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ON λ-COMPACT OPERATORS

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Using the duality theory of sequence spaces, we study in this paper λ-compact operators defined on Banach spaces, corresponding to a sequence space λ. We show that these operators form a quasi-normed operator ideal under suitable restrictions on λ. We also study the relationships of these operators with λ-summing, λ-nuclear and quasi-λ-nuclear operators. The results of this paper generalize the earlier results proved by Sinha and Karn; and also Delgado, Piñeiro and Serrano.

Key words : Banach sequence spaces; operator ideals; λ-compact sets; λ-summing operators; λ-nuclear operators; quasi λ-nuclear operators.

1. INTRODUCTION

As we all know, compactness plays an important role in the structural study of Banach spaces. Ball, being compact, yields the finite dimensionality of the space. Besides, we have numerous examples of compact operators on infinite dimensional

Banach spaces, which are defined as taking bounded sets to relatively compact sets. However, a characterization of relatively compact sets as the sets sitting inside the convex hull of a null sequence, led Sinha and Karn [9], to introduce the notion of p-compact sets in 2002, and consequently p-compact operators. In this paper we introduce the notion of λ-compact sets and λ-compact operators corresponding to a sequence space λ and prove that such operators form a quasi-normed operator ideal when λ is restricted suitably. We also find relations of these operators with λ-summing, λ-nuclear and quasi-λ-nuclear operators for certain class of sequence spaces λ. The results of this paper generalize some of the results proved by Sinha and Karn [9]; and also Delgado, Piñeiro and Serrano [1].

2. PRELIMINARIES

For the rudimentary results and notions of sequence spaces, we essentially follow [3]. Our references for operator ideals, λ-summing operators, λ-nuclear and quasi-λ-nuclear operators are [4-7].

Let ω be the family of all real or complex sequences, which is a vector space with usual pointwise addition and scalar multiplication. We write $e^n (n \geq 1)$ for the n^{th} unit vector in ω, i.e., $e^n = \{\delta_{nj}\}_{j\geq 1}$, where δ_{nj} is the Kronecker delta; and ϕ for the subspace of ω generated by e^n's, $n \geq 1$, i.e., $\phi = sp\{e^n \ : \ n \geq 1\}$. A sequence space λ is a subspace of ω containing ϕ. Members of λ are denoted by the symbols $\bar{\alpha}$, $\bar{\beta}$ etc., where $\bar{\alpha} = \{\alpha_1, \alpha_2, \alpha_3, ...\}$, $\bar{\beta} = \{\beta_1, \beta_2, \beta_3, ...\}$. The n^{th} section of $\bar{\alpha}$ for $n \in \mathbb{N}$ is written as $\bar{\alpha}^{(n)}$ and is defined as $\bar{\alpha}^{(n)} = \{\alpha_1, \alpha_2, ..., \alpha_n, 0, 0, ...\}$ i.e., $\bar{\alpha}^{(n)} = \sum_{i=1}^{n} \alpha_i e^i$.

For a subsequence $J = \{n_i\}$ of $\mathbb{N}$ and a sequence space λ, we define J-stepspace of λ as $\lambda_J = \{\{\alpha_i\} : \ there \ is \ a \ \{\beta_i\} \in \lambda \ with \ \alpha_i = \beta_{n_i}, \ \forall \ n_i \in J\}$. If $\bar{\alpha_J} \in \lambda_J$, then the **canonical preimage** of $\bar{\alpha_J}$ is the sequence $\widetilde{\bar{\alpha_J}}$ which agrees with $\bar{\alpha_J}$ on the indices in J and is zero elsewhere.

A sequence space λ is called (i) **symmetric** if $\bar{\alpha_\sigma} = \{\alpha_{\sigma(i)}\} \in \lambda$ whenever $\bar{\alpha} = \{\alpha_i\} \in \lambda$ and $\sigma \in \Pi$, where Π is collection of all permutations of the set of natural numbers $\mathbb{N}$, (ii) normal or solid if $\bar{\beta} = \{\beta_i\} \in \lambda$ whenever $|\beta_i| \leq |\alpha_i|$,

$i \geq 1$ for some $\bar{\alpha} = \{\alpha_i\} \in \lambda$ and (iii) monotone provided λ contains canonical preimages of all its stepspaces.

Every normal sequence space is always monotone.

The α-dual or cross-dual or Köthe-dual of λ is the space λ^α or $\lambda^\times$ defined as

$$\lambda^\times \equiv \lambda^\alpha = \{\bar{\beta} = \{\beta_i\} \in \omega : \sum_{i\geq 1} |\alpha_i\beta_i| \; converges, \; for \; all \; \bar{\alpha} \in \lambda\}.$$

A sequence space λ is said to be **perfect** if $\lambda = \lambda^{\times\times} = (\lambda^\times)^\times$. Every perfect sequence space is normal.

A Banach sequence space (λ, $\|.\|_\lambda$) is called a **BK-space** provided each of the projection maps $P_i : \lambda \to \mathbb{K}$, $P_i(\bar{\alpha}) = \alpha_i$ is continuous, for $i \geq 1$, where $\mathbb{K}$ is the field of scalars and $\bar{\alpha} = \{\alpha_1, \alpha_2, ...\}$. A BK-space ($\lambda$, $\|.\|_\lambda$) is called an AK-space if $\bar{\alpha}^{(n)} \to \bar{\alpha}$, for each $\bar{\alpha} \in \lambda$.

For a BK-space (λ, $\|.\|_\lambda$), we define the dual-norm on $\lambda^\times$ as follows:

$$\|\bar{\beta}\|_{\lambda^\times} = sup\{\sum_{i\geq 1} |\alpha_i||\beta_i| : \; \bar{\alpha} \in \lambda, \; \|\bar{\alpha}\|_\lambda \leq 1\}.$$

The space ($\lambda^\times$, $\|.\|_{\lambda^\times}$) can easily shown to be a BK-space provided $0 < \sup_n \|e^n\|_\lambda < \infty$. If ($\lambda$, $\|.\|_\lambda$) is an AK-space, it is topologically isomorphic to its topological dual (λ^*, $\|.\|$), where $\|f\| = sup\{|f(\bar{\alpha})| : \bar{\alpha} \in \lambda, \|\bar{\alpha}\|_\lambda \leq 1\}$. The topological isomorphism $R : \lambda^\times \to \lambda^*$, $R(\bar{\beta}) = f_{\bar{\beta}}$, where $f_{\bar{\beta}}(\bar{\alpha}) = \sum_{i\geq 1} \alpha_i\beta_i$, for $\bar{\alpha} \in \lambda$, satisfies the inequality

$$\|f_{\bar{\beta}}\| \leq \|\bar{\beta}\|_{\lambda^\times} \leq M_\lambda \|f_{\bar{\beta}}\| \tag{1}$$

for some constant $M_\lambda \geq 1$. The unit balls of λ and $\lambda^\times$ are symbolized as B_λ and $B_{\lambda^\times}$ respectively.

