Therefore, (11) reduces to the following

$$\pounds_{\hat{X}} K_{jl} = P_{l|j} - nP_{j|l}. \tag{12}$$

$$\pounds_{\hat{X}} K_{lj} = P_{j|l} - nP_{l|j}. \tag{13}$$

From (12) and (13), one can obtain

$$P_{j|l} = \frac{1}{1-n^2} \pounds_{\hat{X}} \Big\{ K_{lj} + nK_{jl} \Big\}. \tag{14}$$

Substituting (14) into (9) and using the assumption $Q_{ij} = 0$, we are led to the following tensor

$$\overline{W}^{\,i}_{jkl} := K^i_{jkl} - \frac{1}{1-n^2} \Big\{ \delta^i_l (K_{kj} + nK_{jk}) - \delta^i_k (K_{lj} + nK_{jl}) \Big\}. \tag{15}$$

If we put $\overline{W}^{\,i}_{\ k} := \overline{W}^{\,i}_{jkl} y^j y^l$, then we have

$$\overline{W}^{\,i}_{\ k} = K^i_{\ k} - \frac{1}{1-n^2} \Big\{ y^i (K_{k0} + nK_{0k}) - \delta^i_k (n+1) K_{00} \Big\}. \tag{16}$$

The tensor $\overline{W}^{\,i}_{\ k}$ is said to be *special projective Weyl curvature* or $\overline{W}$*-curvature*.

Proposition 3.2 — Let X be a C-projective vector field of Finsler metric F. Then we have $\pounds_{\hat{X}} \overline{W}^{\,i}_{hjk} = 0$.

Therefore, special Weyl projective curvature is invariant under C-projective transformations.

4. Proof of Theorem 1.1

First, we prove that the class of Finsler metrics of scalar flag curvature contains the class of Finsler metrics with vanishing $\overline{W}$-*curvature*.

Proposition 4.1 — Let F be a Finsler metric with vanishing $\overline{W}$-*curvature*. Then F is of scalar flag curvature.

PROOF : By assumption, we have the following

$$K^i{}_k - \frac{1}{1-n^2}\Big\{y^i(K_{k0} + nK_{0k}) - \delta^i_k(n+1)K_{00}\Big\} = 0, \tag{17}$$

where by definition $K^i{}_k = K^i_{jkl}y^j y^l$. It is well known that

$$K^i{}_k = 2\partial_k G^i - y^j\partial_j\dot{\partial}_k G^i + 2G^j\dot{\partial}_j\dot{\partial}_k G^i - \dot{\partial}_j G^i\dot{\partial}_k G^j, \tag{18}$$

which implies that $y_i K^i{}_k = 0$. Contracting (17) with y_i and using last relation imply that

$$F^2(K_{k0} + nK_{0k}) - y_k(n+1)K_{00} = 0. \tag{19}$$

Hence

$$K_{k0} + nK_{0k} = F^{-2}(n+1)y_k K_{00}. \tag{20}$$

Plugging (20) into (17), we get

$$K^i_k = \frac{1}{n-1}K_{00}h^i_k, \tag{21}$$

which means that F is of scalar flag curvature. □

To prove Theorem 1.1, we need to find the $\overline{W}$-curvature of Finsler metrics of scalar flag curvature.

Proposition 4.2 — Let F be a Finsler metric of scalar flag curvature λ. Then $\overline{W}$-curvature is given by

$$\overline{W}{}^i_k = a_n F^2 y^i \lambda_k, \tag{22}$$

where $\lambda_k := \dot{\partial}_k\lambda$ and $a_n := \frac{n^2-2n-1}{3n^2-3}$.

PROOF : By assumption, the Riemannian curvature of Berwald connection is in the following form.

$$\begin{aligned}K^i_{jkl} = {} & \lambda(\delta^i_k g_{jl} - \delta^i_l g_{jk}) + \lambda_j F(\delta^i_k F_l - \delta^i_l F_k) + \frac{1}{3}F^2(h^i_{\ k}\lambda_{jl} - h^i_{\ l}\lambda_{jk}) \\ & + \frac{1}{3}\lambda_l F(2\delta^i_{\ k}F_j - \delta^i_{\ j}F_k - g_{jk}\ell^i) \\ & - \frac{1}{3}F\lambda_k(2\delta^i_{\ l}F_j - \delta^i_{\ j}F_l - g_{jl}\ell^i).\end{aligned} \tag{23}$$

where $\lambda_{ij} = \dot{\partial}_j \lambda_i$. Hence, we have

$$K^i_k = \lambda F^2 h^i_k. \tag{24}$$

Then, we get the following relations

$$\begin{aligned}&K_{jl} = (n-1)(\lambda g_{jl} + FF_l\lambda_j) + \frac{n-2}{3}F^2\lambda_{jl} + \frac{2(n-3)}{3}FF_j\lambda_l, \\ &K_{00} = \lambda(n-1)F^2, \quad K_{k0} = \lambda(n-1)FF_k + \frac{2n-1}{3}F^2\lambda_k, \\ &K_{0k} = \lambda(n-1)FF_k + \frac{n-4}{3}F^2\lambda_k.\end{aligned} \tag{25}$$

Plugging (24) and (16) into (16), we get the result. □

Proof of Theorem 1.1 Suppose that (M, F) is a Finsler manifold with dimension $n \geq 3$ and vanishing $\overline{W}$*-curvature*. Then, by Proposition 4.1, F is of scalar flag curvature λ. Then, by Proposition 4.2, we get $\lambda_k = 0$. Since M is connected, λ is a function of positions, i.e., F is of isotropic flag curvature. Now, Schur's lemma, which asserts that every connected Finsler manifold (M, F) of isotropic flag curvature with dimension greater than two is of constant flag curvature, completes the proof.

Conversely, let F be a Finsler metric of constant flag curvature. Then, a straightforward computation shows that $\overline{W}^{\,i}_{hjk} = 0$ and consequently $\overline{W}^{\,i}_{j} = 0$. This completes the proof. □

A Finsler metric F is said to be C-projectively flat if F is C-projectively related to a locally Minkowski metric. It is well known that a Riemannain metric F is C-projectively flat if and only if F is projectively flat [1, 6].

Beltrami's theorem establishes an equivalency between projectively flatness and being of constant sectional curvature of Riemannian metrics [3, 4]. There are many projectively flat Finsler metrics which are not of constant flag curvature [8]. Here, we prove that C-projectively flatness implies constancy of the flag curvature.

Proposition 4.3 — Let F be a C-projectively flat Finsler metric with dimension $n \geq 3$. Then F is of constant flag curvature.

PROOF : Suppose that F is C-projectively flat. Thus, by definition, F is C-projectively related to a locally Minkowski metric. It is easy to see that for a locally Minkowski metric $\overline{W} = 0$. Therefore, by Proposition 3.2, F has vanishing $\overline{W}$-curvature, and, by Theorem 1.1, F is of constant flag curvature.

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