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COMPOSITIONS, DERIVATIONS AND POLYNOMIALS

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Let R be a prime ring with extended centroid C. In this paper, we discuss the case when the composition of a generalized derivation δ and a polynomial map $f(Y) \in C[Y]$ of R is commutative on a non-zero right ideal ρ and a non-commutative Lie ideal L of R respectively, i.e., when the identity $\delta \circ f(x) = f \circ \delta(x)$ holds on ρ or L. As applications of our main theorems, we clarify the generalized derivations which act as n-Jordan homomorphisms (S_n-homomorphisms) on ρ or L.

Key words : Prime ring; generalized derivation; polynomial; S_n-homomorphism; n-Jordan homomorphism.

Throughout this paper, unless stated otherwise, R always denotes a prime ring with extended centroid C and maximal right ring of quotients U (see [4] for details).

Recall that an additive map δ on R is said to act as a homomorphism or an anti-homomorphism on T if either $\delta(ab) = \delta(a)\delta(b)$ or $\delta(ab) = \delta(b)\delta(a)$ for each pair $a, b \in T$, where T is any subset of R. In [5] Bell and Kappe proved that a derivation $\mu : R \to R$ acting as a homomorphism or an anti-homomorphism on a nonzero right ideal ρ of R must be zero. In [3] Asma, Rehman and Shakir proved a similar result for a nonzero Lie ideal L of R in case that char$R \neq 2$ and $u^2 \in L$ for all $u \in L$. Recently Wang and You [16] got rid of the assumption that $u^2 \in L$ for all $u \in L$. De Filippis characterized the generalized derivation acting as a 2-Jordan homomorphism on a noncentral Lie ideal or a more general subset in a prime ring. Generalized derivations and generalized (θ, ϕ)-derivations as homomorphisms or anti-homomorphisms in prime rings, as well as derivations as homomorphisms or anti-homomorphisms in σ-prime rings, have also been discussed in [1, 2, 14, 15].

An additive map δ on R is said to be an S_n-homomorphism on T if for each $(a_1, \ldots, a_n) \in T^n$, there exists a $\sigma \in S_n$ such that

$$\delta(a_1 a_2 \cdots a_n) = \delta(a_{\sigma(1)})\delta(a_{\sigma(2)}) \cdots \delta(a_{\sigma(n)}),$$

where T is any subset of R. An additive map acting as a homomorphism or an anti-homomorphism is exactly the additive map acting as an S_2-homomorphism. Moreover, if an additive map δ on R acts as an S_n-homomorphism on $T \subseteq R$, then it acts as an n-Jordan homomorphism on T, i.e. δ satisfies $\delta(x^n) = \delta(x)^n$ for all $x \in T$. However the additive map δ acting as an n-Jordan homomorphism on T is exactly the additive map such that $\delta \circ f(x) = f \circ \delta(x)$ for all $x \in T$, where f is the map on R induced by the special polynomial $f(Y) = Y^n$, i.e. f can commutatively compose with the polynomial map $f(Y) = Y^n$.

For a general polynomial $f(Y) \in C[Y]$, if an additive map δ on R such that $\delta \circ f(x) = f \circ \delta(x)$ holds on a subset of R, should δ be of some special form? In this paper, we will give an equivalent condition for a generalized derivation to commutatively compose with $f(Y) \in C[Y]$ on a nonzero right ideal, or on a noncommutative Lie ideal, of a prime ring. As applications of our main theorems, we will describe the generalized derivation acting as an n-Jordan homomorphism (S_n-homomorphism) on a nonzero right ideal and noncommutative Lie ideal of a prime ring, respectively.

§1. Preparation

An additive map μ of a ring R is called a derivation of R if $\mu(xy) = \mu(x)y + x\mu(y)$ for all $x, y \in R$. An additive map δ of a ring R is called a generalized derivation of R if there exists a derivation μ of R such that $\delta(xy) = \delta(x)y + x\mu(y)$ for all $x, y \in R$. By Lee [12, Theorem 4], a generalized derivation δ of a semiprime ring R is of the form $\delta(x) = ax + \mu(x)$ for some $a \in U$ and a derivation μ of R. Moreover a and μ are uniquely determined by δ, and μ is usually called the associated derivation of δ. A derivation μ of a semiprime ring R is called X-inner if there exists $a \in U$ such that $\mu(x) = [a, x]$, otherwise μ is called X-outer. Here we use the notation $[x, y] = xy - yx$ for any $x, y \in R$. A generalized derivation is called X-inner if its associated derivation μ is X-inner, otherwise δ is called X-outer.

