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A NOTE ON THE HYPER COCHRANE SUM [1]

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The main purpose of this paper is using the analytic methods to study the upper bound of the hyper Cochrane sum, and obtain an interesting result which generalized the main theorem of Liu [2].

Key words : Dedekind sum; hyper Cochrane sum; upper bound.

1. Introduction

For any positive integer q and n and an arbitrary integer h, the general Dedekind

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sum $S(h,n,q)$ is defined by

$$S(h,n,q)=\sum_{a=1}^{q}\overline{B}_n\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{ha}{q}\right),$$

where

$$\overline{B}_n(x)=\begin{cases} B_n(x-[x]), & \text{if } x \text{ is not an integer };\\ 0, & \text{if } x \text{ is is an integer,}\end{cases}$$

$B_n(x)$ is the Bernoulli polynomial, and $\overline{B}_n(x)$ is the n-th Bernoulli periodic function defined over $\mathbb{R}$.

This sum is very important, since $S(h,1,q)=s(h,q)$ is the famous Dedekind sum defined as

$$s(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ha}{q}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer };\\ 0, & \text{if } x \text{ is an integer .}\end{cases}$$

In reference [3, 8], Professor Zhang has given some mean value properties of $S(h,n,q)$.

In October 2000, during his visit to Xi'an, Professor Todd Cochrane introduced the following sum analogous to the Dedekind sum as

$$C(h,q)={\sum_{a=1}^{q}}'\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{h\overline{a}}{q}\right)\right),$$

where $\sum'$ denotes the summation over all a such that $(a,q)=1$, and $a\overline{a}\equiv 1 \bmod q$. He advised us to study the arithmetical properties and mean value distributive properties of Cochrane sum $C(h,q)$. Zhang and Yi [9] studied the upper bound estimate of it, and obtained that

$$|C(h,q)| \ll q^{\frac{1}{2}}d(q)\ln^2(q),$$

where d(q) is the divisor function, and the implied constant is independent on h. Similarly, Xu and Zhang [4] defined the high-dimensional Cochrane sum as

$$\begin{aligned} C(h,k,q) &= \sum_{a_1=1}^{q}{}' \sum_{a_2=1}^{q}{}' \cdots \sum_{a_k=1}^{q}{}' \left(\left(\frac{a_1}{q}\right)\right)\left(\left(\frac{a_2}{q}\right)\right) \\ &\quad \cdots \left(\left(\frac{a_k}{q}\right)\right)\left(\left(\frac{h\overline{a_1 a_2 \cdots a_k}}{q}\right)\right), \end{aligned}$$

and obtained

$$|C(h,k,q)| \ll \frac{2^{(k+1)^2}}{\pi^{k+1}} q^{\frac{k}{2}} d(q)(2^{k+2}k)^{\omega(q)} \ln^{k+1}(q),$$

where $\omega(q)$ denotes the number of all different prime divisors of q, and the implied constant is independent on h. Soon after that, Liu [2] improved the upper bound with a simple method. And the authors in [6, 7] obtained several interesting mean value theorems involving the hyper Cochrane sum as

$$\begin{aligned} C(h,q;\mathbf{m},k) &= \sum_{a_1=1}^{q}{}' \sum_{a_2=1}^{q}{}' \cdots \sum_{a_k=1}^{q}{}' \overline{B}_{m_1}\left(\frac{a_1}{q}\right)\overline{B}_{m_2}\left(\frac{a_2}{q}\right) \\ &\quad \cdots \overline{B}_{m_k}\left(\frac{a_k}{q}\right)\overline{B}_{m_{k+1}}\left(\frac{h\cdot\overline{a_1 a_2 \cdots a_k}}{q}\right). \end{aligned}$$

In this paper, we shall use the analytic methods to study the upper bound of the hyper Cochrane sum, and obtain an interesting result which generalized the main theorem of Liu [2]. That is to say, we shall prove the following