The norm $\|.\|_\lambda$ is said to be (i) k-symmetric if $\|\bar{\alpha}\|_\lambda = \|\bar{\alpha}_\sigma\|_\lambda$, for all $\sigma \in \Pi$ and (ii) monotone if $\|\bar{\alpha}\|_\lambda \leq \|\bar{\beta}\|_\lambda$ for $\bar{\alpha}$, $\bar{\beta}$ in λ with $|\alpha_i| \leq |\beta_i|$, $\forall \; i \geq 1$.

The space (λ, $\|.\|_\lambda$) is said to have the norm iteration property if for each sequence $\{\bar{\alpha}^n\}$ in λ, $\bar{\alpha}_i = \{\alpha_i^1, \alpha_i^2, \alpha_i^3, ...\} \in \lambda$ for each $i \geq 1$ and $\|\{\|\bar{\alpha}^n\|_\lambda\}_n\|_\lambda = \|\{\|\bar{\alpha}_i\|_\lambda\}_i\|_\lambda$, cf. [6].

Corresponding to a sequence space λ and a Banach space X with its topological dual X^* equipped with the operator norm topology generated by $\|.\|$, the vector valued sequence spaces $\lambda^s(X)$ and $\lambda^w(X)$ defined below, have been introduced and studied earlier in [6, 7], under different notations. Indeed, we have,

$$\lambda^s(X) = \{\bar{x} = \{x_n\} \subset X : \ \{\|x_n\|\} \in \lambda\}$$

and

$$\lambda^w(X) = \{\bar{x} = \{x_n\} \subset X : \ \{f(x_n)\} \in \lambda, \ \forall \ f \in X^*\}.$$

In case $\|.\|_\lambda$ is a monotone norm, the space $\lambda^s(X)$ becomes a normed linear space with respect to the norm defined as

$$\|\bar{x}\|_\lambda^s = \|\{\|x_n\|\}\|_\lambda, \bar{x} = \{x_n\} \in \lambda^s(X).$$

However, for $\overline{x} \in \lambda^w(X)$, the norm on $\lambda^w(X)$ is defined as

$$\|\bar{x}\|_\lambda^w = sup\{\|\{f(x_n)\}\|_\lambda, f \in X^*, \|f\| \leq 1\}, \ \bar{x} = \{x_n\} \in \lambda^w(X),$$

which can be proved to be finite by applying closed graph theorem. For the norm $\|\overline{y}\|_{\lambda^\times}^w$ on $((\lambda^\times)^w(X), \ \|.\|_{\lambda^\times}^w)$, we assume throughout that $0 < \sup_n \|e^n\|_\lambda < \infty$ so that $(\lambda^\times)^w(X)$ equipped with this norm becomes a K-space.

The symbol $\mathcal{L}(X, Y)$ is used for the class of bounded linear operators between any two Banach spaces X and Y; whereas $\mathcal{L}$ denotes the collection of all bounded operators between any pair of Banach spaces.

An operator $T \in \mathcal{L}(X, Y)$ is said to be

(i) absolutely λ-summing if for each $\bar{x} = \{x_i\} \in \lambda^w(X)$, the sequence $\{Tx_i\} \in \lambda^s(Y)$;

(ii) λ-nuclear if T has the representation

$$Tx = \sum_{n \geq 1} \alpha_n < x, f_n > y_n,$$

where $\{f_n\} \subseteq X^*$ with $\|f_n\| \leq 1$, for each n; $\bar{y} = \{y_n\} \in (\lambda^\times)^w(Y)$ with $\|\bar{y}\|_{\lambda^\times}^w \leq 1$ and $\bar{\alpha} = \{\alpha_n\} \in \lambda$; and

(iii) quasi-λ-nuclear if there exists $\{f_n\} \subseteq X^*$ such that $\bar{f} = \{f_n\} \in \lambda^s(X^*)$ and $\|Tx\| \leq \|\{f_n(x)\}\|_\lambda$, for each $x \in X$.

The symbols $\Pi_\lambda(X,Y)$, $N_\lambda(X,Y)$ and $QN_\lambda(X,Y)$ denote respectively the collection of all λ-summing, λ-nuclear and quasi-λ-nuclear operators from X to Y.

Let $\mathcal{A}$ be a subset of $\mathcal{L}$. Writing $\mathcal{A}(X,Y) = \mathcal{A} \cap \mathcal{L}(X,Y)$ where X and Y are Banach spaces, the collection $\mathcal{A}$ is said to be an operator ideal if it satisfies the following conditions:

(i) $\mathcal{A}\, contains\, all\, finite\, rank\, operators$

(ii) $T + S \in \mathcal{A}(X,Y)\ for\ S,T \in \mathcal{A}(X,Y)$

(iii) $if\ T \in \mathcal{A}(X,Y)\ and\ S \in \mathcal{L}(Y,Z), then\ ST \in \mathcal{A}(X,Z)\ and\ also\ if\ T \in \mathcal{L}(X,Y)\ and\ S \in \mathcal{A}(Y,Z), then\ ST \in \mathcal{A}(X,Z)$

The collection $\mathcal{A}(X,Y)$, for a given pair of Banach spaces X and Y, is called a component of $\mathcal{A}$.

