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SEMI-LINEAR LIOUVILLE THEOREMS IN THE GENERALIZED GREINER VECTOR FIELDS[1]

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This paper is devoted to study a class of semi-linear elliptic equation with principal part constructed by generalized Greiner vector fields. Using the idea of vector field method, we introduce a new functional for generalized Greiner vector fields. Through many identity deformations and accurate estimates, a class of Liouville type theorem is given. It improves the Liouville type theorem obtained by Niu etc., which can be seen in *Canad. Math. Bull.*, **47**(3), 417-430 (2004).

Key words : Vector field method; generalized greiner vector fields; Liouville theorem; semi-linear equation.

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1. INTRODUCTION

In $\mathbb{R}^n$, the entire solution of the following semi-linear equation

$$\Delta u + u^p = 0, \quad n \geq 3, \tag{1.1}$$

has received much attention. When $p = \frac{n+2}{n-2}$, (1.1) is the famous Yamabe equation. Gidas, Ni and Nirenberg [9] showed that all the positive solutions of (1.1) are radially symmetric under the condition

$$u = O(|x|^{2-n}), \qquad |x| \to +\infty$$

and given the explicit radial solution. Caffarelli, Gidas and Spruck [3] obtained the same results removing the above conditions. Recently Chen etc. (see [5, 6, 15]) simplified the proof and expand to high order elliptic operators by the method of moving planes. For $1 \leq p < \frac{n+2}{n-2}$, Gidas and Spruck [10] introduced the vector field for semi-linear elliptic equation in $\mathbb{R}^n$ and in some manifolds and proved that all nonnegative solutions of (1.1) are trivial by some identities obtained from the vector field. In 1991, the results had been obtained again by Chen and Li [5] using the method of moving planes. The basic form of vector field appeared in a geometric result of Obata [19] to consider the deformations of the usual metric on S^n. By the idea, Chang, Gursky and Yang [4] classified the entire solutions of a fully nonlinear equation.

Heisenberg group H_n, derived from many objects such as quantum mechanics, complex geometry etc., is the typical representative of noncommutative geometry. The semi-linear equation on the Heisenberg group

$$\Delta_H u + h(\xi) u^p \geq 0, \quad \xi \in H_n \tag{1.2}$$

was also paid much attention, where Δ_H is the sub-Laplacian corresponding to a class of left invariant vector fields. When $p = \frac{2n+4}{2n}$, $h(\xi) \equiv 1$, (1.2) is the famous Yamabe equation. Jerison and Lee [13, 14] proved that the Yamabe equation has a uniqueness cylindrical symmetric solution and gave the explicit solutions. After that, Garofalo and Vassilev [8] obtained the similar results on the H-type group. For

$1 < p \leq \frac{2n+2}{2n}$, Birindelli, Dolcetta and Cutri [1] proved that the only nonnegative solution of (1.2) with $h(\xi) \equiv 1$ is trivial by introducing a class of nonnegative functional. Furthermore, the nonexistence for cylindrical symmetric solutions of (1.2) has been studied by Birindelli and Prajapat [2] with the method of moving planes. But for $\frac{2n+2}{2n} < p < \frac{2n+4}{2n}$, nonexistence results for Eq. (1.2) is still an open problem. Recently Xu [20] introduced the vector field method for Heisenberg group and concluded that when $1 < p \leq 1 + \frac{8n+7}{(2n+1)^2}, h \geq 0$, the following semi-linear equation

$$\Delta_H u + h(\xi)u^p = 0 \tag{1.3}$$

does not have nonnegative solution. Specially, Xu [20] extends the results of Birindelli, Dolcetta and Cutri [1] for $\frac{2n+2}{2n} < 1 + \frac{8n+7}{(2n+1)^2} < \frac{2n+4}{2n}$.

In this paper we will introduce the vector field method for generalized Greiner vector fields. As an application, we can study the Liouville property of the following semi-linear equation

$$\Delta_L u + h(\xi)u^p = 0 \tag{1.4}$$

on the generalized Greiner vector fields.

To state our result, we need to describe some notions and properties about the generalized Greiner vector fields. Take $x, y \in \mathbb{R}^n, t \in \mathbb{R}, z = x + \sqrt{-1}y, \xi = (x, y, t)$. A family of dilations is defined as

$$\delta_r(z, t) = (rz, r^{2k}t) \tag{1.5}$$

and the homogeneous dimension with respect to dilations is $Q = 2n + 2k$. The generalized Greiner vector fields are

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}T, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}T, \tag{1.6}$$

where $j = 1, 2, \ldots, n, T = \frac{\partial}{\partial t}, |z| = [\sum_{j=1}^{n}(x_j^2 + y_j^2)^2]^{\frac{1}{2}}, k \geq 1$. They satisfy the following noncommutative relations:

$$[X_i, X_j] = 2k(2k-2)|z|^{2k-4}(x_i y_j - x_j y_i)T,$$
$$[Y_i, Y_j] = 2k(2k-2)|z|^{2k-4}(x_i y_j - x_j y_i)T,$$
$$[X_i, Y_j] = -2k(2k-2)|z|^{2k-4}(x_i x_j + y_i y_j)T, \ \ i \neq j,$$
$$[X_i, Y_i] = -4k[|z|^{2k-2} + (k-1)|z|^{2k-4}(x_i^2 + y_i^2)]T,$$

where $i, j = 1, 2, \ldots, n$. Denote by

$$\nabla_L = (X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n), \tag{1.7}$$

$$\Delta_L = \sum_{j=1}^{n} (X_j^2 + Y_j^2) \tag{1.8}$$

the generalized gradient and the generalized Greiner operator respectively. We define the norm $|\xi| = (|z|^{4k} + t^2)^{\frac{1}{4k}}$ and the ball of radius R centered at origin

$$B_R = B_R(0) = \{\xi \in \mathbb{R}^{2n+1} : |\xi| < R\}.$$

When $k = 1, \Delta_L$ becomes the sub-Laplacian Δ_{H_n} on the Heisenberg group H_n (see Folland [7]). If $k = 2, 3, \ldots, \Delta_L$ is the Greiner operator (see [11]). As is well known, the vector fields $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ in (1.6) do not possess left translation invariance for $k > 1$ and, if $k \neq 1, 2, 3, \ldots$, they do not meet the Hörmander condition [12]. These are main differences with the left invariant vectors on H_n and bring many difficulties in the application of the vector fields method. We shall overcome them by introducing a special class of cut-off function in the Section 3. Our main result is the following Liouville type theorem.

***Theorem* 1.1** — *Let $u \in C^2(L)$ be a nonnegative solution of*

$$\Delta_L u + h(\xi) u^p = 0, \quad n > 2, \tag{1.9}$$

with $p > 1$. Assume that $a = n + k - 1, b = n + k^2 - 1, e = \frac{n(a^2+2b)[(n+1)a^2-2nb]}{(n-1)(2n+1)^2 a^4}$ and $h(\xi)$ is a nonnegative function such that

$$\Delta_L h(\xi) \geq 0, \tag{1.10}$$

and for $|\xi|$ *large,*

$$|\nabla logh(\xi)| \leq \frac{c}{|\xi|}, \tag{1.11}$$

$$c_1|\xi|^\sigma \leq h(\xi), \quad \sigma > -m-2, \tag{1.12}$$

where $m = \frac{(n+k-2)[4(n+1)e-1]}{2e-1}$. *If* $1 < p < 4(n+1)e$*, then* $u(\xi) \equiv 0$.

Remark 1.2 : When $k = 1$, Theorem 1.1 is the result of Xu [20].

*Remark*1.3 : Condition $1 < p < 4(n+1)e$ implies that constant $k(\geq 1)$ is dependent on the integer n. Combining the software ***Matlab***, we find that k will have a bigger range when n is bigger.

Remark 1.4 : Theorem 1.1 extends partly the Liouville theorems of Niu etc. (see Theorem 5.1 of [17]). Indeed one of the results of [17] was that for $1 < p < \frac{Q}{Q-2}$, the only nonnegative solution of

$$\Delta_L u + u^p \leq 0, \quad in\ R^{2n+1}$$

is trivial. Since $\frac{Q}{Q-2} < 4(n+1)e$ when $k = 1$, we can conclude that $\frac{Q}{Q-2} < 4(n+1)e$ is true in $[1, T]$ (T is a constant), which is a subdomain of the range in Remark 1.3, because of continuity.