***Theorem* 1.1** — *Let $q \geq 3$ and h be integers with $(h,q)=1$, then for any positive integers $m_1, m_2, \cdots, m_{k+1} \geq 2$ such that $m_1 \equiv m_2 \equiv \cdots \equiv m_{k+1} (\bmod\ 2)$, we have the estimate that*

$$\begin{aligned} |C(h,q;\mathbf{m},k)| &\ll \frac{2^{k+1} m_1! m_2! \cdots m_{k+1}! \zeta(m_1)\cdots\zeta(m_{k+1})}{(2\pi)^{m_1+m_2+\cdots+m_{k+1}}} \\ &\quad q^{\frac{k}{2}}(k+1)^{\omega(q)} d(q), \end{aligned}$$

where $\zeta(s)$ denotes the Riemann-zeta function.

***Theorem* 1.2** — *Let $q \geq 3$ and h be integers with $(h, q) = 1$, then for any positive integers $m_1, m_2, \cdots, m_{k+1}$ such that $m_1 \equiv m_2 \equiv \cdots \equiv m_{k+1} (\bmod 2)$, we have*

$$
\begin{aligned}
|C(h, q; \mathbf{m}, k)| \ll \ & \frac{2^{k+1} m_{s+1}! \cdots m_{k+1}! \zeta(m_{s+1}) \cdots \zeta(m_{k+1})}{(2\pi)^{s+m_{s+1}+\cdots+m_{k+1}}} \\
& q^{\frac{k}{2}} (k+1)^{\omega(q)} d(q) \ln^s q,
\end{aligned}
$$

where we assume that $m_1 = m_2 = \cdots = m_s = 1$ while $m_{s+1}, \cdots, m_{k+1} \geq 2$ (without loss of generality).

Taking $s = k+1$ in Theorem 1.2, we may immediately deduce the following

Corollary 1.1 — Let $q \geq 3$ and h be integers with $(h, q) = 1$, then for any fixed positive integer k, we have

$$
|C(h, k, q)| \ll \frac{1}{\pi^{k+1}} q^{\frac{k}{2}} (k+1)^{\omega(q)} d(q) \ln^{k+1} q,
$$

which is the theorem 1.1 of Liu [2].

2. Some Lemmas

To prove the theorems, we need the following several lemmas.

Lemma 2.1 — For the high-dimensional Kloosterman sum $K(r_1, \cdots, r_k, r_{k+1}; q)$ defined as

$$
K(r_1, \cdots, r_k, r_{k+1}; q) = \sum_{a_1=1}^{q}{}' \cdots \sum_{a_k=1}^{q}{}' e\left(\frac{r_1 a_1 + \cdots + r_k a_k + r_{k+1} \cdot \overline{a_1 a_2 \cdots a_k}}{q} \right),
$$

we have the upper bound

$$
K(r_1, \cdots, r_k, r_{k+1}; q) \ll q^{\frac{k}{2}} (k+1)^{\omega(q)} (r_1, r_{k+1}, q)^{\frac{1}{2}} \cdots (r_k, r_{k+1}, q)^{\frac{1}{2}},
$$

where (a, b, c) denotes the greatest divisor of a, b and c.

Proof : This deep lemma is crucial in the following, and the highly nontrivial proof of this bound can be seen in [5].

Lemma 2.2 — For any real number $x \geq 1$, we have the asymptotic formulae

$$\sum_{n \leq x} \frac{1}{n} = \ln x + \gamma + O\left(\frac{1}{x}\right),$$

$$\sum_{n \leq x} \frac{1}{n^t} = \frac{x^{1-t}}{1-t} + \zeta(t) + O\left(x^{-t}\right),$$

where γ is the Euler's constant and $t > 0, t \neq 1$.

PROOF : See Theorem 3.2 of [1].