An **ideal quasi-norm** is a real valued function f defined on an operator ideal $\mathcal{A}$, which satisfies the following properties:

(i) $0 \leq f(T) < \infty, for\ each\ T \in \mathcal{A}\ and\ f(T) = 0\ if\ and\ only\ if\ T = 0$

(ii) there exists a constant $\sigma \geq 1$ such that $f(S+T) \leq \sigma[f(S) + f(T)]\ for\ S,T \in \mathcal{A}(X,Y)$, where $\mathcal{A}(X,Y)$ is any component of $\mathcal{A}$

(iii) a)$f(RS) \leq \|R\| f(S), for\ S \in \mathcal{A}(X,Z), R \in \mathcal{L}(Z,Y)\ and$
$b)\ f(RS) \leq \|S\| f(R), for\ S \in \mathcal{L}(X,Z), R \in \mathcal{A}(Z,Y)$

A quasi-normed operator ideal is an operator ideal equipped with an ideal quasi-norm and a quasi-Banach operator ideal is a quasi-normed operator ideal of which each component is complete with respect to the ideal quasi-norm. It has been shown in [6, 7], that the collection of λ-summing, λ-nuclear and quasi-λ-nuclear operators from X to Y are operator ideals for suitably chosen λ.

For an operator ideal $\mathcal{A}$, the dual operator ideal $\mathcal{A}^d$ is defined as the one of which the component $\mathcal{A}^d(X,Y)$ is given by,

$$\mathcal{A}^d(X,Y) = \{T \in \mathcal{L} : T^* \in \mathcal{A}(Y^*, X^*)\},$$

where T^* denotes the adjoint of the operator T. It is a quasi-Banach operator ideal if $\mathcal{A}$ is so, cf. [5].

Let us recall from [5], p. 73, that an operator ideal $\mathcal{A}$ is said to be surjective if **for every surjection** $Q \in \mathcal{L}(X, X_0)$ **and every operator** $T \in \mathcal{L}(X_0, Y)$, $TQ \in \mathcal{A}(X_0, Y)$ **implies** $T \in \mathcal{A}(X, Y)$.

3. λ-Compact Sets and λ-Compact Operators

Recalling the space $\lambda^s(X)$, where X is a Banach space and λ is a BK sequence space with $0 < \sup_n \|e^n\|_\lambda < \infty$ and $B_{\lambda^\times}$ the unit ball of Banach sequence space $\lambda^\times$ from the previous section, let us introduce the following

Definition 3.1 —

(i) For $\bar{x} = \{x_n\} \in \lambda^s(X)$, the λ-**convex hull** of $\bar{x}$ is written as λ-co$\{x_n\}$ and is defined as

$$\lambda - co\{x_n\} = \{\sum_{n\geq 1} \alpha_n x_n : \ \bar{\alpha} \in B_{\lambda^\times}\};$$

(ii) a set K is said to be λ-compact (resp. weak λ-compact) if there exists $\bar{x} = \{x_n\} \in \lambda^s(X)$(resp. $\{x_n\} \in \lambda^w(X)$) such that $K \subset \lambda - co\{x_n\}$;

(iii) an operator $T \in \mathcal{L}(X,Y)$ is said to be a λ-compact operator (resp. weak λ-compact operator) if T maps bounded sets to λ-compact sets (resp. weak λ-compact sets) in Y, i.e., there exists a $\bar{y} = \{y_n\} \in \lambda^s(Y)$ (resp. $\bar{y} = \{y_n\} \in \lambda^w(Y)$) such that $T(B_X) \subset \lambda - co\{y_n\}$, where B_X is the unit ball of X.

For $\lambda = l^p$, $p \geq 1$, λ-compact sets and λ-compact operators are p-compact sets and p-compact operators studied in [9]. However, illustrating λ-compact sets and λ-compact operators, we have the following.

Proposition 3.2 — Let the sequence spaces λ and μ, equipped with norms $\|.\|_\lambda$ and $\|.\|_\mu$ be *AK-BK* and *AK* respectively such that $0 < \sup_n \|e^n\|_\lambda, \ \sup_n \|e^n\|_\mu < \infty$. Further assume that λ is symmetric and normal; and $l^1 \subseteq \mu$. Then for given $\overline{\alpha} = \{\alpha_n\} \in \lambda$, the set $\{ \{\beta_n \alpha_n\} : \ \{\beta_n\} \in B_{\lambda^\times}\}$ is a λ-compact set in μ and the operator $T : \lambda^\times \rightarrow \mu$ defined as $T(\{\beta_n\}) = \{\beta_n \alpha_n\}$, for $\{\beta_n\} \in \lambda^\times$ is λ-compact.

PROOF : Since λ is normal and (μ, $\|.\|_\mu$) is AK,

$$\|\alpha_n e^n\|_\mu \leq (\sup_n \ \|e^n\|_\mu) \ |\alpha_n|, \ n \geq 1 \ \Rightarrow \ \{\alpha_n e^n\} \in \lambda^s(\mu)$$

for $\bar{\alpha} = \{\alpha_n\} \in \lambda$ and

$$T(\{\beta_n\}) = \sum_{n \geq 1} \beta_n \alpha_n e^n, \ for \ \{\beta_n\} \in \lambda^\times \ \Rightarrow \ T(B_{\lambda^\times}) = \lambda - co\{\alpha_n e^n\}.$$

But

$$T(B_{\lambda^\times}) = \{\{\beta_n \alpha_n\} : \ \{\beta_n\} \in B_{\lambda^\times}\}.$$

$\Rightarrow$ the set $\{\{\beta_n \alpha_n\} : \ \{\beta_n\} \in B_{\lambda^\times}\}$ is λ-compact and the operator T is a λ-compact operator.

Remarks : As particular case, if we consider $\lambda = l_M$, an Orlicz sequence space where M satisfies the Δ_2-condition at zero and $\mu = l^p, \ p \in [1, \infty)$ in the above proposition, the diagonal operator T defined as above for $\overline{\alpha} = \{\alpha_n\} \in l_M$, is an l_M-compact operator from l_N to l^p and the set $\{\{\beta_n \alpha_n\} : \ \{\beta_n\} \in B_{l_N}\}$ is an l_M-compact set in l^p, where N is the complementary Orlicz function to M, cf. [3], p. 297-311, for definitions etc.

Let us denote by $K_\lambda(X, Y)$ (resp. $W_\lambda(X, Y)$), the collection of all λ-compact operators (resp. weak λ-compact operators) from X to Y; and by K_λ (resp. W_λ) as the collection of all λ-compact operators (resp. weak-λ-compact operators) between any pair of Banach spaces.

For $T \in K_\lambda(X, Y)$, let us write

$$k_\lambda(T) = inf\ \|\{y_n\}\|^s_\lambda$$

where the infimum is taken over all possible sequences $\bar{y} = \{y_n\}$ occurring in the definition of λ-compact set $T(B_X)$.