The plan of the paper is as follows. In Section 2, inspired by the vector fields method, we introduce a class of functional, give some quantities through the method of integration by parts and the noncommutative relations of the generalized Greiner vector fields, and then obtain an identity, seen Lemma 2.1, which is the main tool in the proof of Theorem 1.1. The last section is devoted to the proof of our main result. When $k > 1$, the generalized Greiner vector fields are not nilpotent of two step. And furthermore, they are not nilpotent when k isn't integer. These bring more information, such as the twelfth and the thirteenth items in the right of (2.5), in the process of using the noncommutative relations. To overcome them, we introduce a special of cut-off function in Section 3.

2. AN INTEGRAL ABOUT THE GENERALIZED GREINER VECTOR FIELDS

Now let $u \geq 0$ satisfy (1.9). Set $u = v^{-s}(s \neq 0)$, then

$$\Delta_L u = -sv^{-s-1}\Delta_L v + s(s+1)v^{-s-2}|\nabla_L v|^2, \tag{2.1}$$

where $|\nabla_L v|^2 = \sum_{i=1}^{n}[(X_i v)^2 + (Y_i v)^2]$, and v satisfies

$$\Delta_{L^v} = (s+1)v^{-1}|\nabla_L v|^2 + \frac{h}{s}v^{s+1-sp}. \tag{2.2}$$

Denote

$$\begin{aligned}\sum_{i,j=1}^{n}|E_{ij}^u|^2 &= \sum_{i,j=1}^{n}\left[(X_jX_i + Y_jY_i)u - \frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2 + Y_\gamma^2)u\delta_{ij}\right]^2 \\ &+ \sum_{i,j=1}^{n}\left[(X_jY_i - Y_jX_i)u - \frac{1}{n}\sum_{\gamma=1}^{n}[X_\gamma, Y_\gamma]u\delta_{ij}\right]^2.\end{aligned} \tag{2.3}$$

Let $\Omega \subset R^{2n+1}(n > 1)$ a bounded domain and take $\varphi \in C_0^\infty(\Omega)$ satisfying $0 \leq \varphi \leq 1$.

Now we consider the nonnegative integral functional as follows:

$$\sum_{i,j=1}^{n}\int_\Omega |E_{ij}^u|^2\varphi^q v^{r+2s+2}, \tag{2.4}$$

where q, s, r are to be determined. Next we will express (2.4) in two forms and then give the following identity. For convenience, we omit the domain Ω in all integrals below.

Lemma 2.1 — (Main Identity)

$$\begin{aligned}s^2\sum_{i,j=1}^{n}\int |E_{i,j}^v|^2\varphi^q v^r + \lambda_1\int v^{r-2}\varphi^q|\nabla_L v|^4 + \lambda_2\int h^2 v^{r+2s+2-2ps}\varphi^q \\ +\lambda_3\int v^r\varphi^q(Tv)^2|z|^{4k-4} + \lambda_4\sum_{i,j=1}^{n}\int v^r\varphi^q[(X_jX_iv - Y_jY_iv)^2 \\ +(X_jY_iv + Y_jX_iv)^2] + \lambda_5\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2\end{aligned}$$

$$= \lambda_6 \sum_{j=1}^{n} \int v^{r-1} |\nabla_L v|^2 (X_j v X_j \varphi^q + Y_j v Y_j \varphi^q) + \lambda_7 \int v^r |\nabla_L v|^2 \Delta_L \varphi^q$$
$$+ \lambda_8 \sum_{i,j=1}^{n} \int v^r [(X_i X_j \varphi^q + Y_i Y_j \varphi^q)(X_i v X_j v + Y_i v Y_j v) - (Y_i X_j \varphi^q - X_i Y_j \varphi^q)$$
$$(X_i v Y_j v - Y_i v X_j v)] + \lambda_9 \sum_{j=1}^{n} \int v^r (Tv) |z|^{2k-2} (X_j v Y_j \varphi^q - Y_j v X_j \varphi^q)$$
$$+ \lambda_{10} \sum_{i,j=1}^{n} \int v^{r-1} \varphi^q [(X_j Y_i v + Y_j X_i v)(X_j v Y_i v + X_i v Y_j v) + (X_j X_i v - Y_j Y_i v)$$
$$(X_j v X_i v - Y_j v Y_i v)] + \lambda_{11} \sum_{j=1}^{n} \int h v^{r+s+1-ps} (X_j v X_j \varphi^q + Y_j v Y_j \varphi^q)$$
$$+ \lambda_{12} \sum_{j=1}^{n} \int v^r \varphi^q T v |z|^{2k-4} (x_j Y_j v - y_j X_j v) + \lambda_{13} \sum_{i,j=1}^{n} \int v^r \Big[[X_i, X_j] v$$
$$(X_i v X_j \varphi^q + Y_i v Y_j \varphi^q) + [X_i, Y_j] v (X_i v Y_j \varphi^q - Y_i v X_j \varphi^q) \Big], \tag{2.5}$$

where

$$\lambda_1 = s^2(s+1)[(s+1)\left(3 + \frac{1}{n}\right) + 2(r-1)] - \eta s^2 (s+1)(r+s+1)$$
$$- s^2 (r+2s+2)(r+2s+1)$$
$$= s^2 \left[\left(\frac{1}{n} - 1 - \eta \right) (s+1)^2 - (2+\eta)(s+1) r - r(r-1) \right],$$
$$\lambda_2 = -\frac{n-1}{n} - \eta,$$
$$\lambda_3 = 16 k^2 s^2 \left[(n+k-1)^2 \left(1 + \frac{1}{n} - \eta \right) - 2(n+k^2-1)(1+\eta) \right],$$
$$\lambda_4 = s^2 \eta,$$
$$\lambda_5 = 2s \left(1 + \frac{1}{n}\right)(s+1) - \frac{s(r+2s+2)(n-1)}{n} - s(r+2s+2)\eta$$
$$- s(r+2s+2)\left(1 + \frac{1}{n}\right)$$
$$= 2s \left(1 + \frac{1}{n}\right)(s+1) - s(r+2s+2)(2+\eta),$$

$$
\begin{aligned}
\lambda_6 &= -2s^2(s+1) + s^2(r+2s+2) - s^2(r+2s+2)(1-\eta) \\
&\quad + s^2(r+2s+2) \\
&= s^2[(r+2s+2)(1+\eta) - 2(s+1)], \\
\lambda_7 &= s^2, \qquad \lambda_8 = -s^2(1-\eta), \\
\lambda_9 &= -4ks^2\left(1+\frac{1}{n}\right)(n+k-1) + \frac{4}{n}s^2k(n+k-1) \\
&\quad + 4s^2k\eta(n+k-1) - 4s^2k(n+k-1)(1-\eta) \\
&= 8s^2k(n+k-1)(\eta-1), \\
\lambda_{10} &= -2s^2(s+1) - s^2r\eta + s^2(r+2s+2) = s^2r(1-\eta), \\
\lambda_{11} &= \frac{n-1}{n}s + \frac{s}{n} + s\eta - s(1-\eta) = 2s\eta, \\
\lambda_{12} &= -8ks^2(k-1)(2n+2k-1)\eta, \\
\lambda_{13} &= -2s^2(1+\eta).
\end{aligned}
$$

Proof : *Step* 1 : On the one hand, by (2.3) and (2.4),

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\int |E_{ij}^u|^2\varphi^q v^{r+2s+2} \\
&\qquad = \sum_{i,j=1}^{n}\int (X_jX_iu + Y_jY_iu)^2\varphi^q v^{r+2s+2} \\
&\qquad + \sum_{i,j=1}^{n}\int (Y_jX_iu - X_jY_iu)^2\varphi^q v^{r+2s+2} \\
&\qquad - \frac{1}{n}\int\left[\sum_{\gamma=1}^{n}(X_\gamma^2u + Y_\gamma^2u)\right]^2\varphi^q v^{r+2s+2} \\
&\qquad - \frac{1}{n}\int\left(\sum_{\gamma=1}^{n}[X_\gamma, Y_\gamma]u\right)^2\varphi^q v^{r+2s+2} \\
&:= I + II + III + IV.
\end{aligned}
$$

