Indian J. Pure Appl. Math., **44**(4): 473-496, August 2013

COMMUTATIVITY OF RINGS AND NEAR-RINGS WITH GENERALIZED DERIVATIONS

Ahmed A. M. Kamal[*,1] and Khalid H. Al-Shaalan[**]

[*]*Department of Mathematics, Building 4, College of Science,*
King Saud University, P.O. Box 2455
Riyadh 11451, *Kingdom of Saudi Arabia*
[**]*Science Department, King Abdul-aziz Military Academy,*
Kingdom of Saudi Arabia
e-mail: aamkamal_9@hotmail.com

(*Received* 28 *March* 2012; *after final revision* 11 *September* 2012;
accepted 12 *November* 2012)

In this paper we prove some theorems of commutativity for prime rings or 3-prime near-rings with a suitably constrained generalized derivation. As a consequence of the results obtained, we prove several commutativity theorems for 3-prime near-rings and prime rings. Also, we prove some other results about the generalized derivation either on a near-ring or on a ring.

Key words : Rings; near-rings; generalized derivation; primeness; commutativity.

[1]Permanent address for the first author: Department of Mathematics, Faculty of Sciences, Cairo University, Giza, Egypt.

1. INTRODUCTION

Throughout this paper, R will be a left near-ring or a ring and $Z(R)$ is the multiplicative center of R. A near-ring R is called 3-prime (prime if R is a ring) if, for all $x, y \in R$, $xRy = \{0\}$ implies $x = 0$ or $y = 0$. A non-empty subset U of R is called a semigroup right (left) ideal of R, if U satisfies $UR \subseteq U$ ($RU \subseteq U$). A non-empty subset U of R is called a semigroup ideal if it is both a semigroup right and left ideal. A map $d : R \to R$ is a derivation on R if d is an additive mapping and $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. A generalized derivation f on R is an additive mapping $f : R \to R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation on R. An element $x \in R$ is called a left (right) zero divisor in R if there exists a non-zero element $y \in R$ such that $xy = 0$ ($yx = 0$). A zero divisor is either a left or a right zero divisor. A near-ring R is called a constant near-ring, if $xy = y$ for all $x, y \in R$ and is called a zero-symmetric near-ring, if $0x = 0$ for all $x \in R$. By an integral near-ring, we mean a near-ring without non-zero divisors of zero. We need the following notations throughout the paper $[x, y] = xy - yx$, $x \circ y = xy + yx$ and $(x, y) = x + y - x - y$ for all $x, y \in R$. Finally, for subsets $X, Y \subseteq R$, the symbol $[X, Y]$ will denote the set $\{xy - yx | x \in X, y \in Y\}$.

The study of commutativity of 3-prime near-rings by using derivations was initiated by Bell and Mason in 1987 in [4]. After that, Bell generalizes several results of [4] by using one (two) sided semigroup ideal of the near-ring in his work in [2]. In 2006, Golbasi used in [7] and [8] the definition that a map on a near-ring is called a generalized derivation if it is both a left and right generalized derivation on the near-ring. Golbasi used that definition to deduce some results on the class of 3-prime near-rings. In 2008, Bell investigated in [3] possible analogues of three results of [4] and he got only one extension. Recently, many papers studied the commutativity on near-rings by using derivations and generalized derivations, such as [5], [6] and [9]. In this paper, we generalize some results of [3] and prove some theorems of commutativity for prime rings and 3-prime near-rings.

In Section 2 we give some well-known results and we add some new auxiliary results on a near-ring R admitting a non-zero generalized derivation f associated with a derivation d, which will be useful in the sequel. In Proposition 2.2, we show that a near-ring admitting a generalized derivation if and only if it is zero-symmetric. Lemma 2.12 is very useful in the next sections. it states that if R is 3-prime with a non-zero semigroup ideal U and $f(U) = \{0\}$, then $f = 0$. We conclude this section by an example of a non-zero generalized derivation on a near-ring.

In Section 3 we study the commutativity of a 3-prime near-ring (a prime ring) R admitting a non-zero generalized derivation f under the conditions $f(uv) = f(vu)$ and $f(u)x = xf(u)$ for all $x, u \in U, v \in V$, where U and V are suitable subsets of R. As a consequence of the results in this section, we generalize Theorem 2.1 due to Bell [3].

Section 4 is devoted to study the commutativity of a 3-prime near-ring (a prime ring) R admitting non-zero generalized derivations f and g associated with derivations d and h respectively (it will be denoted later by (f, d) and (g, h)) under the conditions $f(x)g(y) = g(y)f(x)$ for all $x \in U, y \in V$, where U and V are suitable subsets of R. In Theorem 4.7, for a prime ring, we give the conditions under which the following statements: (i) $f(nR) = \{0\}$, (ii) $nR = \{0\}$, (iii) $nU = \{0\}$ and (iv) $f(nU) = \{0\}$ are equivalent for all integer $n \geqslant 2$, where U is a non-zero semigroup ideal of R.

Section 5 is focused on studying the commutativity of a 3-prime near-ring (a prime ring) R admitting a non-zero generalized derivation f under the conditions $f(uv) = -f(vu)$ and $f(v)x = -xf(v)$ for all $x \in R, u \in U, v \in V$, where U and V are suitable subsets of R. The results of this section are analogue to results in Section 3 to obtain that R is a commutative ring of characteristic 2.

2. Preliminaries and Some Results

We will start with the following lemma:

Lemma 2.1 [10, Theorem 2.7] — A near-ring R is zero-symmetric if and only if R admits a derivation.

Remark 2.1 : (i) As a consequence of Lemma 2.1, the zero map is not a derivation on a non-zero constant near-ring.

(ii) It is well-known that left multiplication maps by elements of a ring R are generalized derivations on R associated with the zero derivation on R. This is not true in the class of left near-rings, for example the left multiplication is not a generalized derivation on any non-zero constant near-ring by using (i).

Proposition 2.2 — A near-ring R is zero-symmetric if and only if R admits a generalized derivation.

PROOF : Suppose R is zero symmetric, then the zero map is a generalized derivation on R. Conversely, if R admits a generalized derivation, then it admits a derivation. By Lemma 2.1, R is zero-symmetric.

For any near-ring $R \neq \{0\}$ with zero multiplication, the identity map is a non-zero generalized derivation on R associated with any derivation on R (any additive mapping of $(R, +)$ is a derivation on R). The following example shows that any zero-symmetric near-ring in which the multiplication is not the zero multiplication admits a non-zero generalized derivation. In particular, any non-zero 3-prime zero-symmetric near-ring has a non-zero generalized derivation.

Example 2.1 : Let R be a zero-symmetric near-ring in which the multiplication is not the zero multiplication. Define $f : R \to R$ by $f(x) = cx$, where $c \in R$ such that $cR \neq \{0\}$. Observe that f is a non-zero generalized derivation on R associated with the zero derivation.

Lemma 2.3 — (i) [13, Proposition 1] Let R be a near-ring with a derivation d. Then $xd(y) + d(x)y = d(x)y + xd(y)$ for all $x, y \in R$.

(ii) [13, Lemma 2] Let R be a near-ring with a derivation d. If $x \in Z(R)$ then $d(x) \in Z(R)$.

(iii) [4, Lemma 3(i) and (ii)] Let R be a 3-prime near-ring and $z \in Z(R) - \{0\}$. Then, z is not a zero divisor. Moreover, if $z + z \in Z(R)$ then $(R, +)$ is abelian.

(iv) [4, Lemma 1] (the partial distributive law) Let R be a near-ring and d be a derivation on R. For all $x, y, z \in R$ we have $(xd(y)+d(x)y)z = xd(y)z+d(x)yz$.