Before proving the results on λ-compact sets and λ-compact operators, we prove some results on sequence spaces which, besides being of independent interest, we need in the sequel. Recalling the norm $\|.\|_{\lambda^\times}$ and the operator R from Section 2, let us prove

Proposition 3.3 — Let λ be a normal sequence space equipped with a monotone norm $\|.\|_\lambda$ satisfying the condition $0 < \sup_n \|e^n\|_\lambda < \infty$ and $(\lambda,\ \|.\|_\lambda)$ is an *AK-BK* space. Then the topological dual λ^* equipped with the operator norm topology is isometrically isomorphic to the cross-dual $(\ \lambda^\times,\ \|.\|_{\lambda^\times})$.

PROOF : Since (λ, $\|.\|_\lambda$) is an AK space, $\{e^n\}$ is a Schauder basis for λ. For showing that $R^{-1}\ :\ X^* \rightarrow \lambda^\times$ defined as $R^{-1}(f) = \{f(e^n)\}$ is an isometry, consider

$$\begin{aligned} \|f\| &= \sup_{\|\overline{\alpha}\|_\lambda \leq 1} |f(\overline{\alpha})| \leq \sup_{\|\overline{\alpha}\|_\lambda \leq 1} \sum_{n \geq 1} |\alpha_n|\ |f(e^n)| \\ &= \|\{f(e^n)\}\|_{\lambda^\times} = \|R^{-1}(f)\|_{\lambda^\times}. \end{aligned}$$

Also

$$\begin{aligned} \|R^{-1}(f)\|_{\lambda^\times} &= \|\{f(e^n)\}\|_{\lambda^\times} = \sup_{\|\overline{\alpha}\|_\lambda \leq 1} \sum_{n \geq 1} |f(e^n)|\ |\alpha_n| \\ &= \sup_{\|\overline{\alpha}\|_\lambda \leq 1} \sum_{n \geq 1} f(e^n)\ \alpha_n\ \beta_n = \sup_{\|\overline{\alpha}\|_\lambda \leq 1} f(\{\alpha_n\ \beta_n\}) \leq \|f\| \end{aligned}$$

where $\{\beta_n\}$ is a sequence of scalars such that $|\beta_n| = 1$ for each $n \in \mathbb{N}$ and $\|\{\alpha_n\ \beta_n\}\|_\lambda \leq \|\{\alpha_n\}\|_\lambda$, since $\|.\|_\lambda$ is monotone. □

This completes the proof.

If we restrict λ further, we get

Proposition 3.4 — In addition to the hypothesis satisfied by λ and $\|.\|_\lambda$ in Proposition 3.3, if λ is also reflexive, then λ is perfect.

PROOF : As λ is reflexive, $\{e^n\}$ becomes a Schauder basis for $\lambda^\times$ and so $(\lambda^\times)^* = \lambda^{\times\times}$. Again by the Proposition 3.3, we have

$$\lambda^{\times\times} = (\lambda^\times)^* = (\lambda^*)^* = \lambda.$$

$\Rightarrow$ λ is a perfect sequence space. □

Using Proposition 3.3, we can identify $\lambda^w(X)$ with the class of continuous linear maps from $\lambda^\times$ to X as follows:

Proposition 3.5 — Let λ and $\|.\|_\lambda$ be as in Proposition 3.3. Then the map $L : \lambda^w(X) \to \mathcal{L}(\lambda^\times, X)$ defined as $L(\overline{x}) = L_{\overline{x}}$, where

$$L_{\overline{x}}(\overline{\alpha}) = \sum_{n\geq 1} \alpha_n x_n$$

for $\overline{x} = \{x_n\} \in \lambda^w(X)$ and $\overline{\alpha} = \{\alpha_n\} \in \lambda^\times$, is an isometry.

PROOF : The map L is clearly linear. For proving that it is an isometry, consider $\overline{x} = \{x_n\} \in \lambda^w(X)$. Then

$$\begin{aligned} \|L(\overline{x})\| = \|L_{\overline{x}}\| &= \sup_{\|\overline{\beta}\|_{\lambda^\times} \leq 1} \|L_{\overline{x}}(\overline{\beta})\| \\ &= \sup_{\|\overline{\beta}\|_{\lambda^\times} \leq 1} \sup_{f \in B_{X^*}} |f(\sum_{n\geq 1} \beta_n x_n)| \\ &= \sup_{f \in B_{X^*}} \sup_{\|\overline{\beta}\|_{\lambda^\times} \leq 1} | < \{f(x_n)\}, \{\beta_n\} > | \\ &= \sup_{f \in B_{X^*}} \|\{f(x_n\}\|_\lambda, \ \textit{by} \text{ Proposition 3.3} \\ &= \|\overline{x}\|_\lambda^w. \end{aligned}$$

□

Now, we prove some properties of λ-compact sets.

Proposition 3.6 — Let λ and $\|.\|_\lambda$ be as in Proposition 3.4 and X be a Banach space. Then for $\overline{x} = \{x_n\} \in \lambda^s(X)$, λ-convex hull of $\{x_n\}$ i.e., λ-co$\{x_n\}$ is a norm closed set in X.

PROOF : Fix $\overline{x} = \{x_n\} \in \lambda^s(X)$ and define a map $S : \lambda^\times \to X$ as $S(\overline{\alpha}) = \sum_{n\geq 1} \alpha_n \, x_n$, for $\overline{\alpha} \in \lambda^\times$. Then S is a continuous linear map since

$$\|S(\overline{\alpha})\| \leq \|\alpha\|_{\lambda^\times} \, \|\overline{x}\|_\lambda^s, \ \forall \, \overline{\alpha} \in \lambda^\times.$$

As $B_{\lambda^\times}$ is same as B_{λ^*}, cf. Proposition 3.3, $B_{\lambda^\times}$ is weakly compact by Banach-Alaoglu Theorem and reflexivity of λ. Hence λ-co$\{x_n\} = S(B_{\lambda^\times})$ is weakly compact in X, cf. [8], p. 39 and p. 62 and hence norm closed in X, cf. [8], p. 34. This completes the proof. □

Proposition 3.7 — Let λ and $\|.\|_\lambda$ be as in Proposition 3.3 and X be a Banach space. Then the convex hull of a λ-compact set K, i.e., co-K is also λ-compact.