For a vector space V over a field F, we denote by $End(_FV)$ and $End(V_F)$ the right endomorphism ring of V and the left endomorphism ring of V, respectively.

In this paper, we will translate the problems from a prime ring R to the endomorphism ring of some vector space V over a field F by the following Lemma 1 which mainly depends on Martindale [13, Theorem 3], Chuang [6, Theorem 1], the conclusion of Erickson *et al.* [7] and the properties of primitive rings with nonzero socles [4, Theorem 4.3.7]. The following Lemma 1 has been used in the proofs of many papers, for example see Lanski [10] or Lee and Lee [11]. For proof of this lemma, also see [18, Lemma 1] for reference.

Lemma 1 — Let R be a prime GPI ring with extension centroid C. Then there is an extension field F of C and a vector space V over F such that

1. $R \subseteq U \subseteq End(_FV)$,

2. The C-independent elements in U are also F-independent, and

3. R and $End(_FV)$ satisfy the same GPI (with coefficients in $End(_FV)$).

Note that in this lemma, and also in Lemma 2, we have functions in $End(_FV)$ and $Hom_F(V, W)$ act from the right.

The following lemma is an extension of [17, Lemma 3.1 (1)] and is essential in the proof of our main theorems.

Lemma 2 — Let V and W be vector spaces over a field F, $a, b \in Hom_F(V, W)$ be linear mappings from V to W. If for any $v \in V$ there is a $c_v \in F$ such that $vb = c_v va$, then there is $c \in F$ such that $b = ca$.

PROOF : If $a = 0$, then $b = 0$ and the conclusion holds. Suppose $a \neq 0$. For any $v_1, v_2 \in V$, let c, c_1, $c_2 \in F$ such that $(v_1 + v_2)b = c(v_1 + v_2)a$, $v_1 b = c_1 v_1 a$, and $v_2 b = c_2 v_2 a$. Then $(c - c_1)v_1 a + (c - c_2)v_2 a = 0$. In particular, if $v_1 a$ and $v_2 a$ are independent, then $c_1 = c = c_2$. Let $\{v_\alpha a\}_{\alpha \in A}$ be a basis of Va. Then A is not empty since $a \neq 0$. So there is a $c \in F$ such that $v_\alpha b = c v_\alpha a$ for all $\alpha \in A$. For $v \in V$, let $va = \sum_{i=1}^n d_i v_{\alpha_i} a$, where $d_i \in F$ and $\alpha_i \in A$. Then $(v - \sum_{i=1}^n d_i v_{\alpha_i})a = 0$ which implies $(v - \sum_{i=1}^n d_i v_{\alpha_i})b = 0$. So $vb = \sum_{i=1}^n d_i v_{\alpha_i} b = \sum_{i=1}^n d_i (c v_{\alpha_i} a) = c \sum_{i=1}^n d_i v_{\alpha_i} a = cva$. Hence $b = ca$.

For the case of linear polynomials, we have

Proposition 1 — Let R be a prime ring, H a subset of R, $\delta(x) = ax + \mu(x)$ a generalized derivation on R, and $f(Y) = c_1 Y + c_0 \in C[Y]$. If H is a nonzero right ideal or a non-commutative Lie ideal of R, then $\delta \circ f(x) = f \circ \delta(x)$ holds on H if and only if $\delta(c_0) = c_0$ and $\mu(c_1) = 0$.

PROOF : For any $x \in H$,

$$c_1 \delta(x) + c_0 = \delta(c_1 x + c_0) = \delta(x) c_1 + x\mu(c_1) + \delta(c_0).$$

Then $x\mu(c_1) + \delta(c_0) = c_0$ for any $x \in H$. Taking $x = 0$ gives $\delta(c_0) = c_0$ and $H\mu(c_1) = 0$.

If H is a nonzero right ideal, then $\mu(c_1) = 0$ since R is prime; if H is a non-commutative Lie ideal, then we conclude that $[H, R]\mu(c_1) = 0$. Hence $[H, R]R\mu(c_1) = 0$, so $\mu(c_1) = 0$ since H is a non-commutative Lie ideal of a prime ring.

The case when deg$f(Y) > 1$ is discussed in the following sections.

§2. Right Ideal Case

To prove Theorem 1 the following lemma is needed.