Because of $u = v^{-s}$, we have

$$\begin{aligned}
I &= s^2(s+1)^2 \sum_{i,j=1}^{n} \int v^{r-2}\varphi^q (X_j v X_i v + Y_j v Y_i v)^2 \\
&\quad + s^2 \sum_{i,j=1}^{n} \int v^r \varphi^q (X_j X_i v + Y_j Y_i v)^2 \\
&\quad - 2s^2(s+1) \sum_{i,j=1}^{n} \int v^{r-1}\varphi^q (X_j v X_i v + Y_j v Y_i v)(X_j X_i v + Y_j Y_i v) \\
&:= I_1 + I_2 + I_3, \\
II &= s^2(s+1)^2 \sum_{i,j=1}^{n} \int v^{r-2}\varphi^q (Y_j v X_i v - X_j v Y_i v)^2 \\
&\quad + s^2 \sum_{i,j=1}^{n} \int v^r \varphi^q (Y_j X_i v - X_j Y_i v)^2 \\
&\quad - 2s^2(s+1) \sum_{i,j=1}^{n} \int v^{r-1}\varphi^q (Y_j v X_i v - X_j v Y_i v)(Y_j X_i v - X_j Y_i v) \\
&:= II_1 + II_2 + II_3
\end{aligned}$$

and

$$\begin{aligned}
III &= -\frac{s^2(s+1)^2}{n} \int v^{r-2}\varphi^q |\nabla_L v|^4 - \frac{s^2}{n} \int v^r \varphi^q (\Delta_L v)^2 \\
&\quad + \frac{2s^2(s+1)}{n} \int v^{r-1}\varphi^q \Delta_L v |\nabla_L v|^2 \\
&:= III_1 + III_2 + III_3.
\end{aligned}$$

Thus

$$\begin{aligned}
I_1 + II_1 + III_1 + III_3 &= \left(1 + \frac{1}{n}\right) s^2(s+1)^2 \int v^{r-2}\varphi^q |\nabla_L v|^4 \\
&\quad + \frac{2s(s+1)}{n} \int h v^{r+s-sp} \varphi^q |\nabla_L v|^2 \qquad (2.6)
\end{aligned}$$

and

$$I_2 + II_2 + III_2 + IV = s^2 \sum_{i,j=1}^{n} \int |E_{ij}^v|^2 \varphi^q v^r. \qquad (2.7)$$

Using integration by parts, it gives

$$
\begin{aligned}
I_3 + II_3 \;=\;& -2s^2(s+1)\sum_{j=1}^{n}\int v^{r-1}\varphi^q\Big(X_j(|\nabla_L v|^2)X_jv + Y_j(|\nabla_L v|^2)Y_jv\Big)\\
&+2s^2(s+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q\Big[(Y_jvY_iv - X_jvX_iv)(Y_jY_iv - X_jX_iv)\\
&+(Y_jvX_iv + X_jvY_iv)(Y_jX_iv + X_jY_iv)\Big]\\
=\;& 2s^2(s+1)\Big\{(s+r)\int v^{r-2}\varphi^q|\nabla_L v|^4 + \frac{1}{s}\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2\\
&+\sum_{j=1}^{n}\int v^{r-1}|\nabla_L v|^2(X_jvX_j\varphi^q + Y_jvY_j\varphi^q)\Big\}\\
&+2s^2(s+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q\Big[(Y_jvY_iv - X_jvX_iv)(Y_jY_iv - X_jX_iv)\\
&+(Y_jvX_iv + X_jvY_iv)(Y_jX_iv + X_jY_iv)\Big]. \qquad (2.8)
\end{aligned}
$$

From (2.6)-(2.8), we get

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\int |E_{ij}^u|^2\varphi^q v^{r+2s+2}\\
&= s^2\sum_{i,j=1}^{n}\int |E_{ij}^v|^2\varphi^q v^r + 2s(1+\frac{1}{n})(s+1)\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2\\
&\quad + s^2(s+1)[(s+1)(3+\frac{1}{n}) + 2(r-1)]\int v^{r-2}\varphi^q|\nabla_L v|^4\\
&\quad + 2s^2(s+1)\sum_{j=1}^{n}\int v^{r-1}|\nabla_L v|^2(X_jvX_j\varphi^q + Y_jvY_j\varphi^q)\\
&\quad + 2s^2(s+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q\Big[(Y_jvY_iv - X_jvX_iv)(Y_jY_iv - X_jX_iv)\\
&\qquad\qquad + (Y_jvX_iv + X_jvY_iv)(Y_jX_iv + X_jY_iv)\Big]. \qquad (2.9)
\end{aligned}
$$

Step 2 : On the other hand, using the facts that are

$$\sum_{i,j=1}^{n}\big[(X_jX_i+Y_jY_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2+Y_\gamma^2)u\delta_{ij}\big]\delta_{ij}=0,$$
$$\sum_{i,j=1}^{n}\big[(X_jY_i-Y_jX_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2+Y_\gamma^2)u\delta_{ij}\big]\delta_{ij}=0,$$

we easily get

$$\begin{aligned}&\sum_{i,j=1}^{n}|E_{ij}^u|^2\varphi^q v^{r+2s+2}\\&=\sum_{i,j=1}^{n}\Big\{\Big[(X_jX_i+Y_jY_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2+Y_\gamma^2)u\delta_{ij}\Big](X_jX_i+Y_jY_i)u\\&\quad+\Big[(X_jY_i-Y_jX_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}[X_\gamma,Y_\gamma]u\delta_{ij}\big](X_jY_i-Y_jX_i)u\Big\}\varphi^q v^{r+2s+2}.\end{aligned} \tag{2.10}$$

Integrating by parts,

$$\begin{aligned}&\sum_{i,j=1}^{n}\int|E_{ij}^u|^2\varphi^q v^{r+2s+2}\\&=-\sum_{i,j=1}^{n}\int\Big\{X_j[(X_jX_i+Y_jY_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2u+Y_\gamma^2u)\delta_{ij}]X_iu\\&\qquad+Y_j[(X_jX_i+Y_jY_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2u+Y_\gamma^2u)\delta_{ij}]Y_iu\\&\qquad+Y_j[(Y_jX_i-X_jY_i)u+\frac{1}{n}\sum_{\gamma=1}^{n}[X_\gamma,Y_\gamma]u\delta_{ij}]X_iu\\&\qquad-X_j[(Y_jX_i-X_jY_i)u+\frac{1}{n}\sum_{\gamma=1}^{n}[X_\gamma,Y_\gamma]u\delta_{ij}]Y_iu\Big\}\varphi^q v^{r+2s+2}\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i,j=1}^{n}\int\Big\{[(X_jX_i+Y_jY_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2u+Y_\gamma^2u)\delta_{ij}]X_iuX_j\varphi^q\\
&\qquad+[(X_jX_i+Y_jY_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2u+Y_\gamma^2u)\delta_{ij}]Y_iuY_j\varphi^q\\
&\qquad+[(Y_jX_i-X_jY_i)u+\frac{1}{n}\sum_{\gamma=1}^{n}[X_\gamma,Y_\gamma]u\delta_{ij}]X_iuY_j\varphi^q\\
&\qquad-[(Y_jX_i-X_jY_i)u+\frac{1}{n}\sum_{\gamma=1}^{n}[X_\gamma,Y_\gamma]u\delta_{ij}]Y_iuX_j\varphi^q\Big\}v^{r+2s+2}\\
&-(r+2s+2)\sum_{i,j=1}^{n}\int\Big\{[(X_jX_i+Y_jY_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2u+Y_\gamma^2u)\delta_{ij}]X_iuX_jv\\
&\qquad+[(X_jX_i+Y_jY_i)u-\frac{1}{n}\sum_{\gamma=1}^{n}(X_\gamma^2u+Y_\gamma^2u)\delta_{ij}]Y_iuY_jv\\
&\qquad+[(Y_jX_i-X_jY_i)u+\frac{1}{n}\sum_{\gamma=1}^{n}[X_\gamma,Y_\gamma]u\delta_{ij}]X_iuY_jv\\
&\qquad-[(Y_jX_i-X_jY_i)u+\frac{1}{n}\sum_{\gamma=1}^{n}[X_\gamma,Y_\gamma]u\delta_{ij}]Y_iuX_jv\Big\}v^{r+2s+1}\varphi^q\\
&:=V+VI+VII. \qquad (2.11)
\end{aligned}
$$

Exchanging i,j two times through noncommutative relations and calculating delicately, we have

$$
\begin{aligned}
V=&\left(\frac{1}{n}-1\right)\sum_{j=1}^{n}\int[X_j(\Delta_Lu)X_ju+Y_j(\Delta_Lu)Y_ju]\varphi^qv^{r+2s+2}\\
&+\left(1+\frac{1}{n}\right)\sum_{i,j=1}^{n}\int\Big(X_i[X_j,Y_j]uY_iu-Y_i[X_j,Y_j]uX_iu\Big)\varphi^qv^{r+2s+2}\\
&+2\sum_{i,j=1}^{n}\int\Big(X_j[X_i,X_j]uX_iu+Y_j[Y_i,Y_j]uY_iu\\
&\qquad+Y_j[X_i,Y_j]uX_iu-X_j[X_j,Y_i]uY_iu\Big)\varphi^qv^{r+2s+2}. \qquad (2.12)
\end{aligned}
$$