Lemma 2.3 — Let $q \geq 3$ and h be integers with $(h, q) = 1$, then for any positive integers $m_1, m_2, \cdots, m_{k+1}$ such that $m_1 \equiv m_2 \equiv \cdots \equiv m_{k+1} (\bmod\ 2)$, we have

$$\begin{aligned} C(h, q; \mathbf{m}, k) &= \frac{(-2)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1 + \cdots + m_{k+1}} \phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1) = (-1)^{m_j} \\ 1 \leq j \leq k+1}} \overline{\chi}(h) \left(\sum_{r_1 = 1}^{+\infty} \frac{G(r_1, \chi)}{r_1^{m_1}} \right) \\ &\quad \cdots \left(\sum_{r_{k+1} = 1}^{+\infty} \frac{G(r_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right), \end{aligned}$$

where $G(r, \chi) = \sum_{a=1}^{q} \chi(a) \mathrm{e}\left(\frac{ra}{q}\right)$ is the classical Gauss sum.

PROOF : From the orthogonality relation for character sum modulo q we have

$$\begin{aligned} C(h, q; \mathbf{m}, k) &= {\sum_{a_1=1}^{q}}' {\sum_{a_2=1}^{q}}' \cdots {\sum_{a_k=1}^{q}}' \overline{B}_{m_1}\left(\frac{a_1}{q}\right) \overline{B}_{m_2}\left(\frac{a_2}{q}\right) \\ &\quad \cdots \overline{B}_{m_k}\left(\frac{a_k}{q}\right) \overline{B}_{m_{k+1}}\left(\frac{h \cdot \overline{a_1 a_2 \cdots a_k}}{q}\right) \\ &= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \left(\sum_{a_1=1}^{q} \chi(a_1) \overline{B}_{m_1}\left(\frac{a_1}{q}\right) \right) \\ &\quad \cdots \left(\sum_{a_{k+1}=1}^{q} \chi(a_{k+1}) \overline{B}_{m_{k+1}}\left(\frac{h a_{k+1}}{q}\right) \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(h) \left(\sum_{a_1=1}^{q} \chi(a_1) \overline{B}_{m_1} \left(\frac{a_1}{q} \right) \right) \\
&\quad \cdots \left(\sum_{a_{k+1}=1}^{q} \chi(a_{k+1}) \overline{B}_{m_{k+1}} \left(\frac{a_{k+1}}{q} \right) \right).
\end{aligned}
$$

Note the identity (see Theorem 12.19 of reference [1])

$$
\overline{B}_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r \neq 0}}^{+\infty} \frac{e(xr)}{r^n}
$$

holds for $0 < x \leq 1$, we may have

$$
\begin{aligned}
C(h, q; \mathbf{m}, k) &= \frac{(-1)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1 + \cdots + m_{k+1}} \phi(q)} \sum_{\chi \bmod q} \overline{\chi}(h) \\
&\quad \left(\sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \frac{1}{r_1^{m_1}} \sum_{a_1=1}^{q} \chi(a_1) e\left(\frac{r_1 a_1}{q} \right) \right) \\
&\quad \times \cdots \times \left(\sum_{\substack{r_{k+1}=-\infty \\ r_{k+1} \neq 0}}^{+\infty} \frac{1}{r_{k+1}^{m_{k+1}}} \sum_{a_{k+1}=1}^{q} \chi(a_{k+1}) e\left(\frac{r_{k+1} a_{k+1}}{q} \right) \right) \\
&= \frac{(-1)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1 + \cdots + m_{k+1}} \phi(q)} \sum_{\chi \bmod q} \overline{\chi}(h) \left(\sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \frac{G(r_1, \chi)}{r_1^{m_1}} \right) \\
&\quad \cdots \left(\sum_{\substack{r_{k+1}=-\infty \\ r_{k+1} \neq 0}}^{+\infty} \frac{G(r_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}}\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(h) \left(1 + \frac{\chi(-1)}{(-1)^{m_1}}\right) \left(\sum_{r_1=1}^{+\infty} \frac{G(r_1,\chi)}{r_1^{m_1}}\right) \\
&\qquad \times \cdots \times \left(1 + \frac{\chi(-1)}{(-1)^{m_k}}\right) \left(\sum_{r_{k+1}=1}^{+\infty} \frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}}\right) \\
&= \frac{(-2)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}}\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^{m_j} \\ 1\le j\le k+1}} \overline{\chi}(h) \left(\sum_{r_1=1}^{+\infty} \frac{G(r_1,\chi)}{r_1^{m_1}}\right) \\
&\quad \cdots \left(\sum_{r_{k+1}=1}^{+\infty} \frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}}\right).
\end{aligned}
$$

This proves Lemma 2.3.