(v) [2, Lemma 1.2(iii)] (i) Let R be a 3-prime near-ring and $x \in Z(R) - \{0\}$. If either yx or xy in $Z(R)$, then $y \in Z(R)$.

(vi) [1, Theorem 1.1] Let R be a 3-prime near-ring with derivations d_1 and d_2. Suppose that $2R = \{0\}$.Then $d_1 d_2$ is a derivation on R if and only if either $d_1 = 0$ or $d_2 = 0$.

Lemma 2.4 — Let R be a 3-prime near-ring.

(i) [2, Lemma 1.3(i)] Let U be a non-zero semigroup right (left) ideal of R. If $x \in R$ and $Ux = \{0\}$ ($xU = \{0\}$), then $x = 0$.

(ii) [2, Lemma 1.4] Let U be a non-zero semigroup ideal of R. If $x, y \in R$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$. Moreover, suppose d is a non-zero derivation on R and $x \in R$ such that $d(U)x = \{0\}$ ($xd(U) = \{0\}$), so $x = 0$.

(iii) [2, Lemma 3.1] Let U be a non-zero semigroup ideal of R. If d is a non-zero derivation on R such that $d^2(U) = \{0\}$, then $d^2 = 0$.

Lemma 2.5 [7, Lemma 2.2 and Lemma 2.3] — Let R be a near-ring with a generalized derivation f associated with a derivation d. Then

(i) $f(x)y + xd(y) = xd(y) + f(x)y$ for all $x, y \in R$.

(ii) $(f(x)y + xd(y))z = f(x)yz + xd(y)z$ for all $x, y, z \in R$.

Lemma 2.6 (i) [2, Lemma 3.2] — Let R be a 3-prime near-ring with a non-zero semigroup ideal U. Let d be a derivation on R such that $d^2(U) \neq \{0\}$. If $a \in R$ and $[a, d(U)] = \{0\}$, then $a \in Z(R)$.

(ii) [2, Theorem 3.1] Let R be a 3-prime near-ring with a non-zero semigroup ideal U. If R admits a derivation d such that $d^2 \neq 0$ and $[d(U), d(U)] = \{0\}$, then R is a commutative ring.

Lemma 2.7 [2, Theorem 2.1] — Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U. If R admits a non-zero derivation d such that $d(U) \subseteq Z(R)$, then R is a commutative ring.

Lemma 2.8 [2, Lemma 1.3(ii)] — Let R be a left 3-prime near-ring with a non-zero derivation d. If U is either a non-zero semigroup right ideal of R or a non-zero semigroup left ideal of R, then $d(U) \neq \{0\}$.

Lemma 2.9 [11, Proposition 2.8] — Let R be a 3-prime near-ring with a non-zero semigroup ideal U and a non-zero derivation d. Then for any positive integer $n \geqslant 2$, the following statements are equivalent

(i) $d(nR) = \{0\}$.

(ii) $d(nU) = \{0\}$.

(iii) $nU = \{0\}$.

(iv) $nR = \{0\}$.

Lemma 2.10 [2, Lemma 1.3(iii)] — Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U. If x is an element of R which centralizes U, then $x \in Z(R)$.

Lemma 2.11 [2, Lemma 1.5] — Let R be a 3-prime near-ring with a non-zero semigroup right ideal or a non-zero semigroup left ideal U. If $U \subseteq Z(R)$, then R is a commutative ring.

Lemma 2.12 — Let R be a 3-prime near-ring with a non-zero semigroup ideal U and a generalized derivation f such that $f(U) = \{0\}$. Then $f = 0$.

PROOF : Let R be a near-ring. For all $u \in U, x \in R$, we have $0 = f(ux) = f(u)x + ud(x) = ud(x)$. Since $U \neq \{0\}$, we get $d = 0$ by Lemma 2.4(i). It follows that $f(xy) = f(x)y$ for all $x, y \in R$. Replacing y by $u \in U$, we conclude that $0 = f(xu) = f(x)u$. Using Lemma 2.4(i) again, we deduce that $f = 0$.

Lemma 2.13 [11, Corollary 3.2] — Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U and a non-zero semigroup ideal V. If R admits a non-zero derivation d such that $d(vu) = d(uv)$ for all $v \in V, u \in U$, then R is a commutative ring.

Lemma 2.14 [10, Lemma 2.6(i)] — Let R be a non-zero near-ring. If R has no non-zero zero divisors, then R is 3-prime.

Proposition 2.15 — Let R be a 3-prime near-ring with f a generalized derivation associated with a derivation d. If $f = 0$, then $d = 0$.

PROOF : From $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$ and $f = 0$, we have $xd(y) = 0$ for all $x, y \in R$. By Lemma 2.4(i), we have $d = 0$.

The following example shows the existence of a non-zero generalized derivation on a left near-ring which is not a ring.

Example 2.2 : Let R be the zero-symmetric abelian left near-ring with $2R \neq \{0\}$ defined by $R = \{f \in M_o(\mathbb{Z}_n) : (\mathbb{Z}_n - \{1\})f = \{0\}\} = Ann_{M_o(\mathbb{Z}_n)}(\mathbb{Z}_n - \{1\})$ with addition and composition of maps such that $n \geq 3$. Denote the elements of R by f_y to mean $1f_y = y$. Notice that R is not a ring since $(f_1 + f_1)f_1 = f_2 f_1 = f_o \neq f_2 = f_1 + f_1 = f_1 f_1 + f_1 f_1$. Observe that $f_x + f_y = f_{x+y}$ for all $x, y \in \mathbb{Z}_n$, $f_1 f_y = f_y$ and $f_x f_y = f_o$ for all $x \in \mathbb{Z}_n - \{1\}, y \in \mathbb{Z}_n$. Define the map $D : R \to R$ by $D(f_y) = f_{my}$ for all $y \in \mathbb{Z}_n$, where m is a positive integer such that m is not a multiple of n and $(m, n) \neq 1$ (i.e. $f_{my} \neq f_1$ for all $y \in \mathbb{Z}_n$). Thus, we have $D(f_x + f_y) = D(f_{x+y}) = f_{m(x+y)} = f_{mx+my} = f_{mx} + f_{my} = D(f_x) + D(f_y)$ for all $x, y \in \mathbb{Z}_n$ and D is an additive endomorphism of $(R, +)$. Also, $D(f_1 f_y) = D(f_y) = f_1 D(f_y) + f_0 = f_1 D(f_y) + f_m f_y = f_1 D(f_y) + D(f_1) f_y$ for all $y \in \mathbb{Z}_n$ and $D(f_x f_y) = D(f_0) = f_0 = f_0 + f_0 = f_x D(f_y) + D(f_x) f_y$ for all $x \in \mathbb{Z}_n - \{1\}, y \in \mathbb{Z}_n$. Therefore, if n is not a prime number, then D is a non-zero derivation on R. Define $d : R \to R$ by $d(f_y) = f_{sy}$ for all $y \in \mathbb{Z}_n$, where s is a positive integer such that s is not a multiple of n and $(s, n) \neq 1$ and s is congruent to m modulo n. Thus, d is also a non-zero derivation on R. We claim that D is a non-zero generalized derivation on R with associated derivation d. Indeed, for all $1 \neq x, y \in \mathbb{Z}_n$, we have

$$D(f_x f_y) = D(f_o) = f_o = f_o + f_o = D(f_x)f_y + f_x d(f_y).$$

Now, for all $y \in \mathbb{Z}_n$, we get $D(f_1 f_y) = D(f_y) = f_{my}$ and

$$D(f_1)f_y + f_1 d(f_y) = f_m f_y + d(f_y) = 0 + f_{sy} = f_{sy}.$$

Since s is congruent to m modulo n, we obtain $f_{my} = f_{sy}$ and $D(f_1 f_y) = D(f_1)f_y + f_1 d(f_y)$. Therefore, D is a non-zero generalized derivation on R associated with d. Similarly, d is a non-zero generalized derivation on R associated with the derivation D. Observe that $D \neq 0$ if and only if $d \neq 0$, using Proposition 2.15.