PROOF : Let K be a λ-compact set in X. Then there exists $\overline{x} = \{x_n\} \in \lambda^s(X)$ such that $K \subseteq \lambda - co\{x_n\}$. For any $y \in co - K$, $y = \sum_{m=1}^i a_m y_m$, where $y_m \in K$, for $1 \leq m \leq i$ and $a_m \geq 0$ with $\sum_{m=1}^i a_m = 1$. Since $y_m \in K$,

$$y_m = \sum_{n\geq 1} \alpha_n^m \, x_n,$$

for $\{\alpha_n^m\}_n \in B_{\lambda^\times}$, for each $m = 1, 2, ..., i$. Therefore

$$\begin{aligned} y = \sum_{m=1}^{i} a_m y_m &= \sum_{m=1}^{i} a_m \sum_{n\geq 1} \alpha_n^m \, x_n \\ &= \sum_{n\geq 1} (\sum_{m=1}^{i} a_m \alpha_n^m) x_n, \end{aligned}$$

where $\{\sum_{m=1}^i a_m \alpha_n^m\}_n \in B_{\lambda^\times}$. Hence $co - K \subseteq \lambda - co\{x_n\}$ i.e co-K is a λ-compact set in X. □

From Propositions 3.6 and 3.7, we immediately get

Proposition 3.8 — Let λ and $\|.\|_\lambda$ be as in Proposition 3.4 and *X* be a Banach space. Then the closure of co-*K* is λ-compact set whenever *K* is a λ-compact set in *X*.

We now come to prove the main result of this section, namely that K_λ is a quasi-normed operator ideal with respect to the norm k_λ for suitably restricted λ. Let us begin with

***Theorem* 3.9** — *Let λ be a symmetric, monotone sequence space equipped with a k-symmetric, monotone norm $\|.\|_\lambda$, such that (λ, $\|.\|_\lambda$) is a BK-space with $0 < \sup_n \|e^n\|_\lambda < \infty$. Then $\|T\| \leq k_\lambda(T)$ holds for any $T \in K_\lambda(X, Y)$ and $K_\lambda(X, Y)$ becomes a quasi-normed space with respect to the quasi-norm k_λ for any pair of Banach spaces X and Y.*

PROOF : For showing $T + S \in K_\lambda(X, Y)$, whenever $T, S \in K_\lambda(X, Y)$, choose $\bar{y} = \{y_n\},\ \bar{z} = \{z_n\}$ in $\lambda^s(Y)$ such that $T(B_X) \subset \lambda - co\{y_n\}$ and $S(B_X) \subset \lambda - co\{z_n\}$. Fix $x_0 \in X$. Then

$$Tx_0 = \sum_{n\geq 1} \alpha_n y_n \text{ and } Sx_0 = \sum_{n\geq} \beta_n z_n$$

for some $\bar{\alpha} = \{\alpha_n\}$, $\bar{\beta} = \{\beta_n\}$ in $B_{\lambda^\times}$. Define $\{w_n\} \subset Y$ and $\bar{\gamma} = \{\gamma_n\} \subset \mathbb{K}$ as follows:

$$w_{2n} = y_n\ ;\ w_{2n-1} = z_n\ ,\ for\ n = 1, 2, 3, ...$$

and

$$\gamma_{2n} = \alpha_n\ ;\ \gamma_{2n-1} = \beta_n\ ,\ for\ n = 1, 2, 3, ...$$

Since both the spaces λ and $\lambda^\times$ are symmetric and monotone, $\{w_n\} \in \lambda^s(Y)$ and $\{\gamma_n\} \in \lambda^\times$ with $\|\bar{\gamma}\|_{\lambda^\times} \leq 4$. Hence $(T + S)x_0 = \sum_{n\geq 1} \frac{\gamma_n}{4} 4w_n$.

Therefore $(T + S)(B_X) \subset \lambda - co\{4w_n\}$ and so $T + S \in K_\lambda(X, Y)$.

It is easy to check that $\delta T \in K_\lambda(X, Y)$ whenever $T \in K_\lambda(X, Y)$ for any scalar $\delta \in \mathbb{K}$.

For proving $\|T\| \leq k_\lambda(T)$, $T \in K_\lambda(X,Y)$, we have $\bar{y} = \{y_n\} \in \lambda^s(Y)$ such that $T(B_X) \subset \lambda - co\{y_n\}$. Hence for $x \in B_X$, there exists $\{\alpha_n\} \in B_{\lambda^\times}$ such that

$$\begin{aligned} Tx &= \sum_{n\geq 1} \alpha_n y_n \\ &\Rightarrow \ \|Tx\| \leq \sum_{n\geq 1} |\alpha_n| \|y_n\| \leq \|\bar{y}\|_\lambda^s. \\ &\Rightarrow \ \|T\| \leq k_\lambda(T). \end{aligned}$$

In order to prove that k_λ is a quasi-norm on $K_\lambda(X,Y)$, let us note that $k_\lambda(T) \geq 0$ for any $T \in K_\lambda(X,Y)$ and $k_\lambda(T) = 0$ for $T = 0$. Clearly, $k_\lambda(T) = 0$ for $T \in K_\lambda(X,Y) \ \Rightarrow \ T = 0$, since $\|T\| \leq k_\lambda(T)$. Homogeneous property is easy to check. For showing triangular inequality, let us consider $T, S \in K_\lambda(X,Y)$. We can find $\bar{y} = \{y_n\}$, $\bar{z} = \{z_n\}$ in $\lambda^s(Y)$ with

$$T(B_X) \subset \lambda - co\{y_n\} \ ; \ \|\bar{y}\|_\lambda^s < k_\lambda(T) + \epsilon/2$$

and

$$S(B_X) \subset \lambda - co\{z_n\} \ ; \ \|\bar{z}\|_\lambda^s < k_\lambda(S) + \epsilon/2.$$

Now, proceeding as in the proof for vector space structure, we have $(T + S)(B_X) \subset \lambda - co\{4w_n\}$. Since the sequence space λ is equipped with monotone and k-symmetric norm, we have,

$$k_\lambda(T+S) \leq 4\|\{w_n\}\|_\lambda^s \leq 8[\|\bar{y}\|_\lambda^s + \|\bar{z}\|_\lambda^s] \leq 8[k_\lambda(T) + k_\lambda(S) + \epsilon].$$

As $\epsilon > 0$ is arbitrary,

$$k_\lambda(T+S) \leq 8[k_\lambda(T) + k_\lambda(S)].$$

This completes the proof. □

Finally, we prove the main result of this section in

***Theorem* 3.10** — *Let λ be a monotone, symmetric sequence space equipped with a k-symmetric and monotone norm $\|.\|_\lambda$ such that (λ, $\|.\|_\lambda$) is a BK-space and*

$0 < \inf_n \|e^n\|_\lambda \leq \sup_n \|e^n\|_\lambda < \infty$. *Then* K_λ, *the collection of all* λ*-compact operators, becomes a quasi-normed operator ideal with respect to the quasi-norm* k_λ.