Using integration by parts,we obtain

$$\begin{aligned}V &= s\left(1-\frac{1}{n}\right)\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_j\varphi^q+Y_jvY_j\varphi^q)\\&+\left(1-\frac{1}{n}\right)\int h^2v^{r+2s+2-2ps}\varphi^q\\&+s(r+2s+2)\left(1-\frac{1}{n}\right)\int hv^{r+s-ps}|\nabla_Lv|^2\varphi^q\\&+4ks^2\left(1+\frac{1}{n}\right)(n+k-1)\sum_{j=1}^{n}\int|z|^{2k-2}v^rTv(Y_jvX_j\varphi^q-X_jvY_j\varphi^q)\\&+16k^2s^2\left[2(n+k^2-1)-(n+k-1)^2\left(1+\frac{1}{n}\right)\right]\int|z|^{4k-4}v^r\varphi^q(Tv)^2\\&-2s^2\sum_{i,j=1}^{n}\int\Big[[X_i,X_j]v(X_ivX_j\varphi^q+Y_ivY_j\varphi^q)\\&\qquad+[X_i,Y_j]v(X_ivY_j\varphi^q-Y_ivX_j\varphi^q)\Big]v^r. \qquad (2.13)\end{aligned}$$

Calculating through the method of integration by parts and the noncommutative relations,

$$\begin{aligned}VI &= \frac{s}{n}\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_j\varphi^q+Y_jvY_j\varphi^q)+s^2\int v^r|\nabla_Lv|^2\Delta_L\varphi^q\\&+\sum_{i,j=1}^{n}\int[(X_jX_iu-Y_jY_iu)(X_iuX_j\varphi^q-Y_iuY_j\varphi^q)\\&\qquad+(Y_jX_iu+X_jY_iu)(X_iuY_j\varphi^q+Y_iuX_j\varphi^q)]v^{r+2s+2}\\&+\frac{4}{n}s^2k(n+k-1)\sum_{j=1}^{n}\int|z|^{2k-2}v^rTv(X_jvY_j\varphi^q-Y_jvX_j\varphi^q)\\&+s^2(r+2s+2)\sum_{j=1}^{n}\int v^{r-1}|\nabla_Lv|^2(X_jvX_j\varphi^q+Y_jvY_j\varphi^q). \qquad (2.14)\end{aligned}$$

In the above equality, we need to express the term

$$\begin{aligned}&\sum_{i,j=1}^{n}\int[(X_jX_iu-Y_jY_iu)(X_iuX_j\varphi^q-Y_iuY_j\varphi^q)\\&\qquad+(Y_jX_iu+X_jY_iu)(X_iuY_j\varphi^q+Y_iuX_j\varphi^q)]v^{r+2s+2}:=VI'\end{aligned}$$

in two kinds of the form $VI_1^{'}$ and $VI_2^{'}$. The reason will be pointed out in the last part of Section 3. On the one hand, from integration by parts,

$$\begin{aligned}VI_1^{'} = &-\sum_{i,j=1}^{n}\int[X_j(X_jX_iu-Y_jY_iu)X_iu-Y_j(X_jX_iu-Y_jY_iu)Y_iu\\&\quad+Y_j(Y_jX_iu+X_jY_iu)X_iu+X_j(Y_jX_iu+X_jY_iu)Y_iu]\varphi^q v^{r+2s+2}\\&-\sum_{i,j=1}^{n}\int[(X_jX_iu-Y_jY_iu)^2+(Y_jX_iu+X_jY_iu)^2]\varphi^q v^{r+2s+2}\\&-(r+2s+2)\sum_{i,j=1}^{n}\int[(X_jX_iu-Y_jY_iu)(X_iuX_jv-Y_iuY_jv)\\&\qquad+(Y_jX_iu+X_jY_iu)(X_iuY_jv+Y_iuX_jv)]\varphi^q v^{r+2s+1}\\:=&\ ① + ② + ③.\end{aligned}\tag{2.15}$$

Exchanging i, j two times and calculating delicately, we have

$$\begin{aligned}① = &\ s(r+2s+2)\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2+\int h^2v^{r+2s+2-2ps}\varphi^q\\&+2\sum_{i,j=1}^{n}\int([X_i,X_j]X_juX_iu+[Y_i,Y_j]Y_juY_iu\\&\qquad+[X_i,Y_j]Y_juX_iu-[X_j,Y_i]X_juY_iu)\varphi^q v^{r+2s+2}\\&-\sum_{i,j=1}^{n}\int(X_i[X_j,Y_j]uY_iu-Y_i[X_j,Y_j]uX_iu)\varphi^q v^{r+2s-2}\\&+s\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_j\varphi^q+Y_jvY_j\varphi^q)\\=&\ s(r+2s+2)\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2+\int h^2v^{r+2s+2-2ps}\varphi^q\\&+s\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_j\varphi^q+Y_jvY_j\varphi^q)\\&-4ks^2(n+k-1)\sum_{j=1}^{n}\int|z|^{2k-2}v^rTv(Y_jvX_j\varphi^q-X_jvY_j\varphi^q)\end{aligned}$$

$$
\begin{aligned}
&+ 4ks^2(2k-2)(2n+2k-1)\sum_{j=1}^{n}\int |z|^{2k-4}Tv(y_jX_jv - x_jY_jv)\varphi^q v^r\\
&+ 16k^2s^2[2(n+k^2-1)+(n+k-1)^2]\int |z|^{4k-4}v^r\varphi^q(Tv)^2\\
&- 2s^2\sum_{i,j=1}^{n}\int [[X_i,X_j]v(X_ivX_j\varphi^q + Y_ivY_j\varphi^q)\\
&\qquad + [X_i,Y_j]v(X_ivY_j\varphi^q - Y_ivX_j\varphi^q)]v^r. \qquad (2.16)
\end{aligned}
$$

A direct computation gives

$$
\begin{aligned}
② = &-s^2(s+1)^2\int v^{r-2}\varphi^q|\nabla_L v|^4\\
&- s^2\sum_{i,j=1}^{n}\int v^r\varphi^q[(X_jX_iv - Y_jY_iv)^2 + (X_jY_iv + Y_jX_iv)^2]\\
&+ 2s^2(s+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q[(X_jY_iv + Y_jX_iv)(X_jvY_iv + X_ivY_jv)\\
&\qquad + (X_jX_iv - Y_jY_iv)(X_jvX_iv - Y_jvY_iv)] \qquad (2.17)
\end{aligned}
$$

and

$$
\begin{aligned}
③ = &\, s^2(s+1)(r+2s+2)\int v^{r-2}\varphi^q|\nabla_L v|^4\\
&- s^2(r+2s+2)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q[(X_jY_iv + Y_jX_iv)(X_jvY_iv + X_ivY_jv)\\
&\qquad + (X_jX_iv - Y_jY_iv)(X_jvX_iv - Y_jvY_iv)]. \qquad (2.18)
\end{aligned}
$$

Finally, we obtain from (2.15) to (2.18) that

$$
\begin{aligned}
VI_1' = &\, s^2(s+1)(r+s+1)\int v^{r-2}\varphi^q|\nabla_L v|^4\\
&- s^2\sum_{i,j=1}^{n}\int v^r\varphi^q[(X_jX_iv - Y_jY_iv)^2 + (X_jY_iv + Y_jX_iv)^2]\\
&- s^2r\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q[(X_jY_iv + Y_jX_iv)(X_jvY_iv + X_ivY_jv)\\
&\qquad + (X_jX_iv - Y_jY_iv)(X_jvX_iv - Y_jvY_iv)]
\end{aligned}
$$

$$
\begin{aligned}
&+\int h^2 v^{r+2s+2-2ps}\varphi^q + s(r+2s+2)\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2\\
&+s\sum_{j=1}^n\int hv^{r+s+1-ps}(X_jvX_j\varphi^q+Y_jvY_j\varphi^q)\\
&-4ks^2(n+k-1)\sum_{j=1}^n\int |z|^{2k-2}v^rTv(Y_jvX_j\varphi^q-X_jvY_j\varphi^q)\\
&+16k^2s^2[2(n+k^2-1)+(n+k-1)^2]\int |z|^{4k-4}v^r\varphi^q(Tv)^2\\
&+4ks^2(2k-2)(2n+2k-1)\sum_{j=1}^n\int |z|^{2k-4}Tv(y_jX_jv-x_jY_jv)\varphi^q v^r\\
&-2s^2\sum_{i,j=1}^n\int\Big[[X_i,X_j]v(X_ivX_j\varphi^q+Y_ivY_j\varphi^q)\\
&\qquad\qquad+[X_i,Y_j]v(X_ivY_j\varphi^q-Y_ivX_j\varphi^q)\Big]v^r. \qquad (2.19)
\end{aligned}
$$