The following lemmas will be useful in Section 5.

Lemma 2.16 [12, Corollary 2.14(ii)] — Any 3-prime distributive near-ring is a ring.

Lemma 2.17 [11, Lemma 4.1] — Let R be a 3-prime near-ring with a non-zero semigroup right ideal U.

(i) If $ua = (-a)(-u)$ for all $u \in U, a \in A$, where A is a subset of R, then $-A \subseteq Z(R)$.

(ii) If $ua = (-a)(-u)$ for all $u \in U, a \in A$, where A is a subgroup of $(R, +)$, then $A \subseteq Z(R)$.

(iii) If $uj = (-j)(-u)$ for all $u \in U, j \in J$, where J is a non-zero right R-subgroup of R or a non-zero semigroup left ideal of R, then R is a commutative ring.

Lemma 2.18 [11, Corollary 5.5] — Let R be a 3-prime near-ring with non-zero derivations d and g such that $d^2(U) \neq \{0\}$, where U is a non-zero semigroup ideal of R. If $d(u)g(v) = -g(v)d(u)$ for all $u, v \in U$, then R is a commutative ring of characteristic 2.

Lemma 2.19 [11, Corollary 4.4(iii)] — Let R be a 3-prime near-ring with a non-zero derivation d and a non-zero semigroup ideal U. If $d(xy) = -d(yx)$ for all $x \in V, y \in U$ where V is a non-zero right R-subgroup of R. Then R is a commutative ring of characteristic 2.

3. On the Condition $f(xy) = f(yx)$

In this section we will study the commutativity of rings and near-rings, each admitting a generalized derivation f satisfies the conditions $f(xy) = f(yx)$ and $f(y)x = xf(y)$.

The next result is a generalization of Lemma 2.2 of [3].

Proposition 3.1 — Let R be a 3-prime near-ring with a non-zero-semigroup ideal U and f be a generalized derivation on R associated with a derivation d. Then $d(f(U)) = \{0\}$ if and only if $f(d(U)) = \{0\}$.

PROOF : Observe that R is zero-symmetric by Proposition 2.2. If $d = 0$, then the result is true. So suppose $d \neq 0$.

(i) Suppose $d(f(U)) = \{0\}$. For all $u \in U, y \in R$ and using Lemma 2.3(i), we have

$$\begin{aligned} 0 &= d(f(uy)) = d(f(u)y + ud(y)) = d(f(u)y) + d(ud(y)) \\ &= d(f(u))y + f(u)d(y) + d(u)d(y) + ud^2(y). \end{aligned}$$

That is for all $u \in U, y \in R$

$$f(u)d(y) + d(u)d(y) + ud^2(y) = 0 \tag{2.1}$$

as $d(f(u))y = 0$. Applying d again, we get

$$\begin{aligned} 0 &= d(f(u)d(y)) + d(d(u)d(y)) + d(ud^2(y)) \\ &= d(f(u))d(y) + f(u)d^2(y) + d^2(u)d(y) + d(u)d^2(y) + d(u)d^2(y) + ud^3(y) \end{aligned}$$

and we deduce that for all $u \in U, y \in R$

$$f(u)d^2(y) + d^2(u)d(y) + d(u)d^2(y) + d(u)d^2(y) + ud^3(y) = 0. \tag{2.2}$$

Taking $d(y)$ instead of y in (2.1) gives $f(u)d^2(y) + d(u)d^2(y) + ud^3(y) = 0$, hence (2.2) yields

$$d^2(u)d(y) + d(u)d^2(y) = 0 \text{ for all } u \in U, y \in R. \tag{2.3}$$

Now, for all $v \in U + U, u \in U$, we have $df(v) = 0$ and then $0 = d(f(vu)) = f(v)d(u) + d(v)d(u) + vd^2(u) = 0$ by the same way above in (2.1). Observe that $d(tw) = td(w) + d(w)t \in U + U$ for all $t, w \in U$. Replacing v by $d(tw)$ in the last equation, we get

$$f(d(tw))d(u) + d^2(tw)d(u) + d(tw)d^2(u) = 0 \text{ for all } t, w, u \in U. \tag{2.4}$$

Replacing u by tw and y by u in (2.3), we have

$$d^2(tw)d(u) + d(tw)d^2(u) = 0 \text{ for all } t, w, u \in U. \tag{2.5}$$

From (2.4) and (2.5) it is clear that $f(d(tw))d(u) = 0$ for all $t, w, u \in U$. Using Lemma 2.4(ii), we conclude that $f(d(tw)) = 0$ for all $t, w \in U$. Therefore, $f(d(U^2)) = \{0\}$.

Finally,

$$\begin{aligned} 0 &= f(d(tw)) = f(td(w) + d(t)w) = f(t)d(w) + td^2(w) + d(t)d(w) + f(d(t))w \\ &= f(t)d(w) + d(td(w)) + f(d(t))w = f(t)d(w) + d(t)d(w) + td^2(w) + f(d(t))w \\ &= f(d(t))w \end{aligned}$$

using (2.1). Using Lemma 2.4(i), we deduce that $f(d(U)) = \{0\}$.

(ii) Suppose $f(d(U)) = \{0\}$. For all $u \in U, y \in R$ we have by the same way above in (i) that

$$0 = f(d(uy)) = f(u)d(y) + ud^2(y) + d(u)d(y) = 0 \tag{2.6}$$

Applying f again, we get for all $u \in U, y \in R$ that

$$f^2(u)d(y) + f(u)d^2(y) + f(u)d^2(y) + ud^3(y) + d(u)d^2(y) = 0. \tag{2.7}$$

Taking $d(y)$ instead of y in (2.6) gives $f(u)d^2(y) + ud^3(y) + d(u)d^2(y) = 0$, hence (2.7) yields

$$f^2(u)d(y) + f(u)d^2(y) = 0 \text{ for all } u \in U, y \in R. \qquad (2.8)$$

Since $f(uv) = f(u)v + ud(v) \in U + U$ for all $u, v \in U$, we get by the same way above in (i) that

$$f(f(uv))d(w) + f(uv)d^2(w) + d(f(uv))d(w) = 0 \text{ for all } u, v, w \in U \qquad (2.9)$$

and replacing u by uv and y by w in (2.8), where $v \in U$, we have

$$f^2(uv)d(w) + f(uv)d^2(w) = 0 \text{ for all } u, v, w \in U. \qquad (2.10)$$

From (2.9) and (2.10) it is clear that $d(f(uv))d(w) = 0$ for all $u, v, w \in U$. Using Lemma 2.4(ii), we conclude that $d(f(uv)) = 0$ for all $u, v \in U$. Therefore, $d(f(U^2)) = \{0\}$.

Finally, $0 = d(f(uv)) = d(f(u))v$ for all $u, v \in U$ by the same way in (i). Using Lemma 2.4(i), we deduce that $d(f(U)) = \{0\}$.

The following result generalizes Theorem 2.1 of [3].