PROOF : In order to prove that K_λ is an operator ideal, for any pair of Banach spaces X and Y, consider the component $K_\lambda(X,Y)$. Let us note that finite rank operators belong to $K_\lambda(X,Y)$, since if $T \in \mathcal{L}(X,Y)$ be such that $rank(T) = 1$, then $T = f_0 \otimes y_0$, for some $f_0 \in B_{X^*}$ and $y_0 \in Y$.Write $\bar{y} = \{y_0, 0, 0, ...\}$ and $\bar{\alpha_x} = \{f_0(x), 0, 0, ...\}$ for $x \in B_X$. Clearly, $\bar{y} \in \lambda^s(Y)$ and $\bar{\alpha_x} \in \lambda^\times$.

If $C = \frac{1}{\inf_n \|e^n\|_\lambda}$, then

$$\|\bar{\alpha_x}\|_{\lambda^\times} = \sup_{\|\bar{\beta}\|_{\lambda^\times} \leq 1} |f_0(x)||\beta_1| \leq C$$

since

$$\|\beta_1 e^1\|_\lambda \leq \|\bar{\beta}\|_\lambda \leq 1 \ \Rightarrow \ |\beta_1| \leq 1/\|e^1\|_\lambda \leq C.$$

Hence, $\bar{\alpha_x}/C \in B_{\lambda^\times}$ and so $T(B_X) \subseteq \lambda - co\{C\bar{y}\}$. Consequently, $T \in K_\lambda(X,Y)$.

For showing the ideal property of K_λ, let $T \in K_\lambda(X,Y)$, and $R \in \mathcal{L}(Z,X)$. Then for some $\bar{y} = \{y_n\} \in \lambda^s(Y)$, $T(B_X) \subseteq \lambda - co\{y_n\}$. Thus for $z \in B_Z$, $Rz/\|R\| \in B_X$ and so $T(Rz/\|R\|) = \sum_{n\geq 1} \alpha_n y_n$, for some $\{\alpha_n\} \in B_{\lambda^\times}$. $\Rightarrow \ TR(B_Z) \subseteq \lambda - co\{\|R\|y_n\}$ and $TR \in K_\lambda(Z,Y)$. One can similarly prove that $ST \in K_\lambda(X,Z)$ for $T \in K_\lambda(X,Y)$ and $S \in \mathcal{L}(Y,Z)$. Thus (K_λ, k_λ) is an operator ideal. Since $\|.\|_\lambda$ is k-symmetric and monotone, K_λ becomes a quasi-normed operator ideal with respect to the quasi-norm k_λ by Theorem 3.9.

This completes the proof. □

4. RELATIONSHIPS OF K_λ AND K_λ^d WITH N_λ, Π_λ AND QN_λ

In this section we find the relations of members of K_λ and K_λ^d with λ-summing, λ-nuclear and quasi-λ-nuclear operators for suitably restricted sequence space λ. Let us begin with

***Theorem* 4.1** *Let (λ, $\|.\|_\lambda$) be a normal, symmetric, BK sequence space with* $0 < \inf_n \|e^n\|_\lambda \leq \sup_n \|e^n\|_\lambda < \infty$. *Then for any pair of Banach spaces X and Y,* $N_\lambda(X,Y) \subset K_\lambda^d(X,Y)$.

PROOF : Let $T \in N_\lambda(X,Y)$. Then there exists $\bar{\alpha} = \{\alpha_n\} \in \lambda$ and sequences $\{f_n\} \subset X^*$ and $\bar{y} = \{y_n\} \subset Y$ such that

$$Tx = \sum_{n\geq 1} \alpha_n f_n(x) y_n, \; for\ each\ x \in X$$

where $\{f_n\}$ is bounded in X^* and $\{g(y_n)\} \in \lambda^\times$, for $g \in Y^*$. Then for $g \in B_{Y^*}$ and $x \in X$, we have

$$T^*g(x) = g(Tx) = \sum_{n\geq 1} \alpha_n f_n(x) g(y_n).$$

For $n \in \mathbb{N}$, define $x_n^* \in X^*$ as follows

$$x_n^* = \|\{y_n\}\|_{\lambda^\times}^w \alpha_n f_n$$

$\Rightarrow$ $\{x_n^*\} \in \lambda^s(X^*)$ as λ is normal and

$$T^*g = \sum_{n\geq 1} \frac{g(y_n)}{\|\{y_n\}\|_{\lambda^\times}^w} x_n^*$$

Thus $T^*(B_{Y^*}) \subset \lambda - co\{x_n^*\}$ and so $T^* \in K_\lambda(Y^*, X^*)$. This completes the proof. □

If $\Pi_\lambda(X,Y)W_\lambda(Z,X) = \{T_2 \circ T_1 : T_2 \in \Pi_\lambda(X,Y)\ and\ T_1 \in W_\lambda(Z,X)\}$, where X, Y, Z are Banach spaces, one can easily prove

Proposition 4.2 — Let (λ, $\|.\|_\lambda$) be a monotone symmetric BK space. Then $\Pi_\lambda(X,Y)W_\lambda(Z,X) \subset K_\lambda(Z,Y)$.

PROOF : Trivial and so omitted. □

Proposition 4.3 — Let (λ, $\|.\|_\lambda$) be a BK-space such that $\lambda = \{\bar{\alpha} \in \omega : \|\bar{\alpha}\|_\lambda < \infty\}$ and $0 < \sup_n \|e^n\|_\lambda < \infty$. If λ is also perfect and symmetric and its norm $\|.\|_\lambda$ has norm iteration property, then

(i) $K_\lambda(X,Y) \subset \Pi^d_\lambda(X,Y)$

(ii) $K^d_\lambda(X,Y) \subset \Pi_\lambda(X,Y)$

hold for any pair of Banach spaces X and Y with respective duals X^* and Y^* which are equipped with the operator norm topology.