Lemma 3 — Let R be a prime ring with a nonzero right ideal ρ and $a \in U$. If as and s are C-dependent for each $s \in \rho$, then there is a unique $\lambda \in C$ such that $as = \lambda s$ for all $s \in \rho$.

Proof : Suppose for any $0 \neq s \in \rho$, there is a $\lambda_s \in C$ such that $as = \lambda_s s$. Then λ_s is uniquely determined by s since C is a field. For any nonzero $s, t \in \rho$ and $s \neq t$, if s and t are C-dependent, then $\lambda_s = \lambda_t$; if s and t are C-independent, then $\lambda_s s - \lambda_t t = a(s-t) = \lambda_{s-t}(s-t)$, i.e.

$$(\lambda_s - \lambda_{s-t})s + (\lambda_{s-t} - \lambda_t)t = 0$$

which implies $\lambda_s = \lambda_t$. So there is a unique $\lambda \in C$ such that $as = \lambda s$ for all $s \in \rho$.

***Theorem* 1** — *Let R be a prime ring with a nonzero right ideal ρ. Let δ be a generalized derivation on R, and $f(Y) = \sum_{i=0}^{n} c_i Y^i \in C[Y]$ such that $|C| \geq \deg f(Y) = n > 1$ and $f(\xi) \neq c_1\xi + c_0$ for some $\xi \in C$. Then $\delta \circ f(x) = f \circ \delta(x)$ for all $x \in \rho$ if and only if there are $A, B \in U$ and $\omega \in C$ such that $\delta(x) = Ax + xB$ for all $x \in R$, $A\rho = (B - \omega)\rho = 0$, $(A + B - 1)c_0 = 0$ and $c_i(w^{i-1} - 1)B = 0$ for each $2 \leq i \leq n$.*

Proof : Firstly, we check the sufficiency. For each $x \in \rho$, we have

$$\begin{aligned} \delta \circ f(x) &= c_0(A+B) + c_1(Ax + xB) + c_2(Ax^2 + x^2B) + \cdots + c_n(Ax^n + x^nB) \\ &= c_0(A+B) + c_1xB + c_2x^2B + \cdots + c_nx^nB. \end{aligned}$$

On the other hand, for each $x \in \rho$, we have

$$\begin{aligned} f \circ \delta(x) &= c_0 + c_1(Ax + xB) + c_2(Ax + xB)^2 + \cdots + c_n(Ax + xB)^n \\ &= c_0 + c_1xB + c_2(xB)^2 + \cdots + c_n(xB)^n \\ &= c_0 + c_1xB + c_2\omega xB^2 + \cdots + c_n\omega^{n-1}x^nB. \end{aligned}$$

Comparing the above two expressions and using the conditions in the theorem, we obtain $\delta \circ f(x) = f \circ \delta(x)$ for all $x \in \rho$.

Secondly, we prove the necessity. $\delta(f(0)) = f(\delta(0))$ shows $\delta(c_0) = c_0$. Let $g(Y) = f(Y) - c_0$. Then $\delta(g(x)) = g(\delta(x))$ for all $x \in \rho$. Suppose δ is X-outer. Let $\delta(x) = ax + \mu(x)$, where $a \in U$ and μ is an X-outer derivation on R (see [12]). Then for any $0 \neq s \in \rho$ and $x \in R$,

$$\delta(g(sx)) = \sum_{i=1}^{n} \big(ac_i(sx)^i + \mu(c_i)(sx)^i + c_i \sum_{j=0}^{i-1} (sx)^j(\mu(s)x + s\mu(x))(sx)^{i-1-j}\big),$$

and

$$g(\delta(sx)) = \sum_{i=1}^{n} c_i(asx + \mu(s)x + s\mu(x))^i.$$

From [9, Theorem 2], we get

$$\begin{aligned}&\sum_{i=1}^{n} \big(ac_i(sx)^i + \mu(c_i)(sx)^i + c_i \sum_{j=0}^{i-1} (sx)^j(\mu(s)x + sy)(sx)^{i-1-j}\big)\\ &= \sum_{i=1}^{n} c_i(asx + \mu(s)x + sy)^i \end{aligned} \tag{1}$$