On the other hand, exchanging i,j and integration by parts for subindex i,

$$
\begin{aligned}
VI_2' = &-\sum_{j=1}^n\int\Big[(\Delta_L u)(X_juX_j\varphi^q+Y_juY_j\varphi^q)\\
&\qquad-\sum_{i=1}^n[X_i,Y_i]u(X_juY_j\varphi^q-Y_juX_j\varphi^q)\Big]v^{r+2s+2}\\
&-\sum_{i,j=1}^n\int[(X_iX_j\varphi^q+Y_iY_j\varphi^q)(X_iuX_ju+Y_iuY_ju)\\
&\qquad-(Y_iX_j\varphi^q-X_iY_j\varphi^q)(X_iuY_ju-Y_iuX_ju)]v^{r+2s+2}\\
&-(r+2s+2)\sum_{i,j=1}^n\int(X_iuX_iv+Y_iuY_iv)(X_juX_j\varphi^q+Y_juY_j\varphi^q)v^{r+2s+1}\\
=&-s\sum_{j=1}^n\int hv^{r+s+1-ps}(X_jvX_j\varphi^q+Y_jvY_j\varphi^q)\\
&-s^2\sum_{i,j=1}^n\int v^r[(X_iX_j\varphi^q+Y_iY_j\varphi^q)(X_ivX_jv+Y_ivY_jv)\\
&\qquad-(Y_iX_j\varphi^q-X_iY_j\varphi^q)(X_ivY_jv-Y_ivX_jv)]
\end{aligned}
$$

$$- s^2(r+2s+2)\sum_{j=1}^{n}\int v^{r-1}|\nabla_L v|^2(X_jvX_j\varphi^q + Y_jvY_j\varphi^q)$$
$$+ 4ks^2(n+k-1)\sum_{j=1}^{n}\int |z|^{2k-2}v^rTv(Y_jvX_j\varphi^q - X_jvY_j\varphi^q). \tag{2.20}$$

Thus combining (2.19) and (2.20) with (2.14),

$$\begin{aligned} VI = {} & \frac{s}{n}\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_j\varphi^q + Y_jvY_j\varphi^q) + s^2\int v^r|\nabla_L v|^2\Delta_L\varphi^q \\ & + s^2(r+2s+2)\sum_{j=1}^{n}\int v^{r-1}|\nabla_L v|^2(X_jvX_j\varphi^q + Y_jvY_j\varphi^q) \\ & - \frac{4}{n}ks^2(n+k-1)\sum_{j=1}^{n}\int |z|^{2k-2}v^rTv(Y_jvX_j\varphi^q - X_jvY_j\varphi^q) \\ & + \eta VI_1' + (1-\eta)VI_2', \end{aligned} \tag{2.21}$$

where η is to be determined.

Finally in Step II, we calculate VII as follows

$$\begin{aligned} VII = {} & -s^2(s+1)(r+2s+2)\int v^{r-2}\varphi^q|\nabla_L v|^4 \\ & + \frac{s(r+2s+2)}{n}\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2 \\ & - (r+2s+2)\sum_{j=1}^{n}\int [X_j(|\nabla_L u|^2)X_jv + Y_j(|\nabla_L u|^2)Y_jv]\varphi^q v^{r+2s+1} \\ & + s^2(r+2s+2)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q[(X_jY_iv + Y_jX_iv)(X_jvY_iv + X_ivY_jv) \\ & \qquad + (X_jX_iv - Y_jY_iv)(X_jvX_iv - Y_jvY_iv)]. \end{aligned}$$

Through integration by parts, we obtain

$$\begin{aligned}VII &= s^2(r+2s+2)(r+2s+1)\int v^{r-2}\varphi^q|\nabla_L v|^4\\&\quad + s(r+2s+2)\left(1+\frac{1}{n}\right)\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2\\&\quad + s^2(r+2s+2)\sum_{j=1}^{n}\int v^{r-1}|\nabla_L v|^2(X_jvX_j\varphi^q+Y_jvY_j\varphi^q)\\&\quad + s^2(r+2s+2)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q[(X_jY_iv+Y_jX_iv)(X_jvY_iv+X_ivY_jv)\\&\qquad + (X_jX_iv-Y_jY_iv)(X_jvX_iv-Y_jvY_iv)]. \qquad (2.22)\end{aligned}$$

From Steps I and II, we obtain the following identity

$$I+II+III+IV=V+VI+VII$$

and then complete the proof. □

3. Proof of Theorem 1.1

Now we deal with $\lambda_5\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2$ firstly. Since

$$\begin{aligned}&\sum_{j=1}^{n}\int[X_j(hu^p)X_ju+Y_j(hu^p)Y_ju]\varphi^q v^{r+2s+2}\\&=\sum_{j=1}^{n}\int(X_juX_jh+Y_juY_jh)u^p\varphi^q v^{r+2s+2}\\&\quad+\sum_{j=1}^{n}\int[X_juX_ju^p+(Y_juY_ju^p)]h\varphi^q v^{r+2s+2}\\&=ps^2\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2-s\sum_{j=1}^{n}\int v^{r+s+1-ps}\varphi^q\\&\quad(X_jvX_jh+Y_jvY_jh) \qquad (3.1)\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{n}\int[X_j(hu^p)X_ju+Y_j(hu^p)Y_ju]\varphi^q v^{r+2s+2}\\
&=-(r+2s+2)\sum_{j=1}^{n}\int(X_juX_jv+Y_juY_jv)hv^{r+2s+1-ps}\varphi^q\\
&\quad-\int h\Delta_L u\varphi^q v^{r+2s+2-ps}-\sum_{j=1}^{n}\int(X_j\varphi^q X_ju+Y_j\varphi^q Y_ju)hv^{r+2s+2-ps}\\
&=\int h^2v^{r+2s+2-2ps}\varphi^q+s\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_j\varphi^q X_jv+Y_j\varphi^q Y_jv)\\
&\quad+s(r+2s+2)\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2, \qquad (3.2)
\end{aligned}$$

we have

$$\begin{aligned}
&\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2\\
&=-\frac{1}{r+2s+2-ps}\sum_{j=1}^{n}\int v^{r+s+1-ps}\varphi^q(X_jvX_jh+Y_jvY_jh)\\
&\quad-\frac{1}{s(r+2s+2-ps)}\int h^2v^{r+2s+2-2ps}\varphi^q\\
&\quad-\frac{1}{r+2s+2-ps}\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_j\varphi^q+Y_jvY_j\varphi^q). \qquad (3.3)
\end{aligned}$$

Putting (3.3) into $\lambda_5\int hv^{r+s-ps}\varphi^q|\nabla_L v|^2$, Main Identity (2.5) is changed into

$$\begin{aligned}
&s^2\sum_{i,j=1}^{n}\int|E_{i,j}^v|^2\varphi^q v^r+\lambda_1\int v^{r-2}\varphi^q|\nabla_L v|^4\\
&+\lambda_2'\int h^2v^{r+2s+2-2ps}\varphi^q+\lambda_3\int v^r\varphi^q(Tv)^2|z|^{4k-4}\\
&+\lambda_4\sum_{i,j=1}^{n}\int v^r\varphi^q[(X_jX_iv-Y_jY_iv)^2+(X_jY_iv+Y_jX_iv)^2]
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_6 \sum_{j=1}^{n} \int v^{r-1} |\nabla_L v|^2 (X_j v X_j \varphi^q + Y_j v Y_j \varphi^q) + \lambda_7 \int v^r |\nabla_L v|^2 \Delta_L \varphi^q \\
&\quad + \lambda_8 \sum_{i,j=1}^{n} \int v^r [(X_i X_j \varphi^q + Y_i Y_j \varphi^q)(X_i v X_j v + Y_i v Y_j v) \\
&\quad - (Y_i X_j \varphi^q - X_i Y_j \varphi^q)(X_i v Y_j v - Y_i v X_j v)] \\
&\quad + \lambda_9 \sum_{j=1}^{n} \int v^r (Tv) |z|^{2k-2} (X_j v Y_j \varphi^q - Y_j v X_j \varphi^q) \\
&\quad + \lambda_{10} \sum_{i,j=1}^{n} \int v^{r-1} \varphi^q [(X_j Y_i v + Y_j X_i v)(X_j v Y_i v + X_i v Y_j v) \\
&\quad + (X_j X_i v - Y_j Y_i v)(X_j v X_i v - Y_j v Y_i v)] + \lambda_{11}' \sum_{j=1}^{n} \int h v^{r+s+1-ps} \\
&\quad (X_j v X_j \varphi^q + Y_j v Y_j \varphi^q) + \lambda_{11}'' \sum_{j=1}^{n} \int v^{r+s+1-ps} \varphi^q (X_j v X_j h + Y_j v Y_j h) \\
&\quad + \lambda_{12} \sum_{j=1}^{n} \int v^r \varphi^q T v |z|^{2k-4} (x_j Y_j v - y_j X_j v) + \lambda_{13} \sum_{i,j=1}^{n} \int v^r \Big[[X_i, X_j] v \\
&\quad (X_i v X_j \varphi^q + Y_i v Y_j \varphi^q) + [X_i, Y_j] v (X_i v Y_j \varphi^q - Y_i v X_j \varphi^q) \Big], \qquad (3.4)
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_2' &= \lambda_2 - \frac{1}{s(r+2s+2-ps)} \lambda_5 \\
&:= \lambda_2 + \lambda_5' = \frac{(1+\frac{1}{n})r + (1+\eta-\frac{1}{n})ps}{r+2s+2-ps}, \\
\lambda_{11}' &= \lambda_{11} - s\lambda_5', \\
\lambda_{11}'' &= -s\lambda_5'
\end{aligned}
$$

and the others as before.