PROOF : (i) Let $T \in K_\lambda(X,Y)$. Then $T(B_X) \subset \lambda - co\{y_n\}$ for some $\bar{y} = \{y_n\} \in \lambda^s(Y)$ and so for $x \in B_X$, $Tx = \sum_{n\geq 1} \alpha_n y_n$, where $\bar{\alpha} = \{\alpha_n\} \in B_{\lambda^\times}$. For proving $T \in \Pi^d_\lambda(X,Y)$, consider $\{g_k\} \in \lambda^w(Y^*)$, i.e., $\{F(g_k)\} \in \lambda$, $\forall F \in Y^{**}$. In order to show that $\{T^* g_k\} \in \lambda^s(X^*)$, consider

$$\|T^* g_k\| = sup_{x\in B_X}|T^* g_k(x)| = sup_{x\in B_X}|g_k(Tx)|.$$

Now, fix $x_0 \in B_X$. Then there exists $\bar{\beta} = \{\beta_n\} \in B_{\lambda^\times}$, such that $Tx_0 = \sum_{n\geq 1} \beta_n y_n$. Hence

$$\begin{aligned} |g_k(T(x_0)| &\leq \sum_{n\geq 1} |\beta_n||g_k(y_n)| \leq \|\{g_k(y_n)\}_{n\geq 1}\|_\lambda \\ & \sum_{n\geq 1} |\beta_n| \frac{|g_k(y_n)|}{\|\{g_k(y_n)\}_{n\geq 1}\|_\lambda} \leq \|\{g_k(y_n)\}_{n\geq 1}\|_\lambda. \end{aligned}$$

As the right hand side of the above inequality is independent of $x_0 \in B_X$, we get

$$\|T^* g_k\| \leq \|\{g_k(y_n)\}_{n\geq 1}\|_\lambda,\ k \in \mathbb{N}.$$

Next, consider the double sequence $\{g_k(y_n)\}_{k,n\geq 1}$ and write y_n^{**} for the canonical image of y_n in Y^{**} for $n \geq 1$. Since

$$\begin{aligned} \sum_{n\geq 1} |\beta_n|\|\{g_k(y_n)\}_{k\geq 1}\|_\lambda &= \sum_{n\geq 1} |\beta_n|\|y_n^{**}\|\|\{\frac{y_n^{**}}{\|y_n^{**}\|}\,(g_k)\}_{k\geq 1}\|_\lambda \\ &\leq \|\{g_k\}\|^w_\lambda \sum_{n\geq 1} |\beta_n|\|y_n\| < \infty, \end{aligned}$$

for any $\bar{\beta} = \{\beta_n\} \in \lambda^\times$, it follows from the perfectness of λ that $\{\|\{g_k(y_n)\}_{k\geq 1}\|_\lambda\}_{n\geq 1} \in \lambda$. Further by the norm iteration property of λ, we have

$$\|\{\|\{g_k(y_n)\}_{n\geq 1}\|_\lambda\}_{k\geq 1}\|_\lambda = \|\{\|\{g_k(y_n)\}_{k\geq 1}\|_\lambda\}_{n\geq 1}\|_\lambda < \infty$$

and so $\{\{\|\{g_k(y_n)\}_{n\geq 1}\|_\lambda\}_{k\geq 1}\} \in \lambda$. Consequently, $\{\|T^* g_k\|\}_{k\geq 1} \in \lambda$. This completes the proof.

(ii) Omitted as it is similar to that of (i). □

Recalling the map $L : \lambda^w(X) \to \mathcal{L}(\lambda^\times, X)$ defined in Proposition 3.5 we prove the following

Proposition 4.4 — For an *AK-BK* normal symmetric reflexive sequence space λ equipped with a monotone norm $\|.\|_\lambda$ and a Banach space X, the following inclusions hold

(i) $\Pi_\lambda(\lambda^\times, X) \subseteq L(\lambda^s(X)) \subseteq N^d_\lambda(\lambda^\times, X)$

(ii) $\Pi^d_\lambda(X, \lambda) \subseteq {}^dL(\lambda^s(X)) \subseteq N_\lambda(X, \lambda)$,

where ${}^dL(\lambda^s(X)) = \{(L_{\overline{f}})_* : \overline{f} = \{f_n\} \in \lambda^s(X^*)\}$ and $(L_{\overline{f}})_*$ denotes the preadjoint of $L_{\overline{f}}$.

PROOF : (i) Let $T \in \Pi_\lambda(\lambda^\times, X)$. Since $\{e^n\} \in \lambda^w(\lambda^\times)$, cf. Proposition 3.3, $\{T(e^n)\} \in \lambda^s(X)$. We consider $x_n = T(e^n)$, for each n. Then $T = L_{\overline{x}}$, where $\overline{x} = \{x_n\}$. Consequently $T \in L(\lambda^s(X))$ and $\Pi_\lambda(\lambda^\times, X) \subseteq L(\lambda^s(X))$.

For the second inclusion, let $\overline{x} = \{x_n\} \in \lambda^s(X)$ and $L_{\overline{x}} \in L(\lambda^s(X))$. For any $f \in X^*$ and $\overline{\alpha} \in \lambda^\times$

$$L^*_{\overline{x}}(f)(\overline{\alpha}) = \sum_{n\geq 1} \alpha_n f(x_n) = \sum_{n\geq 1} x^{**}_n(f)\alpha_n,$$

where x^{**}_n is the canonical image of x_n from X to its double dual X^{**}, for each $n \in \mathbb{N}$. Therefore

$$L^*_{\overline{x}}(f) = \sum_{n\geq 1} x^{**}_n(f) e^n.$$

Hence $L(\lambda^s(X)) \subseteq N_\lambda^d(\lambda^\times, X)$.

(ii) Consequence of (i) and hence omitted. □

For the result stating the relationship of λ-compactness of an operator with its quasi-λ-nuclearity, we make use of the following

Proposition 4.5 — Let (λ, $\|.\|_\lambda$) be a normal, *BK* sequence space with $0 < \sup_n \|e^n\|_\lambda < \infty$ and $T \in \mathcal{L}(X, Y)$. Then we have

(i) If $T(B_X) \subset \overline{\lambda - co\{y_n\}}$, for $\bar{y} = \{y_n\} \in \lambda^w(Y)$, then $\|T^*g\| \leq \|\{g(y_n)\}\|_\lambda$, for each $g \in Y^*$.