Lemma 2.4 — Let $q \geq 3$ and h be integers with $(h,q) = 1$, then for any odd numbers $m_1, m_2, \cdots, m_{k+1} \geq 2$, we have the estimate that

$$
\begin{aligned}
&\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\sum_{r_1=1}^{\infty} \frac{G(r_1,\chi)}{r_1^{m_1}}\right) \cdots \left(\sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}}\right) \\
&\quad \ll \zeta(m_1) \cdots \zeta(m_{k+1}) \phi(q) q^{\frac{k}{2}} (k+1)^{\omega(q)} d(q),
\end{aligned}
$$

where $\zeta(s)$ denotes the Riemann-zeta function.

PROOF : For any parameter $N \geq q$ and any non-principal character χ modulo q, applying Abel's identity we have

$$
\sum_{r=1}^{+\infty} \frac{G(r,\chi)}{r^m} = \sum_{1\le r\le N} \frac{G(r,\chi)}{r^m} + m \int_N^{+\infty} \frac{\sum_{N<r\le y} G(r,\chi)}{y^{m+1}} dy. \tag{2.1}
$$

Then from the estimate of Gauss sum that $G(r,\chi) \ll q^{\frac{1}{2}}(r,q)$ and Lemma 2.3, we may have

$$\begin{aligned}\sum_{1\le r\le N}\frac{G(r,\chi)}{r^m} &\ll q^{\frac12}\sum_{1\le r\le N}\frac{(r,q)}{r^m}=q^{\frac12}\sum_{t|q}\sum_{1\le r\le N/t}\frac{1}{r^m t^{m-1}}\\ &\ll \zeta(m)q^{\frac12}d(q). \end{aligned}\tag{2.2}$$

On the other hand, from the properties of trigonometric sum we have

$$\begin{aligned}\sum_{N<r\le y}G(r,\chi) &= \sum_{a=1}^{q}\chi(a)\sum_{N<r\le y}e\left(\frac{ar}{q}\right)\\ &= \sum_{a=1}^{q}\chi(a)\frac{e\left(\frac{(N+1)a}{q}\right)-e\left(\frac{([y]+1)a}{q}\right)}{1-e\left(\frac{a}{q}\right)}\\ &\ll \sum_{a=1}^{q}\frac{1}{\left|\sin\frac{\pi a}{q}\right|}\ll\sum_{a=1}^{q}\frac{q}{a}\ll q\ln q. \end{aligned}$$

So we have

$$\int_N^{+\infty}\frac{\sum_{N<r\le y}G(r,\chi)}{y^{m+1}}dy\ll\frac{q\ln q}{N^m}.$$

Combining the above we have

$$\begin{aligned}&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(h)\left(\sum_{r_1=1}^{\infty}\frac{G(r_1,\chi)}{r_1^{m_1}}\right)\cdots\left(\sum_{r_{k+1}=1}^{\infty}\frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}}\right)\\ &=\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(h)\left(\sum_{1\le r_1\le N}\frac{G(r_1,\chi)}{r_1^{m_1}}\right)\cdots\left(\sum_{1\le r_{k+1}\le N}\frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}}\right)\\ &+O\left(\frac{lq^{\frac{k}{2}+1}d^k(q)\ln q}{N^l}\right), \end{aligned}\tag{2.3}$$

where $l=\min\{m_1,\cdots,m_{k+1}\}$.