for all $x, y \in R$. Setting $x = 0$ in (1) gives

$$\sum_{i=2}^{n} c_i(sy)^i = 0 \tag{2}$$

for all $y \in R$. Obviously, (2) is a nontrivial GPI on R, then Lemma 1 implies that there is an extension field F of C and a vector space V over F such that $End({}_FV)$ (acting on V from the right) satisfies (2). There must be some $v \in V$ such that $vs \neq 0$, because $s \neq 0$. Choose $y_0 \in End({}_FV)$ such that $vsy_0 = \xi v$. Then taking $y = y_0$ in (2) and acting on v lead to the following contradiction

$$0 \neq (f(\xi) - c_1\xi - c_0)v = v \sum_{i=2}^{n} c_i(sy)^i = 0$$

which means that δ must be X-inner. Let $\delta(x) = ax + xb$ on R, where $a, b \in U$. Then

$$\sum_{i=2}^{n} c_i\big((asx + sxb)^i - (a(sx)^i + (sx)^i b)\big) = 0 \tag{3}$$

for all $0 \neq s \in \rho$ and $x \in R$. Suppose there is a $s \in \rho$ such that as and s are C-independent. Then (3) is a nontrivial GPI on R and Lemma 1 implies that there is an extension field F of C and a vector space V over F such that $End({}_FV)$ satisfies (3) and as and s are F-independent.

Case 1 : There is a $v \in V$ such that $vs = 0$ and $vas \neq 0$. Choose $x_0 \in End({}_FV)$ such that $vasx_0 = \xi v$. Taking $x = x_0$ in (3) and acting on v lead to a contradiction

$$0 \neq (f(\xi) - c_1\xi - c_0)v = v\sum_{i=2}^{n} c_i\Big((asx_0 + sx_0b)^i - (a(sx_0)^i + (sx_0)^i b)\Big) = 0.$$

Case 2 : For any $v \in V$, if $vs = 0$, then $vas = 0$. Lemma 2 implies that there is a $v \in V$ such that vas and vs are C-independent. Choose $x_0 \in End({}_FV)$ such that $vsx_0 = 0$ and $vasx_0 = \xi v$. Taking $x = x_0$ in (3) and acting on v lead to a contradiction

$$0 \neq (f(\xi) - c_1\xi - c_0)v = v\Big(\sum_{i=2}^{n} c_i((asx_0 + sx_0b)^i - (a(sx_0)^i + (sx_0)^i b))\Big) = 0.$$

Then we arrive at the conclusion that as and s are C-dependent for any $s \in \rho$. Using Lemma 3, there is a unique $\lambda \in C$ such that $as = \lambda s$ for all $s \in \rho$. Then (3) can be written as

$$\sum_{i=2}^{n} c_i\big((sx(\lambda + b))^i - (sx)^i(\lambda + b)\big) = 0, \quad x \in R. \tag{4}$$

If there is a $s \in \rho$ such that $(\lambda + b)s$ and s are C-independent, then (4) is a nontrivial GPI on R and Lemma 1 implies that there is an extension field F of C and a vector space V over F such that $End(V_F)$ (acting on V from the left) satisfies (4) , and $(\lambda + b)s$ and s are F-independent. Multiplying (4) by $\lambda + b$ and s from different sides yields

$$\sum_{i=2}^{n} c_i\Big(((\lambda + b)sx)^i(\lambda + b)s - (\lambda + b)sx(sx)^{i-1}(\lambda + b)s\Big) = 0. \tag{5}$$

Applying the similar discussion of Case 2 to (5) will also imply a contradiction. This means that $(\lambda + b)s$ and s are C-dependent for any $s \in \rho$. Then Lemma 3 yields that there is a unique $\omega \in C$ such that $(\lambda + b)s = \omega s$ for all $s \in \rho$. Setting $A = a - \lambda$ and $B = b + \lambda$, we have that $\delta(x) = Ax + xB$ for all $x \in R$. In fact, we have proved that $A\rho = (B - \omega)\rho = 0$. Checking $\delta \circ f(0) = f \circ \delta(0)$, we have that $(A + B - 1)c_0 = 0$. Putting $\delta(x) = Ax + xB$ with $x \in \rho$ in $\delta \circ f(x) = f \circ \delta(x)$, we have that

$$\sum_{i=2}^{n} c_i(w^{i-1} - 1)(sx)^i B = 0. \tag{6}$$

If $B = 0$ or $c_i(w^{i-1} - 1) = 0$ for every $2 \leq i \leq n$, then we are done. Assuming that $B \neq 0$ and there is some $2 \leq i \leq n$ such that $c_i(w^{i-1} - 1) \neq 0$, we proceed to obtain a contradiction. Then (6) is a nontrivial GPI on R and Lemma 1 implies that there is an extension field F of C and a vector space V over F such that $End(_F V)$ satisfies (6). There exists $\xi \in C$ such that $h(\xi) \neq 0$, because $|C| \geq n$, where $h(Y) = \sum_{i=2}^{n} c_i(w^{i-1} - 1)Y^i \in C[Y]$ is a nontrivial polynomial.