Proof of Theorem 1.1 : Now we choose

$$s \neq 0, \quad p > 1, \quad h \geq 0.$$

It is easy to prove $u \equiv 0$ if we get $v \equiv 0$ concluded from the identity (3.4). To realize our goal, we shall make the left-hand side of (3.4) to be positive, which need $\lambda_1, \lambda_2', \lambda_3$ and λ_4 all be nonnegative, and at the same time, the exponent of v in positive integrals to be nonnegative at least, which needs $r \geq 0$ at least sine the first term is positive for $s \neq 0$. On the other hand, we need that the right-hand side of (3.4) can be controlled by the terms in the left-hand side and the cut-off function φ with its derivatives. In the sequel, we will use repeatedly the Young's inequality

$$ab \leq \epsilon^t \frac{a^t}{t} + \frac{1}{\epsilon^s}\frac{b^s}{s}, \frac{1}{t} + \frac{1}{s} = 1, \tag{3.5}$$

where a, b, t and ϵ are positive numbers. As the first application, we have

$$\begin{aligned}
&\lambda_6 \sum_{j=1}^{n} \int v^{r-1}|\nabla_L v|^2 (X_j v X_j \varphi^q + Y_j v Y_j \varphi^q)\\
&= q\lambda_6 \sum_{j=1}^{n} \int v^{r-1}|\nabla_L v|^2 \varphi^{q-1}(X_j v X_j \varphi + Y_j v Y_j \varphi)\\
&\leq \epsilon \int v^{r-2}\varphi^q |\nabla_L v|^4 + C \int v^{r+2}\varphi^{q-4}|\nabla_L \varphi|^4,
\end{aligned} \tag{3.6}$$

where ϵ is a sufficiently small positive number and C is a positive constant which may vary from line to line. Since $\varphi \leq 1$, then $\varphi^{q-1} \leq \varphi^{q-2}$ and

$$\begin{aligned}
&\lambda_7 \int v^r |\nabla_L v|^2 \Delta_L \varphi\\
&= \lambda_7 \int v^r |\nabla_L v|^2 [q\varphi^{q-1}\Delta_L \varphi + 2q(q-1)\varphi^{q-2}|\nabla_L \varphi|^2]\\
&\leq \epsilon \int v^{r-2}\varphi^q |\nabla_L v|^4 + C \int v^{r+2}\varphi^{q-4}(|\Delta_L \varphi| + |\nabla_L \varphi|^2)^2.
\end{aligned} \tag{3.7}$$

Similarly

$$\begin{aligned}
&\lambda_8 \sum_{i,j=1}^{n} \int v^r [(X_i X_j \varphi^q + Y_i Y_j \varphi^q)(X_i v X_j v + Y_i v Y_j v)\\
&- (Y_i X_j \varphi^q - X_i Y_j \varphi^q)(X_i v Y_j v - Y_i v X_j v)]\\
&\leq \epsilon \int v^{r-2}\varphi^q |\nabla_L v|^4 + C \int v^{r+2}\varphi^{q-4}(|\nabla_L^2 \varphi| + |\nabla_L \varphi|^2)^2.
\end{aligned} \tag{3.8}$$

Owing to

$$\lambda_3 = 16k^2s^2[(n+k-1)^2(1+\frac{1}{n}-\eta) - 2(n+k^2-1)(1+\eta)] \geq 0$$

and

$$\lambda_4 = s^2\eta \geq 0,$$

we get

$$0 \leq \eta \leq \frac{(1+\frac{1}{n})(n+k-1)^2 - 2(n+k^2-1)}{(n+k-1)^2 + 2(n+k^2-1)} < 1.$$

At the same time we need $\lambda_1 > 0$ to control the terms $\epsilon \int v^{r-2}\varphi^q |\nabla_L v|^4$ both in (3.7) and (3.8). Thus $r > 0$ and then $\lambda_{10} \neq 0$. Using (3.5),

$$\begin{aligned} &\lambda_{10} \sum_{i,j=1}^{n} \int v^{r-1}\varphi^q[(X_jY_iv + Y_jX_iv)(X_jvY_iv + X_ivY_jv) \\ &+ (X_jX_iv - Y_jY_iv)(X_jvX_iv - Y_jvY_iv)] \\ &\leq a' \sum_{i,j=1}^{n} \int v^r\varphi^q[(X_jX_iv - Y_jY_iv)^2 + (X_jY_iv + Y_jX_iv)^2] \\ &+ b' \int v^{r-2}\varphi^q |\nabla_L v|^4 \end{aligned} \tag{3.9}$$

valid for some $a' > 0, b' > 0$, especially we choose $a' = s^2\eta, b' = \frac{s^2r^2(1-\eta)^2}{4\eta}$. To control the positive term

$$a' \sum_{i,j=1}^{n} \int v^r\varphi^q[(X_jX_iv - Y_jY_iv)^2 + (X_jY_iv + Y_jX_iv)^2],$$

it needs $\lambda_4 > 0$. So

$$0 < \eta \leq \frac{(1+\frac{1}{n})(n+k-1)^2 - 2(n+k^2-1)}{(n+k-1)^2 + 2(n+k^2-1)} < 1$$

and then $\lambda_9 \neq 0$. By (3.5), we have

$$\begin{aligned}
&\lambda_9 \sum_{j=1}^{n} \int v^r (Tv)|z|^{2k-2}(X_j v Y_j \varphi^q - Y_j v X_j \varphi^q) \\
&\leq \epsilon \int v^r \varphi^q (Tv)^2 |z|^{4k-4} + C \int v^r \varphi^{q-2} |\nabla_L v|^2 |\nabla_L \varphi|^2 \\
&\leq \epsilon \int v^r \varphi^q (Tv)^2 |z|^{4k-4} + \epsilon \int v^{r-2} \varphi^q |\nabla_L v|^4 + C \int v^{r+2} \varphi^{q-4} |\nabla_L \varphi|^4.
\end{aligned} \tag{3.10}$$

We need $\lambda_3 > 0$ to control $\epsilon \int v^r \varphi^q (Tv)^2 |Z|^{4k-4}$, and then

$$0 < \eta < \frac{(1+\frac{1}{n})(n+k-1)^2 - 2(n+k^2-1)}{(n+k-1)^2 + 2(n+k^2-1)} < 1, \tag{3.11}$$

which tells us why we should express the term VI' as $\eta VI_1' + (1-\eta)VI_2'$. Applying (3.5), we deduce that

$$\begin{aligned}
&\lambda_{12} \sum_{j=1}^{n} \int v^r \varphi^q Tv |z|^{2k-4} (x_j Y_j v - y_j X_j v) \\
&\leq \epsilon \int |z|^{4k-4} v^r \varphi^q (Tv)^2 + C \int |z|^{-2} v^r \varphi^q |\nabla_L v|^2 \\
&\leq \epsilon \int |z|^{4k-4} v^r \varphi^q (Tv)^2 + \epsilon \int v^{r-2} \varphi^q |\nabla_L v|^4 + C \int |z|^{-4} v^{r+2} \varphi^q
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\lambda_{13} \sum_{i,j=1}^{n} \int v^r \Big[[X_i, X_j] v (X_i v X_j \varphi^q + Y_i v Y_j \varphi^q) \\
&\qquad + [X_i, Y_j] v (X_i v Y_j \varphi^q - Y_i v X_j \varphi^q) \Big] \\
&\leq \epsilon \int |z|^{4k-4} v^r \varphi^q (Tv)^2 + C \int v^r \varphi^{q-2} |\nabla_L v|^2 |\nabla_L \varphi|^2 \\
&\leq \epsilon \int |z|^{4k-4} v^r \varphi^q (Tv)^2 + \epsilon \int v^{r-2} \varphi^q |\nabla_L v|^4 \\
&\quad + C \int v^{r+2} \varphi^{q-4} |\nabla_L \varphi|^4.
\end{aligned} \tag{3.13}$$