***Theorem* 3.2** — *Let R be a 3-prime near-ring with a non-zero generalized derivation f such that $f(U)$ centralizes U, where U is a non-zero semigroup ideal of R. Then R is a commutative ring.*

PROOF : R is zero-symmetric by using Proposition 2.2. $f(U)$ centralizes U implies $f(U) \subseteq Z(R)$ by Lemma 2.10. From $f \neq 0$ and Lemma 2.12, there exists $a \in U$ such that $0 \neq f(a) \in Z(R)$. Suppose $d^2 = 0$. If $d = 0$, then $f(uy) = f(u)y \in Z(R)$ for all $u \in U, y \in R$. So $f(a)y \in Z(R)$ for all $y \in R$ and we get that $y \in Z(R)$ by Lemma 2.3(v). Hence, R is a commutative ring. If $d \neq 0$, then $f(ud(x)) = f(u)d(x) + ud(d(x)) = f(u)d(x) \in Z(R)$ for all $u \in U, x \in R$. Thus, $f(a)d(x) \in Z(R)$ implies $d(x) \in Z(R)$ for all $x \in R$ by Lemma 2.3(v) and $f(a) \neq 0$. Therefore, R is a commutative ring by Lemma 2.7.

Now, assume that $d^2 \neq 0$. From $f(uf(v)) \in Z(R)$ for all $u, v \in U$, we have $f(u)f(v) + udf(v) \in Z(R)$. Using Lemma 2.5, we conclude that $xf(u)f(v) +$

$xudf(v) = f(u)f(v)x + udf(v)x$ for all $x \in R$. But $xf(u)f(v) = f(u)f(v)x$ implies $xudf(v) = udf(v)x$ for all $u, v \in U, x \in R$. It follows that $df(v)xu = df(v)ux$ by Lemma 2.3(ii). Thus, $df(v)(xu - ux) = 0$ for all $u, v \in U, x \in R$. Using Lemma 2.3(iii) either $df(U) = \{0\}$ or $xu = ux$ for all $u \in U, x \in R$. If $df(U) \neq \{0\}$, then $U \subseteq Z(R)$ and R is a commutative ring by Lemma 2.11.

Now, suppose $d(f(U)) = \{0\}$. That implies $f(d(U)) = \{0\}$ by Proposition 3.1. If there exists $e \in R$ such that $f(e) = 0$, then $f(eu) = ed(u) \in Z(R)$ for all $u \in U$. In particular, since $f(d(u)) = 0$ for all $u \in U$, we have $f(d(u)w) = d(u)d(w), f(d(w)u) = d(w)d(u) \in Z(R)$ for all $u, w \in U$. If $d(r)d(s) = 0$ for some $r, s \in U$, then $(d(s)d(r))^2 = d(s)d(r)d(s)d(r) = d(s)0d(r) = 0$. But $d(s)d(r) \in Z(R)$, so $(d(s)d(r))^2 = 0$ implies that $d(s)d(r) = 0$ by Lemma 2.3(iii). If $d(r)d(s) \neq 0$, then $d(s)d(r) \neq 0$ and both of them are not zero divisors by Lemma 2.3(iii). Thus, $d(r)(d(r)d(s)) = (d(r)d(s))d(r)$ implies $d(r)(d(r)d(s) - d(s)d(r)) = 0$. It follows that $d(s)d(r)(d(r)d(s) - d(s)d(r)) = 0$ which gives us $d(r)d(s) = d(s)d(r)$ for all $r, s \in U$, since $d(s)d(r) \neq 0$ is not a zero divisor. So in both cases, $d(r)d(s) \neq 0$ and $d(r)d(s) = 0$, we have $d(r)d(s) = d(s)d(r)$ for all $r, s \in U$. Therefore, R is a commutative ring by Lemma 2.6 since $d^2 \neq 0$.

Proposition 3.3 — Let R be a 3-prime near-ring with a generalized derivation f associated with the zero derivation. If $f(uv) = f(vu)$ for all $u \in U, v \in V$, where U is a non-zero semigroup ideal of R and V is a non-zero subset of R, then either $f(V) = \{0\}$ or $V \subseteq Z(R)$.

PROOF : Replace u by vu in $f(uv) = f(vu)$ to get $f(vuv) = f(vvu)$ for all $u \in U, v \in V$. It follows that $f(v)uv = f(v)vu$. Replacing u by uw where $w \in U$, we have $f(v)uwv = f(v)vuw = f(v)uvw$ and hence $f(v)u(wv - vw) = 0$ for all $u, w \in U, v \in V$. Using Lemma 2.4(ii), we get for all $v \in V$ either $f(v) = 0$ or v centralizes U. If $f(V) \neq \{0\}$, then there exists $a \in V$ such that $f(a) \neq 0$. Thus, a centralizes U and then $a \in Z(R)$ by Lemma 2.10. Now, for all $u \in U, v \in V$ we have $f((au)v) = f(v(au)) = f(avu)$ which means $f(a)uv = f(a)vu$. Replacing u by uw and by the same way above, we get $f(a)u(vw - wv) = 0$ for all $u, w \in U, v \in V$. Using Lemma 2.4(ii) and $f(a) \neq 0$, we have $vw = wv$ for all $v \in V, w \in U$. So $V \subseteq Z(R)$ by Lemma 2.10.

Corollary 3.4 — Let R be a 3-prime near-ring with a non-zero generalized derivation f associated with the zero derivation. If $f(uv) = f(vu)$ for all $u \in U, v \in V$, where U is a non-zero semigroup ideal of R and V is a non-zero semigroup left ideal of R, then R is a commutative ring.

PROOF : From Proposition 3.3, either $f(V) = \{0\}$ or $V \subseteq Z(R)$. If $f(V) = \{0\}$, then $0 = f(xv) = f(x)v$ for all $x \in R, v \in V$. Lemma 2.4(i) implies that $f = 0$, a contradiction. Therefore, $V \subseteq Z(R)$ and R is a commutative ring by Lemma 2.11.

***Theorem* 3.5** — *Let R be a prime ring with a non-zero generalized derivation f associated with a derivation d. If $f(uv) = f(vu)$ for all $u \in U, v \in V$, where U is a non-empty subset of R and V is a non-zero semigroup ideal of R, then either $d(U) = \{0\}$ or $U \subseteq Z(R)$.*

PROOF : Replacing v by vu in $f(uv) = f(vu)$, we have $0 = f((uv-vu)u) = (uv - vu)d(u)$ for all $u \in U, v \in V$. Replacing v by vw, where $w \in V$, we have $uvwd(u) = v(wud(u)) = v(uwd(u))$ and hence $(uv - vu)wd(u) = 0$. Lemma 2.4(ii) implies that for each $u \in U$ either $d(u) = 0$ or u centralizes V. If $d(U) \neq \{0\}$, then there exists $a \in U$ such that $d(a) \neq 0$, $a \in Z(R)$ (by Lemma 2.10). Replacing v by va in $f(uv) = f(vu)$, we have $0 = f((uv - vu)a) = (uv - vu)d(a)$ for all $u \in U, v \in V$. But $d(a) \in Z(R) - \{0\}$ is not a zero divisor by using Lemma 2.3(ii), (iii). Thus, $uv = vu$ for all $u \in U, v \in V$ and U centralizes V. Therefore, $U \subseteq Z(R)$ by Lemma 2.10.

Corollary 3.6 — Let R be a prime ring with a non-zero generalized derivation f. If $f(uv) = f(vu)$ for all $u \in U, v \in V$, where U is a non-zero semigroup ideal of R and V is a non-zero semigroup left ideal of R, then R is a commutative ring.

PROOF : Direct from Corollary 3.4, Theorem 3.5, Lemma 2.8 and Lemma 2.11.