(ii) Let X^* be equipped with the dual norm topology and $T^*(B_{Y^*}) \subset \overline{\lambda - co\{x_n^*\}}$, for $\{x_n^*\} \in \lambda^w(X^*)$. Then $\|Tx\| \leq \|\{x_n^*(x)\}\|_\lambda$, for every $x \in B_X$.

PROOF : (i) Given any $\epsilon > 0$ and $g \in B_{Y^*}$, choose $x_0 \in B_X$ with $\|T^*g\| < |T^*g(x_0)| + \epsilon/2$. Now, for this $\epsilon > 0$ and $x_0 \in B_X$, we can find some $\bar{\alpha} = \{\alpha_n\} \in B_{\lambda^\times}$ with $\|Tx_0 - \sum_{n\geq 1} \alpha_n y_n\| < \epsilon/2$. Therefore,

$$\|T^*g\| < |T^*g(x_0)| + \epsilon/2 \leq |g(Tx_0 - \sum_{n\geq 1} \alpha_n y_n)| + |\sum_{n\geq} \alpha_n g(y_n)| + \epsilon/2$$
$$\leq \sum_{n\geq 1} |\alpha_n||g(y_n)| + \epsilon \leq \|\{g(y_n)\}\|_\lambda + \epsilon.$$

As $\epsilon > 0$ is arbitrary, we get $\|T^*g\| \leq \|\{g(y_n\}\|_\lambda$.

(ii) The proof is similar to that of (i) and so omitted. □

From the definition of λ-compact and quasi-λ-nuclear operator, the above proposition immediately leads to

***Theorem* 4.6** — *Let λ be a normal, BK sequence space equipped with the norm $\|.\|_\lambda$ and $0 < \sup_n \|e^n\|_\lambda < \infty$. If $T \in \mathcal{L}(X, Y)$ is a λ-compact operator, then $T^* \in \mathcal{L}(Y^*, X^*)$ is a quasi-λ-nuclear operator.*

Recalling the constant M_λ occurring in the inequality (1), the converse of Proposition 4.5, is proved in the following form.

Proposition 4.7 —- Let (λ, $\|.\|_\lambda$) be a normal, AK, BK-sequence space with $0 < \sup_n \|e^n\|_\lambda < \infty$ and $T \in \mathcal{L}(X, Y)$. Then the following assertions hold.

(i) If $\|T^*g\| \leq M_\lambda^{-1} \|\{g(y_n)\}\|_\lambda$, for $\bar{y} = \{y_n\} \in \lambda^w(Y)$ and $g \in Y^*$, then $T(B_X) \subset \overline{\lambda - co\{y_n\}}$.

(ii) Let X^* be equipped with w^*-topology and $\{x_n^*\} \in \lambda^w(X^*)$. If $\|Tx\| \leq M_\lambda^{-1}\|\{x_n^*(x)\}\|_\lambda$, $\forall\, x \in X$, then $T^*(B_{Y^*}) \subset \overline{\lambda - co\{x_n^*\}}^{w^*}$.

PROOF : (i) Let us assume that, there exists $x_0 \in B_X$ such that $Tx_0 \notin \overline{\lambda - co\{y_n\}}$. Then by a consequence of Hann-Banach theorem, we can find $\delta > 0$ and $g \in Y^*$ such that $|g(Tx_0)| > \delta$ and $|g(z)| < \delta$, for all $z \in \lambda - co\{y_n\}$, cf. [8], p. 30. Consequently, in view of (1)

$$\begin{aligned} \delta < |g(Tx_0)| &\leq \|T^*g\| \leq M_\lambda^{-1}\|\{g(y_n)\}\|_\lambda \\ &\leq \sup_{\|\bar{\beta}\|_{\lambda^\times} \leq 1} \sum_{n\geq 1} |\beta_n||g(y_n)| \\ &= \sup_{\|\bar{\beta}\|_{\lambda^\times} \leq 1} g(\sum_{n\geq 1} \beta_n \gamma_n y_n) \leq \delta \end{aligned}$$

where $\{\gamma_n\} \subseteq \mathbb{K}$ is such that $|\gamma_n| = 1$ and $|\beta_n||g(y_n)| = \beta_n\gamma_n g(y_n)$ for each $n \in \mathbb{N}$. This contradiction proves the result.

(ii) The proof is omitted as it is analogous to that of (i). □

Let us recall from [1] the following two operators, defined corresponding to a bounded set K in a Banach space X as follows

$$u_K : l_K^1 \to X;\ \ u_K(\{\eta_x\}_{x\in K}) = \sum_{x\in K} \eta_x\, x \tag{4.1}$$

and

$$j_K : X^* \to l_K^\infty;\ j_K(f) = \{f(x)\}_{x\in K}. \tag{4.2}$$

One can easily check that $u_K^* = j_K$.

Now we have

Proposition 4.8 — Let X be a Banach space and λ be a symmetric, normal sequence space equipped with a monotone norm $\|.\|_\lambda$ such that it is also AK-BK and reflexive. Then for a subset K of X, following conditions are equivalent

(i) K is a λ-compact set in X.

(ii) The map $u_k : l^1_K \to X$ is λ-compact.

PROOF : Immediate from the inclusion $K \subseteq u_K(B_{l^1_K}) \subseteq \; closure \; of \; co - K$ and Proposition 3.8. ☐

Since $u^*_K = j_K$, a consequence of this proposition and Theorem 4.6 is

Proposition 4.9 — Let $(\lambda, \|.\|_\lambda)$ and X be as in Proposition 4.8. Then the map $j_K : X^* \to l^\infty_K$ is a quasi-λ-nuclear map provided K is a λ compact set in X.

Finally, let us note from the definition of surjective operator ideal given in Section 2, that an operator ideal $\mathcal{A}$ is surjective if and only if $\mathcal{A} = \mathcal{A}^{sur}$, where the component of $\mathcal{A}^{sur}$ is defined by

$$\mathcal{A}^{sur}(X, Y) = \{T \in \mathcal{L}(X, Y) : \; Tu_{B_X} \in \mathcal{A}(l^1_{B_X}, Y)\}$$

where u_{B_X} is the map defined in (2) for $K = B_X$. Thus, it follows from Proposition 4.8 that K_λ is a surjective operator ideal.

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