Using integration by parts,

$$
\begin{aligned}
&\lambda_{11}'\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_j\varphi^q+Y_jvY_j\varphi^q)\\
&=\frac{\lambda_{11}'}{r+s+2-ps}\sum_{j=1}^{n}\int h[X_j\varphi^qX_j(v^{r+s+2-ps})+Y_j\varphi^qY_j(v^{r+s+2-ps})]\\
&=-\frac{\lambda_{11}'}{r+s+2-ps}\int v^{r+s+2-ps}[\sum_{j=1}^{n}(X_jhX_j\varphi^q+Y_jhY_j\varphi^q)+h\Delta_L\varphi^q]\\
&=-\frac{\lambda_{11}'q}{r+s+2-ps}\sum_{j=1}^{n}\int v^{r+s+2-ps}h\varphi^{q-1}[X_j\varphi X_j(\log h)+Y_j\varphi Y_j(\log h)]\\
&\quad-\frac{\lambda_{11}'}{r+s+2-ps}\int v^{r+s+2-ps}h[q\varphi^{q-1}\Delta_L\varphi+q(q-1)\varphi^{q-2}|\nabla_L\varphi|^2].
\end{aligned}
\tag{3.14}
$$

If

$$
\frac{\lambda_{11}''}{r+s+2-ps}\Delta_Lh\geq 0, \tag{3.15}
$$

we have

$$
\begin{aligned}
&\lambda_{11}''\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_jh+Y_jvY_jh)\\
&\quad=-\frac{\lambda_{11}''q}{r+s+2-ps}\sum_{j=1}^{n}\int v^{r+s+2-ps}h\varphi^{q-1}\\
&\qquad[X_j\varphi X_j(\log h)+Y_j\varphi Y_j(\log h)]\\
&\quad-\frac{2\lambda_{11}''}{r+s+2-ps}\int v^{r+s+2-ps}\varphi^q\Delta_Lh\\
&\qquad\leq-\frac{\lambda_{11}''q}{r+s+2-ps}\sum_{j=1}^{n}\int v^{r+s+2-ps}h\varphi^{q-1}\\
&\qquad[X_j\varphi X_j(\log h)+Y_j\varphi Y_j(\log h)].
\end{aligned}
\tag{3.16}
$$

Noting $\lambda_2<0$ and $\lambda_{11}''=-s\lambda_5'$, we make $\lambda_5'\geq-\lambda_2>0$ to ensure $\lambda_2=\lambda_5+\lambda_2'\geq 0$. Then assumption (3.15) is equivalent to

$$
\frac{s}{r+s+2-ps}\Delta_Lh\leq 0. \tag{3.17}
$$

Under the condition (3.17), we obtain

$$\begin{aligned}
&\lambda_{11}^{'}\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_j\varphi^q+Y_jvY_j\varphi^q)\\
&+\lambda_{11}^{''}\sum_{j=1}^{n}\int hv^{r+s+1-ps}(X_jvX_jh+Y_jvY_jh)\\
&\leq C\int v^{r+s+2-ps}h\varphi^{q-2}(|\nabla_L logh||\nabla_L\varphi|+|\Delta_L\varphi|+|\nabla_L\varphi|^2). \qquad (3.18)
\end{aligned}$$

Putting (3.6)-(3.18) into (3.4), we get

$$\begin{aligned}
&s^2\sum_{i,j=1}^{n}\int |E_{i,j}^v|^2\varphi^q v^r+(\lambda_1-b'-\epsilon)\int v^{r-2}\varphi^q|\nabla_L v|^4\\
&+\lambda_2^{'}\int h^2v^{r+2s+2-2ps}\varphi^q+(\lambda_3-\epsilon)\int v^r\varphi^q(Tv)^2|z|^{4k-4}\\
&+(\lambda_4-a')\sum_{i,j=1}^{n}\int v^r\varphi^q[(X_jX_iv-Y_jY_iv)^2+(X_jY_iv+Y_jX_iv)^2]\\
&\leq C\int v^{r+s+2-ps}h\varphi^{q-2}(|\nabla_L logh||\nabla_L\varphi|+|\Delta_L\varphi|+|\nabla_L\varphi|^2)\\
&+C\int v^{r+2}\varphi^{q-4}[(|\nabla_L^2\varphi|+|\nabla_L\varphi|^2)^2+(|\Delta_L\varphi|+|\nabla_L\varphi|^2)^2+|\nabla_L\varphi|^4]\\
&+C\int |z|^{-4}v^{r+2}\varphi^q. \qquad (3.19)
\end{aligned}$$

Following, we deal with the right-hand side of (3.19) by Youngs inequality again. Choosing $t_1=\frac{r+2s+2-2ps}{r+s+2-ps}>1$, $s_1=\frac{r+2s+2-2ps}{s-ps}>1$, we have

$$\begin{aligned}
&\int v^{r+s+2-ps}h\varphi^{q-2}(|\nabla_L logh||\nabla_L\varphi|+|\Delta_L\varphi|+|\nabla_L\varphi|^2)\\
&\leq \epsilon\int h^2v^{r+2s+2-2ps}\varphi^q\\
&+C\int h^{-\frac{s+2}{s-ps}}\varphi^{\mu_1}[|\nabla_L logh||\nabla_L\varphi|+|\Delta_L\varphi|+|\nabla_L\varphi|^2]^{s_1},
\end{aligned}$$

where $\mu_1=(q-2-\frac{q}{t_1})s_1$.

Choosing $t_2 = \frac{r+2s+2-2ps}{r+2} > 1, s_2 = \frac{1}{2}s_1 = \frac{r+2s+2-2ps}{2(s-ps)} > 1$, it gives

$$\begin{aligned}\int v^{r+2}\varphi^{q-4}[(|\nabla_L^2\varphi| + |\nabla_L\varphi|^2)^2 + (|\Delta_L\varphi| + |\nabla_L\varphi|^2)^2 + |\nabla_L\varphi|^4]&\\ \leq \epsilon \int h^2 v^{r+2s+2-2ps}\varphi^q&\\ + C\int h^{-\frac{r+2}{s-ps}}\varphi^{\mu_2}[(|\nabla_L^2\varphi| + |\nabla_L\varphi|^2)^2&\\ + (|\Delta_L\varphi| + |\nabla_L\varphi|^2)^2 + |\nabla_L\varphi|^4]^{s_2},&\end{aligned}$$

where $\mu_2 = \left(q - 4 - \dfrac{q}{t_2}\right) s_2 = \mu_1$. Here, to ensure $t_1, s_1, t_2, s_2 > 1$, we need $s < 0$. And moreover, condition (3.17), i.e. condition (3.15), is equivalent to (1.10) when $s < 0$.

The third term in the right side of (3.19) can be changed into

$$\begin{aligned}\int |z|^{-4}v^{r+2}\varphi^q \leq \epsilon \int h^2 v^{r+2s+2-2ps}\varphi^q&\\ + C\int h^{-\frac{r+2}{s-ps}}\varphi^q|z|^{-\frac{2(r+2s+2-2ps)}{s-ps}}.&\end{aligned}$$

Here, we need the condition

$$2n - \frac{2(r+2s+2-ps)}{s(1-p)} > 0 \tag{3.20}$$

to ensure the integrability of the second integral in the right side of the above formula. A direct computation shows that (3.20) is equivalent to $0 < r+2 < (n-2)(1-p)s$, and then equivalent to $n > 2$ under the condition $s < 0$.