Noting the identity that

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a) = \begin{cases} \frac{\phi(q)}{2}, & \text{if } a \equiv 1 (\bmod\ q); \\ -\frac{\phi(q)}{2}, & \text{if } a \equiv -1 (\bmod\ q); \\ 0, & \text{otherwise,} \end{cases}$$

then by Lemma 2.1 we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\sum_{1 \le r_1 \le N} \frac{G(r_1, \chi)}{r_1^{m_1}} \right) \cdots \left(\sum_{1 \le r_{k+1} \le N} \frac{G(r_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right) \\
&= \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\sum_{1 \le r_1 \le N} \frac{1}{r_1^{m_1}} \sum_{a_1=1}^{q} \chi(a_1) e\left(\frac{a_1 r_1}{q} \right) \right) \\
&\quad \times \cdots \times \left(\sum_{1 \le r_{k+1} \le N} \frac{1}{r_{k+1}^{m_{k+1}}} \sum_{a_1=1}^{q} \chi(a_{k+1}) e\left(\frac{a_{k+1} r_{k+1}}{q} \right) \right) \\
&= \sum_{1 \le r_1 \le N} \cdots \sum_{1 \le r_{k+1} \le N} \frac{1}{r_1^{m_1} \cdots r_{k+1}^{m_{k+1}}} \sum_{a_1=1}^{q} \cdots \sum_{a_{k+1}=1}^{q} \\
&\quad e\left(\frac{a_1 r_1 \cdots + a_{k+1} r_{k+1}}{q} \right) \times \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a_1 \cdots a_{k+1} \overline{h}) \\
&= \frac{\phi(q)}{2} \sum_{1 \le r_1 \le N} \cdots \sum_{1 \le r_{k+1} \le N} \frac{1}{r_1^{m_1} \cdots r_{k+1}^{m_{k+1}}} \sum_{a_1=1}^{q} \cdots \sum_{\substack{a_{k+1}=1 \\ a_1 \cdots a_{k+1} \equiv h (\bmod\ q)}}^{q} \\
&\quad e\left(\frac{a_1 r_1 \cdots + a_{k+1} r_{k+1}}{q} \right) \\
&\quad - \frac{\phi(q)}{2} \sum_{1 \le r_1 \le N} \cdots \sum_{1 \le r_{k+1} \le N} \frac{1}{r_1^{m_1} \cdots r_{k+1}^{m_{k+1}}} \sum_{a_1=1}^{q} \cdots \sum_{\substack{a_{k+1}=1 \\ a_1 \cdots a_{k+1} \equiv -h (\bmod\ q)}}^{q} \\
&\quad e\left(\frac{a_1 r_1 \cdots + a_{k+1} r_{k+1}}{q} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\phi(q)}{2} \sum_{1\le r_1\le N} \cdots \sum_{1\le r_{k+1}\le N} \frac{1}{r_1^{m_1}\cdots r_{k+1}^{m_{k+1}}} \sum_{a_1=1}^{q}\cdots \sum_{\substack{a_{k+1}=1\\ a_1\cdots a_{k+1}\equiv 1(\bmod q)}}^{q} \\
&\quad e\left(\frac{a_1r_1\cdots + a_kr_k + ha_{k+1}r_{k+1}}{q}\right) \\
&\quad -\frac{\phi(q)}{2} \sum_{1\le r_1\le N} \cdots \sum_{1\le r_{k+1}\le N} \frac{1}{r_1^{m_1}\cdots r_{k+1}^{m_{k+1}}} \sum_{a_1=1}^{q}\cdots \sum_{\substack{a_{k+1}=1\\ a_1\cdots a_{k+1}\equiv -1(\bmod q)}}^{q} \\
&\quad e\left(\frac{a_1r_1\cdots + a_kr_k - ha_{k+1}r_{k+1}}{q}\right) \\
&\ll \phi(q)q^{\frac{k}{2}}(k+1)^{\omega(q)} \sum_{1\le r_1\le N} \cdots \sum_{1\le r_{k+1}\le N} \\
&\quad \frac{(r_1, hr_{k+1}, q)^{\frac12}\cdots(r_k, hr_{k+1}, q)^{\frac12}}{r_1^{m_1}\cdots r_{k+1}^{m_{k+1}}} \\
&\quad +\phi(q)q^{\frac{k}{2}}(k+1)^{\omega(q)} \sum_{1\le r_1\le N} \cdots \sum_{1\le r_{k+1}\le N} \\
&\quad \frac{(r_1, -hr_{k+1}, q)^{\frac12}\cdots(r_k, -hr_{k+1}, q)^{\frac12}}{r_1^{m_1}\cdots r_{k+1}^{m_{k+1}}},
\end{aligned}
$$

where we used the sharp bounds for hyper Kloosterman sum.