***Theorem* 3.7** — *Let R be an integral near-ring with a non-zero generalized* *derivation f. If $f(uv) = f(vu)$ for all $u, v \in U$, where U is a non-zero two-sided R-subgroup of R, then R is a commutative ring.*

PROOF : Let d be the associated derivation with f. If $d = 0$, then R is a commutative ring by using Lemma 2.14 and Corollary 3.4. Let $d \neq 0$. Replace v by uv in $f(uv) = f(vu)$ to get $f(uuv) = f(uvu)$ for all $u, v \in U$. It follows that $0 = f(u(uv - vu)) = f(u)(uv - vu) + ud(uv - vu)$. Replace u by $[u, w]$, where $w \in U$. So $0 = f([u, w])([u, w]v - v[u, w]) + [u, w]d([u, w]v - v[u, w]) = [u, w]d([u, w]v - v[u, w])$ for all $u, v, w \in U$ as $f([u, w]) = 0$. Write $c \in U$ for $[u, w]$. That means $cd(cv - vc) = 0$ for all $c \in B, v \in U$, where $B = \{[u, w] : u, w \in U\}$. Since R is without zero divisors, either $c = 0$ or $d(cv - vc) = 0$. But $c = 0$ implies $d(cv - vc) = 0$. So in both cases

$$d(cv - vc) = 0 \text{ for all } c \in B, v \in U. \tag{3.1}$$

Putting cv instead of v and using (3.1), we have $0 = d(ccv - cvc) = d(c(cv - vc)) = cd(cv - vc) + d(c)(cv - vc) = d(c)(cv - vc)$. For all $c \in B$ either $d(c) = 0$ or $cv = vc$ for all $v \in U$. If $d(c) = 0$ for all $c \in B$, then by Lemma 2.13 R is a commutative ring. If $d(c_o) \neq 0$ for some $c_o \in B$, then $c_o v = vc_o$ for all $v \in U$ and then $c_o \in Z(R)$ by Lemma 2.10. Now, for all $u, v \in U$ we have $f((uc_o)v) = f(v(uc_o))$. So $0 = f((uv - vu)c_o) = f(uv - vu)c_o + (uv - vu)d(c_o) = (uv - vu)d(c_o)$ and hence $uv = vu$ for all $u, v \in U$. Therefore, $U \subseteq Z(R)$ by Lemma 2.10 and R is a commutative ring by Lemma 2.11.

Corollary 3.8 — Let R be an integral near-ring with a non-zero generalized derivation f. If $f(xy) = f(yx)$ for all $x, y \in R$, then R is a commutative ring.

4. ON THE CONDITION $f(xy) = f(yx)$

In this section we will study the commutativity of rings and near-rings, each admitting a generalized derivation f satisfies the condition $f(x)f(y) = f(y)f(x)$.

The following result is a partial generalization of Lemma 3(iv) of [4].

Proposition 4.1 — Let R be an integral near-ring with $2R \neq \{0\}$ and a generalized derivation f such that $f^2(U) = \{0\}$, where U is a non-zero semigroup ideal of R which is closed under addition. Then $f = 0$ or $d = 0$.

PROOF : For all $u \in U, y \in R$ we have

$$\begin{aligned} 0 &= f^2(uy) = f(f(uy)) = f(f(u)y + ud(y)) \\ &= f^2(u)y + f(u)d(y) + f(u)d(y) + ud^2(y) = 2f(u)d(y) + ud^2(y). \end{aligned}$$

Since $f(uv) \in U$, Replace u by $f(uv)$ where $v \in U$. Thus, $f(uv)d^2(y) = 0$ for all $u, v \in U, y \in R$. If $f \neq 0$, then $f(U^2) \neq \{0\}$ by Lemma 2.12 (since U^2 is also a semigroup ideal of R) and there exist $a, b \in U$ such that $f(ab)$ is not a zero divisor. It follows that $d^2 = 0$ and hence $d = 0$ by Lemma 2.3(vi).

Proposition 4.2 — Let R be a 3-prime near-ring with a non-zero generalized derivation f associated with the zero derivation and a non-zero generalized derivation g such that $f(u)g(v) = g(v)f(u)$ for all $u \in U, v \in V$, where U and V are non-zero semigroup ideals of R. Then R is a commutative ring.

PROOF : For all $u, s \in U, v \in V$, we have $f(us)g(v) = g(v)f(us)$. It follows that $f(u)sg(v) = g(v)f(u)s = f(u)g(v)s$. Replacing s by sr, where $r \in R$, we get $f(u)srg(v) = f(u)g(v)sr = f(u)sg(v)r$ for all $u, s \in U, v \in V, r \in R$. Thus, $f(u)s(rg(v) - g(v)r) = 0$. Since $f \neq 0$, there exists $a \in U$ such that $f(a) \neq 0$ by Lemma 2.12. Using Lemma 2.4(ii) and $f(a) \neq 0$, we have $rg(v) = g(v)r$ for all $v \in V, r \in R$. Therefore, $g(V) \subseteq Z(R)$ and R is a commutative ring by Theorem 3.2.

***Theorem* 4.3** — *Let R be a prime ring such that $2R \neq \{0\}$ with two non-zero generalized derivations (f, d) and (g, h) such that $0 \neq f(a) \in Z(R)$ for some $a \in V$, where V is a non-zero semigroup ideal of R. If $f(x)g(y) = g(y)f(x)$ for all $x \in V, y \in U$, where U is a non-zero semigroup ideal of R, then R is a commutative ring.*

PROOF : If $h = 0$ or $d = 0$, then R is commutative by Proposition 4.2. Suppose $h \neq 0 \neq d$. Then for all $x, y \in U, z \in V$ we have $f(z)g(xg(y)) = g(xg(y))f(z)$ and hence $f(z)g(x)g(y) + f(z)xh(g(y)) = g(x)g(y)f(z) + xh(g(y))f(z)$. But $f(z)g(x)g(y) = g(x)g(y)f(z)$, so $f(z)xh(g(y)) = xh(g(y))f(z)$ for all $x, y \in$

$U, z \in V$. Replacing x by xr, where $r \in U$, it yields $f(z)xrh(g(y)) = xrh(g(y))$ $f(z) = xf(z)rh(g(y))$ and then $(f(z)x - xf(z))rh(g(y)) = 0$ for all $x, y \in U, z \in V$. If $h(g(U)) \neq \{0\}$, then $f(V)$ centralizes U and $f(V) \subseteq Z(R)$ by Lemma 2.10. Using Theorem 3.2, we obtain that R is a commutative ring. If $h(g(U)) = \{0\}$, then $g(h(U)) = \{0\}$ by Proposition 3.1(i). So for all $y, u \in U, v \in V$, we have $g(h(u)y)f(v) = f(v)g(h(u)y)$ which implies $h(u)h(y)f(v) = f(v)h(u)h(y)$. Replacing v by a, we get $(h(u)h(y) - h(y)h(u))f(a) = 0$. Therefore, $h(u)h(y) = h(y)h(u)$ for all $u, y \in U$ by Lemma 2.3(iii). So we obtain that R is a commutative ring or $h^2(U) = 0$ by Lemma 2.6(ii). If $h^2(U) = 0$, then $h^2 = 0$ by Lemma 2.4(iii) and $h = 0$ by Lemma 2.3(vi), a contradiction.