Combining the above three estimates of the right side of (3.19), we have

$$\begin{aligned}&s^2\sum_{i,j=1}^{n}\int |E_{i,j}^v|^2\varphi^q v^r + (\lambda_1 - b' - \epsilon)\int v^{r-2}\varphi^q|\nabla_L v|^4\\ &+ (\lambda_2' - \epsilon)\int h^2 v^{r+2s+2-2ps}\varphi^q + (\lambda_3 - \epsilon)\int v^r\varphi^q(Tv)^2|Z|^{4k-4}\\ &+ (\lambda_4 - a')\sum_{i,j=1}^{n}\int v^r\varphi^q[(X_jX_iv - Y_jY_iv)^2 + (X_jY_iv + Y_jX_iv)^2]\end{aligned}$$

$$\leq C\int h^{-\frac{r+2}{s-ps}}\varphi^{\mu_1}(|\nabla_L logh||\nabla_L\varphi|+|\Delta_L\varphi|+|\nabla_L\varphi|^2)^{s_1}$$
$$+C\int h^{-\frac{r+2}{s-ps}}\varphi^{\mu_2}[(|\nabla_L^2\varphi|+|\nabla_L\varphi|^2)^2+(|\Delta_L\varphi|+|\nabla_L\varphi|^2)^2+|\nabla_L\varphi|^4]^{s_2}$$
$$+C\int h^{-\frac{r+2}{s-ps}}\varphi^q|z|^{-\frac{2(r+2s+2-2ps)}{s-ps}}. \tag{3.21}$$

We choose $\Omega = B_R$ and $\psi \in C_0^\infty(B_R)$ with

$$\psi(\xi) = 1 \quad \text{for } |\xi| \leq \frac{R}{2},$$
$$0 \leq \psi(\xi) \leq 1 \quad \text{for } \frac{R}{2} < |\xi| \leq R,$$
$$|\nabla_L\psi| \leq \frac{C}{R},\ \ |\nabla_L^2\psi| \leq \frac{C}{R^2} \quad \text{for } \frac{R}{2} \leq |\xi| \leq R.$$

Set $\varphi = \psi(1-\psi)$, then $\varphi \in C_0^\infty(B_R\backslash\overline{B_{\frac{R}{2}}})$ has the following properties

$$0 \leq \varphi(\xi) \leq 1 \quad \text{for } \frac{R}{2} < |\xi| \leq R, \tag{3.22}$$
$$|\nabla_L\varphi| \leq \frac{C}{R},\ \ |\nabla_L^2\varphi| \leq \frac{C}{R^2} \quad \text{for } \frac{R}{2} \leq |\xi| \leq R. \tag{3.23}$$

Let q large enough such that $\mu_1 = \mu_2 \geq 0$ to make $\varphi^{\mu_1} = \varphi^{\mu_2} \leq 1$. Applying the polar coordinate transformation on generalized Greiner vector fields referred to Niu, Ou and Han [18] (a general formula see Luo [16]), combining (1.10), (1.11) and (1.12), we get

$$\text{the right side of (3.21)} \leq CR^{-2s_1-\frac{r+2}{s-ps}\sigma+2n+2k}, \tag{3.24}$$

on the assumptions $s<0$. Taking $a' = s^2\eta, b' = \frac{s^2r^2(1-\eta)^2}{4\eta}$ in (3.21), we have

$$s^2\sum_{i,j=1}^n\int|E_{i,j}^v|^2\varphi^q v^r + [\lambda_1 - \frac{s^2r^2(1-\eta)^2}{4\eta} - \epsilon]\int v^{r-2}\varphi^q|\nabla_L v|^4$$
$$+(\lambda_2' - \epsilon)\int h^2v^{r+2s+2-2ps}\varphi^q + (\lambda_3-\epsilon)\int v^r\varphi^q(Tv)^2|Z|^{4k-4}$$
$$\leq CR^{-2s_1-\frac{r+2}{s-ps}\sigma+2n+2k}. \tag{3.25}$$

From above it gives $v \equiv 0$ as $R \to \infty$, by the conditions

$$\lambda_1 - \frac{s^2 r^2 (1-\eta)^2}{4\eta} > 0, \quad \lambda_2' > 0, \quad \lambda_3 > 0, \tag{3.26}$$

$$-2s_1 - \frac{r+2}{s-ps}\sigma + 2n + 2k < 0. \tag{3.27}$$

In order to complete the proof of the theorem, we only need to pick suitable $k < 0, r > 0, p > 1, \eta$ and then σ so that (3.26) and (3.27) hold. Set $y = 1 + \frac{1}{s}, \delta = -\frac{r}{s} > 0$, then $\lambda_1 - \frac{s^2 r^2 (1-\eta)^2}{4\eta} > 0$ and $\lambda_2' > 0$ are respectively equivalent to the following two inequalities:

$$\Big(1 + \eta - \frac{1}{n}\Big) y^2 - (1+\eta)\delta y + \Big[\frac{(1+\eta)^2}{4\eta}\delta^2 - \delta\Big] < 0, \tag{3.28}$$

$$\Big[-\Big(1 + \frac{1}{n}\Big)\delta + \Big(1 + \eta - \frac{1}{n}\Big)p\Big] / (-\delta + 2y - p) > 0. \tag{3.29}$$

It is easy to know that (3.29) is equivalent to

$$\delta > \frac{n + n\eta - 1}{n+1} p, \qquad \text{if} \quad 2y < \delta + p,$$
$$\delta < \frac{n + n\eta - 1}{n+1} p, \qquad \text{if} \quad 2y > \delta + p.$$

It is needed to choose suitable δ such that the discriminant of the quadratic polynomial more than zero to realize (3.28), that is

$$\Delta_y := (1+\eta)^2 \delta^2 - 4\Big(1 + \eta - \frac{1}{n}\Big)\Big[\frac{(1+\eta)^2}{4\eta}\delta^2 - \delta\Big] > 0, \tag{3.30}$$

which deduces that

$$0 < \delta < \frac{4(n + n\eta - 1)\eta}{(n-1)(1+\eta)^2}. \tag{3.31}$$

At the same time, y must be between the two roots of the quadratic polynomial, i.e.

$$y_1 := \frac{(1+\eta)\delta - \sqrt{\Delta_y}}{2(1 + \eta - \frac{1}{n})} < y < \frac{(1+\eta)\delta + \sqrt{\Delta_y}}{2(1 + \eta - \frac{1}{n})} := y_2.$$

Let $W_1 = 2\Big(\frac{\delta+p}{2}\Big)\Big(1+\eta-\frac{1}{n}\Big) - (1+\eta)\delta$. From (3.31),

$$W_1 = p\Big(1+\eta-\frac{1}{n}\Big) - \frac{\delta}{n}$$

$$> p\Big(1+\eta-\frac{1}{n}\Big)\Big[1-\frac{4\eta}{pn(1-\frac{1}{n})(1+\eta)^2}\Big].$$

Let $\alpha = \dfrac{(1+\frac{1}{n})(n+k-1)^2 - 2(n+k^2-1)}{(n+k-1)^2+2(n+k^2-1)}$ and note that α is monotonically decreasing w.r.t. k. So, $\eta \le \frac{n-1}{n+2}$ is true because of (3.11). Also set $\beta = \dfrac{\eta}{(1-\frac{1}{n})(1+\eta)^2}$ and find that β is monotonically increasing w.r.t. η. Thus, $\beta < \dfrac{n(n+2)}{(2n+1)^2}$ and then

$$\begin{aligned} W_1 &> \Big(1+\eta-\frac{1}{n}\Big)\Big[1-\frac{4(n+2)}{(2n+1)^2}\Big] \\ &= \frac{(1+\eta-\frac{1}{n})}{(2n+1)^2}(4n^2-7) \\ &> 0 > -\sqrt{\Delta_y}, \end{aligned}$$

which implies $y_1 < \dfrac{\delta+p}{2}$. Set

$$\begin{aligned} W_2 &= W_1^2 - \Delta_y \\ &= \Big[\frac{1}{n^2} + \Big(1-\frac{1}{n}\Big)\frac{(1+\eta)^2}{\eta}\Big]\delta^2 - \Big(1+\eta-\frac{1}{n}\Big)\Big(4+\frac{2p}{n}\Big)\delta \\ &\quad + p^2\Big(1+\eta-\frac{1}{n}\Big)^2. \end{aligned}$$

Similarly, we can conclude $y_2 < \frac{\delta+p}{2}$. Till now we can choose y s.t. $y_1 < y < y_2 < \frac{\delta+p}{2}$, and any δ satisfying (3.31). Finally we obtain the range of p as in

Theorem 1.1

$$1 < p < \frac{n+1}{n+n\eta-1}\delta < \frac{4(n+1)\eta}{(n-1)(1+\eta)^2}$$

$$= 4\frac{n+1}{n}\beta$$

$$< \frac{4n(n+1)(a^2+2b)[(n+1)a^2-2nb]}{(n-1)(2n+1)^2a^4}$$

$$:= 4(n+1)e \tag{3.32}$$

with $a = n+k-1, b = n+k^2-1$. Indeed we can choose $\delta = \dfrac{8(na^2-b)}{a^2+2b}e-\theta$, where θ is sufficiently small, $y = \dfrac{2(2n+1)a^2}{a^2+2b}e$, $s = \dfrac{a^2+2b}{2(2n+1)a^2e-(a^2+2b)}$, and $r = -\dfrac{8(na^2-b)}{2(2n+1)a^2e-(a^2+2b)}e+\dfrac{a^2+2b}{2(2n+1)a^2e-(a^2+2b)}\theta$. At last, the range of σ as in Theorem 1.1 is that $\sigma > -\dfrac{(n+k-2)[4(n+1)e-1]}{2e-1}$ $-2 := -m-2$. □

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