Noting that $(h, q) = 1$, the identity

$$(r_i, hr_{k+1}, q) = (r_i, (hr_{k+1}, q)) = (r_i, (r_{k+1}, q))$$

holds for any integer i such that $1 \le i \le k$. Then by Lemma 2.2 we have

$$
\begin{aligned}
&\sum_{1\le r_1\le N} \cdots \sum_{1\le r_{k+1}\le N} \frac{(r_1, hr_{k+1}, q)^{\frac12}\cdots(r_k, hr_{k+1}, q)^{\frac12}}{r_1^{m_1}\cdots r_{k+1}^{m_{k+1}}} \\
&= \sum_{\substack{1\le r_{k+1}\le N\\ (r_{k+1},q)=t}} \frac{1}{r_{k+1}^{m_{k+1}}} \left(\sum_{1\le r_1\le N} \frac{(r_1,t)^{\frac12}}{r_1^{m_1}}\right)\cdots\left(\sum_{1\le r_k\le N} \frac{(r_k,t)^{\frac12}}{r_k^{m_k}}\right)
\end{aligned}
$$

$$= \sum_{t|q} \sum_{1\le r_{k+1}\le N/t} \frac{1}{(r_{k+1}t)^{m_{k+1}}} \left(\sum_{t_1|t} \sum_{1\le r_1\le N/t} \frac{1}{r_1^{m_1} t_1^{m_1-\frac{1}{2}}} \right)$$
$$\cdots \left(\sum_{t_k|t} \sum_{1\le r_k\le N/t} \frac{1}{r_k^{m_k} t_k^{m_k-\frac{1}{2}}} \right)$$

$$\ll \zeta(m_1)\cdots\zeta(m_k) \sum_{t|q} \sum_{1\le r_{k+1}\le N/t} \frac{1}{(r_{k+1}t)^{m_{k+1}}}$$
$$\left(\sum_{t_1|t} \frac{1}{t_1^{m_1-\frac{1}{2}}} \right) \cdots \left(\sum_{t_k|t} \frac{1}{t_k^{m_k-\frac{1}{2}}} \right)$$
$$\ll \zeta(m_1)\cdots\zeta(m_{k+1}) \sum_{t|q} \frac{d^k(t)}{t^{m_{k+1}}}$$
$$\ll \zeta(m_1)\cdots\zeta(m_{k+1}) d(q).$$

Using the similar method, we may obtain

$$\textstyle\sum_{1\le r_1\le N} \cdots \sum_{1\le r_{k+1}\le N} \frac{(r_1,-hr_{k+1},q)^{\frac{1}{2}} \cdots (r_k,-hr_{k+1},q)^{\frac{1}{2}}}{r_1^{m_1}\cdots r_{k+1}^{m_{k+1}}}$$
$$\ll \zeta(m_1)\cdots\zeta(m_{k+1}) d(q).$$

Now taking $N = q$ in the above, we may immediately obtain

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\sum_{r_1=1}^{\infty} \frac{G(r_1,\chi)}{r_1^{m_1}} \right) \cdots \left(\sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}} \right)$$
$$\ll \zeta(m_1)\cdots\zeta(m_{k+1}) \phi(q) q^{\frac{k}{2}} (k+1)^{\omega(q)} d(q).$$

This proves Lemma 2.4.