***Theorem* 4.4** — *Let R be a 3-prime near-ring with two non-zero generalized derivations (f, d) and (g, h) such that $f(x)g(y) = g(y)f(x)$ for all $x \in U, y \in V$, where U and V are non-zero semigroup ideals of R. Suppose that $f(U) \subseteq U$ and there exists $a \in U$ such that $f(a)$ is not a left zero divisor in R and $f^2(a) = 0$. Then R is a commutative ring.*

PROOF : From Proposition 2.2, R is zero-symmetric. For all $x \in U, y \in R, z \in V$, we have $f(f(x)y)g(z) = g(z)f(f(x)y)$. Using Lemma 2.5(ii), we get $f^2(x)yg(z) + f(x)d(y)g(z) = g(z)f^2(x)y + g(z)f(x)d(y)$. Putting $x = a$, we obtain $f(a)d(y)g(z) = g(z)f(a)d(y) = f(a)g(z)d(y)$. That is, for all $y \in R, z \in V$, we conclude that

$$0 = f(a)(g(z)d(y) - d(y)g(z)). \tag{4.1}$$

Since $f(a)$ is not a left zero divisor in R, we deduce that $g(z)d(y) = d(y)g(z)$ for all $z \in V, y \in R$. Lemma 2.6(i) and Lemma 2.4(iii) imply that $d^2 = 0$ or $g(V) \subseteq Z(R)$. If $d^2 \neq 0$, then R is a commutative ring by Theorem 3.2. If $d^2 = 0$ and $2R \neq \{0\}$, then $d = 0$ by Lemma 2.3(vi) and R is a commutative ring by Proposition 4.2. Now, suppose that $d^2 = 0$ and R is of characteristic 2. So for all $x, y \in R$, we have

$$\begin{aligned} f^2(xy) &= f(f(x)y + xd(y)) = f^2(x)y + f(x)d(y) + f(x)d(y) + xd^2(y) \\ &= f^2(x)y + 2f(x)d(y) + xd^2(y) = f^2(x)y \end{aligned}$$

and f^2 is a non-zero generalized derivation on R associated with the zero derivation. Since $f(x) \in U$ for all $x \in U$, we get $f^2(x)g(z) = g(z)f^2(x)$ for all $x \in U, z \in V$. Proposition 4.2 again implies that R is a commutative ring.

Corollary 4.5 — Let R be a 3-prime near-ring with two non-zero generalized derivations (f, d) and (g, h) such that $f(x)g(y) = g(y)f(x)$ for all $x, y \in R$ and there exists $a \in R$ such that $f(a)$ is not a left zero divisor in R and $f^2(a) = 0$. Then R is a commutative ring.

For the final theorem of this section, we need the following result:

Lemma 4.6 — (i) Let R be a 3-prime near-ring with a non-zero semigroup ideal U and a non-zero generalized derivation f associated with a non-zero derivation d. If $af(U) = \{0\}$ for some $a \in R$, then $a = 0$.

(ii) Let R be a 3-prime near-ring with a non-zero semigroup ideal U and a non-zero generalized derivation f associated with the zero derivation. If $f(U)a = \{0\}$ for some $a \in R$, then $a = 0$. Moreover, If $af(U) = \{0\}$, then $af(R) = \{0\}$.

PROOF : (i) For all $u, v \in U$, we have $0 = af(uv) = af(u)v + aud(v) = aud(v)$. Using Lemma 2.4(ii) and Lemma 2.8, we get $a = 0$.

(ii) For all $u \in U, x \in R$, we have $0 = f(xu)a = f(x)ua$. Using Lemma 2.4(ii) and $f \neq 0$, we get $a = 0$. Now, suppose $af(U) = \{0\}$. It follows that $0 = af(xu) = af(x)u$ for all $x \in R, u \in U$. Using Lemma 2.4(i), we have $af(R) = \{0\}$.

***Theorem* 4.7** — *Let R be a prime ring with a non-zero semigroup ideal U and a non-zero generalized derivation f. Then for any positive integer $n \geqslant 2$, the following statements are equivalent*

(i) $f(nR) = \{0\}$.

(ii) $nR = \{0\}$.

(iii) $nU = \{0\}$.

(iv) $f(nU) = \{0\}$.

PROOF : (i) $\Rightarrow$ (ii). For all $x, y \in R$ we have $0 = f(nxy) = f((nx)y) = f(nx)y + (nx)d(y) = (nx)d(y)$. Using Lemma 2.4(ii), either $nR = \{0\}$ or $d = 0$. Suppose $d = 0$. Then for all $x, y \in R$ we get $0 = f(nxy) = f(x(ny)) = f(x)(ny)$. Using Lemma 4.6(ii), we conclude that $nR = \{0\}$.

It is clear that (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). For all $u \in U, x, y \in R$ we have $0 = f(nxuy) = f(nxu)y + (nxu)d(y) = (nx)ud(x)$. Using Lemma 2.4(ii), either $nR = \{0\}$ or $d = 0$. Suppose $d = 0$. For all $u \in U, x \in R$ we get $0 = f(nxu) = f(nx)u$. Using Lemma 2.4(i), we deduce that $f(nR) = \{0\}$.

5. ON THE CONDITION $f(xy) = -f(yx)$

In this section we will study the commutativity of rings and near-rings, each admitting a generalized derivation f satisfies the conditions $f(xy) = -f(yx)$ and $f(y)x = -xf(y)$. Firstly, we start with the following result.

Lemma 5.1 — Let R be a 3-prime zero-symmetric near-ring and a be a non-zero element of R. If $ax = -xa$ for all $x \in R$, then a is not a zero divisor.

PROOF : Let R be a near-ring. Suppose a is a right zero divisor. Then there exists $0 \neq b \in R$ such that $ba = 0$. Thus, $0 = bax = b(-xa) = bx(-a)$ for all $x \in R$. Since R is 3-prime, either $a = 0$ or $b = 0$, a contradiction. Now, suppose a is a left zero divisor. Then there exists $0 \neq b \in R$ such that $ab = 0$. Thus, $0 = xab = (-ax)b = a(-x)b$ for all $x \in R$. So either $a = 0$ or $b = 0$, a contradiction.

Lemma 5.2 — Let R be a 3-prime near-ring with a derivation d. If for some $a \in R$, $ax = -xa$ for all $x \in R$, then $d(a)x = -xd(a)$.

PROOF : Observe that $d(ax) = ad(x) + d(a)x$ and $d(xa) = xd(a) + d(x)a$. Since $d(ax) = -d(xa)$ and $ad(x) = -d(x)a$ we have $d(a)x = -xd(a)$ for all $x \in R$.

Lemma 5.3 — Let R be a zero-symmetric near-ring such that $(x+y)a = ya + xa$ for some $a \in R$ and for all $x, y \in A$, where A is a subgroup of R. Then $(-x)a = -xa$ for all $x \in A$.

PROOF : Let R be a near-ring. Since $0 = 0a = (y + (-y))a = ya + (-y)a$ for all $y \in A$, we have $(-y)a = -ya$.

***Theorem* 5.4** — *Let R be a 3-prime near-ring with a non-zero generalized derivation f such that $f(u)x = -xf(u)$ for all $x \in R, u \in U$, where U is a non-zero semigroup ideal of R. Then R is a commutative ring of characteristic 2.*

PROOF : Firstly, R is zero-symmetric by Proposition 2.2. If R is a commutative near-ring, then it is of characteristic 2. Indeed, $f(u)x = -xf(u) = -f(u)x$ for all $x \in R, u \in U$. Thus, $f(U)(2R) = \{0\}$. From $f \neq 0$ and Lemma 2.12, there exists $a \in U$ such that $0 \neq f(a)$ which is not a zero divisor by Lemma 5.1. Hence, $2R = \{0\}$. Therefore, It is enough to prove that R is commutative. Suppose $d^2 = 0$. If $d = 0$, then $f(xy) = f(x)y$ for all $x, y \in R$. So $f(ay) = f(a)y$ and $f(ay)x = -xf(ay) = -xf(a)y = -(-f(a)x)y = -f(a)(-x)y = f(a)(-x)(-y)$ for all $x, y \in R$. It follows that $f(a)(yx - (-x)(-y)) = 0$ and then $yx = (-x)(-y)$ for all $x, y \in R$ by Lemma 5.1 since $f(a)$ is not a zero divisor. Using Lemma 2.17(iii), R is a commutative ring. If $d \neq 0$, then $f(ud(x)) = f(u)d(x) + ud(d(x)) = f(u)d(x)$ for all $u \in U, x \in R$. Thus, for all $x, y \in R$

$$\begin{aligned} f(a)d(x)y &= f(ad(x))y = -yf(ad(x)) = -yf(a)d(x) \\ &= yf(a)(-d(x)) = (-f(a)y)(-d(x)) = f(a)(-y)(-d(x)). \end{aligned}$$

It follows that $f(a)(d(x)y - (-y)(-d(x))) = 0$ and then $d(x)y = (-y)(-d(x))$ for all $x, y \in R$ by Lemma 5.1. Using Lemma 2.17(ii), $d(R) \subseteq Z(R)$ and R is a commutative ring by Lemma 2.7.

Now, assume that $d^2 \neq 0$. From $f(uf(v))x = -xf(uf(v))$ for all $u, v \in U, x \in R$ and using Lemma 2.5(i) and (ii), we have $f(u)f(v)x + udf(v)x = -(xudf(v) + xf(u)f(v)) = -xf(u)f(v) - xudf(v)$. But for all $u, v \in U, x \in R$

$$\begin{aligned} f(u)f(v)x &= f(u)(-xf(v)) = -f(u)xf(v) = -(-xf(v)f(u)) \\ &= xf(v)f(u) = x(-f(u)f(v)) = -xf(u)f(v). \end{aligned}$$

So $udf(v)x = -xudf(v)$ for all $u, v \in U, x \in R$. Lemma 5.2 implies that $df(u)x = -xdf(u)$ for all $u \in U, x \in R$. It follows that $udf(v)x = u(-xdf(v)) = -uxdf(v) = df(v)ux$ and $-xudf(v) = df(v)xu$. Thus, $df(v)ux = df(v)xu$ for all $u, v \in U, x \in R$. Replacing u by uy where $y \in R$, we get $df(v)uyx = df(v)xuy = df(v)uxy$ and then $df(v)u(yx - xy) = 0$ for all $u, v \in U, x, y \in R$. Using Lemma 2.4(ii), either $df(U) = \{0\}$ or $yx - xy$ for all $x, y \in R$. If $df(U) \neq \{0\}$, then R is commutative near-ring and hence a commutative ring by Lemma 2.16.

Now, suppose $d(f(U)) = \{0\}$. That implies $f(d(U)) = \{0\}$ by Proposition 3.1. By the same way of the proof of Theorem 3.2, we deduce that $d(r)d(s)x = -xd(r)d(s)$ for all $r, s \in U, x \in R$. If $d(r)d(s) = 0$ for some $r, s \in U$, then $(d(s)d(r))^2 = d(s)d(r)d(s)d(r) = d(s)0d(r) = 0$. Now, either $d(s)d(r) = 0$ or not. But $(d(s)d(r))^2 = 0$ implies that $d(s)d(r) = 0$ by Lemma 5.1. If $d(r)d(s) \neq 0$, then $d(s)d(r) \neq 0$ and both of them are not zero divisors in R. Thus, $(d(r)d(s))d(r) = -d(r)(d(r)d(s)) = d(r)(-d(r)d(s))$ implies $d(r)(d(s)d(r) + d(r)d(s)) = 0$. It follows that $d(s)d(r)(d(r)d(s) + d(s)d(r)) = 0$ which gives us $d(r)d(s) = -d(s)d(r)$ for all $r, s \in U$. Since $d^2 \neq 0$, we have $d^2(U^2) \neq \{0\}$ by Lemma 2.4(iii). Therefore, R is a commutative ring by Lemma 2.18.

Proposition 5.5 — Let R be a 3-prime near-ring with a generalized derivation f associated with the zero derivation. If $f(uv) = -f(vu)$ for all $u \in U, v \in V$, where U is a non-zero semigroup ideal of R and V is a subset of R, then either $f(V) = \{0\}$ or $-V \subseteq Z(R)$.

PROOF : Replace u by vu in $f(uv) = f(vu)$ to get $f(vuv) = -f(vvu)$ for all $u \in U, v \in V$. It follows that $f(v)uv = -f(v)vu$. Replacing u by uw where $w \in U$, we have $f(v)uwv = -f(v)vuw = -(-f(v)uv)w = (-f(v)uv)(-w) = f(v)u(-v)(-w)$ and hence $f(v)u(wv - (-v)(-w)) = 0$ for all $u, w \in U, v \in V$. Using Lemma 2.4(ii), we get for each $v \in V$ either $f(v) = 0$ or $wv = (-v)(-w)$ for all $w \in U$. If $f(V) \neq \{0\}$, then there exists $a \in V$ such that $f(a) \neq 0$. Thus, $wa = (-a)(-w)$ for all $w \in U$ and hence $-a \in Z(R)$ by Lemma 2.17(i). So for all $u \in U, v \in V$ we have $f(((-a)u)v) = -f(v((-a)u)) = -f((v(-a))u) = -f((-a)vu)$ which means $f(-a)uv = -f(-a)vu$. Replacing u by uw and by the same way above, we get $f(-a)u(wv-(-v)(-w))=0$ for all $u,w\in U, v\in V$. Using Lemma 2.4(ii) and $f(a)\neq 0$, we have $wv=(-v)(-w)$ for all $v\in V, w\in U$ and hence $-V\subseteq Z(R)$ by Lemma 2.17(i).

Corollary 5.6 — Let R be a 3-prime near-ring with a non-zero generalized derivation f associated with the zero derivation. If $f(uv)=-f(vu)$ for all $u\in U, v\in V$, where U and V are non-zero semigroup ideals of R, then R is a commutative ring of characteristic 2.

PROOF : From Proposition 5.5, either $f(V)=\{0\}$ or $-V\subseteq Z(R)$. From Lemma 2.12, we deduce that $-V\subseteq Z(R)$. Since $-V$ is a non-zero semigroup left ideal of R, we have that R is a commutative ring by Lemma 2.11. Now, $f(V)\neq\{0\}$ implies $f(a)\neq 0$ for some $a\in V$ and then $f(a)\in Z(R)-\{0\}$. For all $v\in V, u\in U$ we have $f(a(vu))=f((av)u)=-f(u(av))=-f((ua)v)=-f((au)v)=-f(a(uv))$. Therefore, $f(a(vu+uv))=0$ and then $0=f(a)(vu+uv)$. It follows that $uv=-vu$ for all $v\in V, u\in U$ by Lemma 2.3(iii). Commutativity of R implies $uv=-vu=-uv=u(-v)$ and hence $0=u(v+v)$. Thus, $v+v=0$ for all $v\in V$ by Lemma 2.4(i) which means $2V=\{0\}$. Using Lemma 2.9, R is of characteristic 2.