Lemma 2.5 — Let $q \ge 3$ and h be integers with $(h,q) = 1$, then for any positive integers $m_1, m_2, \cdots, m_{k+1}$, we have the estimate that

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\sum_{r_1=1}^{\infty} \frac{G(r_1,\chi)}{r_1^{m_1}} \right) \cdots \left(\sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}} \right)$$
$$\ll \zeta(m_{s+1}) \cdots \zeta(m_{k+1}) \phi(q) q^{\frac{k}{2}} (k+1)^{\omega(q)} d(q) \ln^s q,$$

where we assume that $m_1 = m_2 = \cdots = m_s = 1$ while $m_{s+1}, \cdots, m_{k+1} \geq 2$ (without loss of generality).

PROOF : Assuming that $m_1 = m_2 = \cdots = m_s = 1$ while $m_{s+1}, \cdots, m_{k+1} \geq 2$ (without loss of generality), similar with the proof of Lemma 2.4, taking $m = 1$ in (2.1), the estimate in (2.2) can be replaced by

$$\sum_{1 \leq r \leq N} \frac{G(r,\chi)}{r} \ll q^{\frac{1}{2}} \sum_{1 \leq r \leq N} \frac{(r,q)}{r} = q^{\frac{1}{2}} \sum_{t|q} \sum_{1 \leq r \leq N/t} \frac{1}{r} \ll q^{\frac{1}{2}} d(q) \ln N.$$

And (2.3) can be replaced by

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\sum_{r_1=1}^{\infty} \frac{G(r_1,\chi)}{r_1} \right) \cdots \left(\sum_{r_s=1}^{\infty} \frac{G(r_s,\chi)}{r_s} \right)$$
$$\left(\sum_{r_{s+1}=1}^{\infty} \frac{G(r_{s+1},\chi)}{r_{s+1}^{m_{s+1}}} \right) \cdots \left(\sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}} \right)$$
$$= \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\sum_{r_1=1}^{N} \frac{G(r_1,\chi)}{r_1} \right) \cdots \left(\sum_{r_s=1}^{N} \frac{G(r_s,\chi)}{r_s} \right)$$
$$\left(\sum_{r_{s+1}=1}^{N} \frac{G(r_{s+1},\chi)}{r_{s+1}^{m_{s+1}}} \right) \cdots \left(\sum_{r_{k+1}=1}^{N} \frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}} \right)$$
$$+O\left(\frac{q^{\frac{k}{2}+1} d^k(q) \ln^s N \ln q}{N} \right).$$

So we can obtain the result of Lemma 2.5 without any difficulty.

3. PROOF OF THEOREMS

In this section, we shall complete the proof of the theorems. First we come to prove Theorem 1.1. Note that $C(h,k;\mathbf{m},q)=0$ if there exists $1\le i,j\le k+1$ such that $m_i\not\equiv m_j(\bmod 2)$. Then for any odd numbers $m_1,m_2,\cdots,m_{k+1}\ge 3$, from Lemmas 2.3 and 2.4 we have

$$\begin{aligned}
&|C(h,k;\mathbf{m},q)|\\
&=\left|\frac{(-2)^{k+1}m_1!\cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}}\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(h)\left(\sum_{r_1=1}^{+\infty}\frac{G(r_1,\chi)}{r_1^{m_1}}\right)\right.\\
&\qquad\left.\cdots\left(\sum_{r_{k+1}=1}^{+\infty}\frac{G(r_{k+1},\chi)}{r_{k+1}^{m_{k+1}}}\right)\right|\\
&\ll\frac{2^{k+1}m_1!m_2!\cdots m_{k+1}!\zeta(m_1)\cdots\zeta(m_{k+1})}{(2\pi)^{m_1+m_2+\cdots+m_{k+1}}}q^{\frac{k}{2}}(k+1)^{\omega(q)}d(q).
\end{aligned}$$

For any even numbers $m_1,m_2,\cdots,m_{k+1}ge\ 2$, we can obtain the same upper bound similarly. This completes the proof of Theorem 1.1.

Using the similar method of proving Theorem 1.1 by replacing Lemma 2.4 with 2.5, we can prove Theorem 1.2.

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