***Theorem* 5.7** — *Let R be a prime ring with a non-zero generalized derivation f associated with a derivation d. If $f(uv)=-f(vu)$ for all $u\in U, v\in V$, where U is a subset of R and V is a non-zero semigroup ideal of R, then either $d(U)=\{0\}$ or $U\subseteq Z(R)$.*

PROOF : Replace v by vu in $f(uv)=-f(vu)$ to get $f(uvu)=-f(vuu)$ for all $u\in U, v\in V$. It follows that $0=f((uv+vu)u)=f(uv+vu)u+(uv+vu)d(u)=(uv+vu)d(u)$. That means $uvd(u)=-vud(u)$ for all $u\in U, v\in V$. Replacing v by vw where $w\in V$, we have $uvwd(u)=-vwud(u)=-v(-uwd(u))=vuwd(u)$ and hence $(uv-vu)wd(u)=0$. Lemma 2.4(ii) implies that for all $u\in U$ either $d(u)=0$ or $uv=vu$ for all $v\in V$. If $d(U)\neq\{0\}$, then there exists $a\in U$ such that $d(a)\neq 0$. Thus, $av=va$ for all $v\in V$ and hence $a\in Z(R)$ by Lemma 2.17(i). So $f(u(va)))=-f((va))u)=-f(vua)$ which means $0=f((uv+vu)a)=(uv+vu)d(a)$. Replacing v by vw where

$w \in V$, we have by the same way above $uvwd(a) = vuwd(a)$ and hence $(uv - vu)wd(a) = 0$. Using Lemma 2.3(ii) and (iii), $d(a) \in Z(R) - \{0\}$ is not a zero divisor. Thus, $uv = vu$ for all $u \in U, v \in V$. Lemma 2.17(i) implies that $U \subseteq Z(R)$.

Corollary 5.8 — Let R be a prime ring with a non-zero generalized derivation f associated with a derivation d. If $f(uv) = -f(vu)$ for all $u \in U, v \in V$, where U and V are non-zero semigroup ideals of R, then R is a commutative ring of characteristic 2.

PROOF : If $d = 0$ then R is a commutative ring of characteristic 2 by Corollary 5.6. So suppose $d \neq 0$. From Theorem 5.7 and Lemma 2.8 we have $V \subseteq Z(R)$. So R is a commutative ring by Lemma 2.11. Let $a \in V$ such that $d(a) \in Z(R) - \{0\}$. So $d(a)$ is not a zero divisor by Lemma 2.3(iii). For all $v \in V, u \in U$ we have $f((uv)a) = f(u(va)) = -f((va)u) = -f(v(au)) = -f(v(ua)) = -f((vu)a)$. Therefore, $f((uv + vu)a) = 0$ and then $0 = (uv + vu)d(a)$. It follows that $uv = -vu$ for all $v \in V, u \in U$. Commutativity of R implies $vu = -uv = -vu$ and hence $0 = v(u + u)$. Thus, $u + u = 0$ for all $u \in U$ by Lemma 2.4(i) which means $2U = \{0\}$. Using Lemma 2.9, we have $2R = \{0\}$. Hence, R is a commutative ring of characteristic 2.

***Theorem* 5.9** — *Let R be an integral near-ring with a non-zero generalized derivation f. If $f(uv) = -f(vu)$ for all $u, v \in U$, where U is a non-zero two-sided R-subgroup of R, then R is a commutative ring of characteristic 2.*

PROOF : Let d be the associated derivation with f. We divide the proof into two cases:

(i) $d = 0$. It is clear using Lemma 2.14 and the proof of Corollary 5.6.

(ii) $d \neq 0$. Replace v by uv in $f(uv) = -f(vu)$ to get $f(uuv) = -f(uvu)$ for all $u, v \in U$. It follows that $0 = f(u(uv + vu)) = f(u)(uv + vu) + ud(uv + vu)$.

Replacing u by $u \circ w$ where $w \in U$, we have

$$\begin{aligned} 0 &= f(u \circ w)((u \circ w)v + v(u \circ w)) + (u \circ w)d((u \circ w)v + v(u \circ w)) \\ &= (u \circ w)d((u \circ w)v + v(u \circ w)) \end{aligned}$$

for all $u, v, w \in U$ as $f(u \circ w) = 0$. That means $cd(cv + vc) = 0$ for all $c \in B, v \in U$, where $B = \{u \circ w : u, w \in U\}$. Since R is without zero divisors, either $c = 0$ or $d(cv + vc) = 0$. But $c = 0$ implies $d(cv + vc) = 0$. So in both cases

$$d(cv + vc) = 0 \text{ for all } c \in B, v \in U. \tag{5.1}$$

Putting cv instead of v and using (5.1), we have $0 = d(ccv + cvc) = d(c(cv + vc)) = cd(cv + vc) + d(c)(cv + vc) = d(c)(cv + vc)$. For all $c \in B$ either $d(c) = 0$ or $cv = vc$ for all $v \in U$. If $d(c) = 0$ for all $c \in B$, then by Lemma 2.19 R is a commutative ring of characteristic 2. If $d(c_o) \neq 0$ for some $c_o \in B$, then $c_o v = -vc_o$ for all $v \in U$. Now, $(u+v)c_o = c_o(-v-u) = -c_o v - c_o u = vc_o + uc_o$ for all $u, v \in U$ and then $(-v)c_o = -vc_o$ for all $v \in U$ by Lemma 5.3. It follows that

$$\begin{aligned} f(c_o(xy)) &= -f((xy)c_o) = -f(x(yc_o)) = -f(x(-c_o y)) \\ &= f(x(c_o y)) = -f((c_o y)x) = -f(c_o(yx)) \end{aligned}$$

for all $x, y \in U$. So $0 = f(c_o(xy + yx)) = f(c_o)(xy + yx) + c_o d(xy + yx) = c_o d(xy + yx)$. As $c_o \neq 0$, we have $d(x \circ y) = 0$ for all $x, y \in U$. Using Lemma 2.19, R is a commutative ring of characteristic 2.

References

1. K. I. Beidar, Y. Fong and X. K. Wang, Posner and Herstein theorems for derivations of 3-prime near-rings, *Comm. Algebra*, **24**(5) (1996), 1581-1589.

2. H. E. Bell, *On derivations in near-rings II*, Proceeding of Near-rings and Near-Fields and K-loops, Hamburg, 1995, Kluwer Acad. Publ., 1997, pp. 191-197.

3. H. E. Bell**,** On prime near-rings with generalized derivation, *Int. J. Math. Math. Sci.,* **2008**, Art. ID 490316, 5 pp.

4. H. E. Bell and G. Mason, *On derivations in near-rings*, Proceeding of Near-rings and Near-Fields, Tübingen, 1985, North-Holland Mathematical Studies, 1987, pp. 31-35.

5. A. Boua and L. Oukhtite, Derivations on prime near-rings, *Int. J. Open Probl. Comput. Sci. Math.,* **4**(2) (2011), 162-167.

6. A. Boua and L. Oukhtite, Generalized derivations and commutativity of prime near-rings, *J. Adv. Res. Pure Math.,* **3**(2) (2011), 120-124.

7. O. Golbasi**,** Notes on prime near-rings with generalized derivation, *Southeast Asian Bull. Math.,* **30**(1) (2006), 49-54.

8. O. Golbasi**,** On generalized derivations of prime near-rings, *Hacet. J. Math. Stat.,* **35**(2) (2006), 173-180.

9. O. Golbasi**,** On prime and semiprime near-rings with generalized derivation, *Quaest. Math.,* **33**(3) (2010), 387-390.

10. A. A. M. Kamal and K. H. Al-shaalan**,** Commutativity of near-rings with derivations, to appear in *Algebra Colloq*.

11. A. A. M. Kamal and K. H. Al-shaalan**,** Commutativity of near-rings with derivations by using algebraic substructures, *Indian J. Pure Appl. Math.*, **43**(3) (2012), 211-225.

12. A. A. M. Kamal and K. H. Al-shaalan**,** Existence of derivations on near-rings, *Math. Slovaca*, **63**(3) (2013), 431-448.

13. X. K. Wang, Derivations in prime near-rings, *Proc. Amer. Math. Soc.*, **121** (1994) 361-366.