If there is $v \in V$ such that $vs \neq 0$ and $vB \neq 0$, Choose $x_0 \in End(_F V)$ such that $vsx_0 = \xi v$. Taking $x = x_0$ in (6) and acting on v lead to a contradiction

$$0 \neq h(\xi)vB = v\sum_{i=2}^{n} c_i(w^{i-1} - 1)(sx_0)^i B = 0.$$

Hence, for every $v \in V$, if $vs \neq 0$ then $vB = 0$. In fact there exists $v_0 \in V$ such that $v_0 s \neq 0$, since $\rho \neq 0$. On the other hand, if there is a $u \in V$ such that $us = 0$ and $uB \neq 0$, then $(u + v_0)s \neq 0$ but $(u + v_0)B \neq 0$ gives a contradiction. This means that for every $u \in V$, if $us = 0$ then $uB = 0$. Then we get the conclusion that $vB = 0$ for all $v \in V$, i.e. $B = 0$, a contradiction.

If $|C| < n$, Theorem 1 will not be correct. For example

Example 1 : Let $R = \mathrm{M}_2(\mathbb{Z}_2)$, $\delta(x) = 2x = xB = \omega x$, $\rho = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \in R \,\middle|\, u, v \in \mathbb{Z}_2 \right\}$, and $f(x) = x^6 + x^4 + x^3 + x^2$. Then $\delta \circ f(x) = f \circ \delta(x)$ for all $x \in L$, but $c_k(\omega^{k-1} - 1)B \neq 0$ for $k \neq 3$.

If the generalized derivation in Theorem 1 is actually a derivation, then we will get the equivalent condition without any restriction on $|C|$.

Corollary 1 — Let R be a prime ring with a nonzero right ideal ρ and a derivation μ on R, and let $f(Y) = \sum_{i=0}^{n} c_i Y^i \in C[Y]$ such that $\deg f(Y) = n > 1$ and $f(\xi) \neq c_1\xi + c_0$ for some $\xi \in C$. Then $\mu \circ f(x) = f \circ \mu(x)$ for all $x \in \rho$ if and only if $c_0 = 0$ and $\mu = 0$.

PROOF : We only deal with the necessity. Recalling the proof of Theorem 1 before $|C| \geq n$ is applied, we have showed that $\delta(x) = Ax + xB$ for all $x \in R$, $A\rho = (B - \omega)\rho = 0$, $(A + B - 1)c_0 = 0$. But $B = -A$, since δ is a derivation. Then $w = 0$, since $\rho \neq 0$. Then (6) will be

$$\sum_{i=2}^{n} c_i(0^{i-1} - 1)(sx)^i B = 0. \tag{7}$$

Obviously $c_n(0^{n-1} - 1) = -c_n \neq 0$. Assuming that $B \neq 0$, we proceed to get a contradiction. Since $f(\xi) \neq c_1\xi + c_0$ for some $\xi \in C$, $h(\xi) = -(f(\xi) - c_1\xi - c_0) \neq 0$ in the proof of Theorem 1. Then the proof of Theorem 1 (after $|C| \geq n$ is applied) works.

In particular, if $f(Y) = Y^n (n > 1)$, then $h(Y) = (\omega^{n-1} - 1)Y^n$ in the proof of Theorem 1. If $\omega^{n-1} - 1 \neq 0$, then $h(1) \neq 0$. So the proof of Theorem 1 also works. Hence we can get the following corollaries on generalized derivations acting as n-Jordan homomorphisms (S_n-homomorphisms).

Corollary 2 — Let R be a prime ring, ρ a nonzero right ideal of R, δ a generalized derivation on R, and $n > 1$ an integer. Then δ is an n-Jordan homomorphism (S_n-homomorphism) on ρ if and only if there are $A, B \in U$ and $\omega \in C$ such that $\delta(x) = Ax + xB$ on R and $A\rho = (B - \omega)\rho = (w^{n-1} - 1)B = 0$.

By Corollary 2, we have

Corollary 3 — Let R be a prime ring. For $n > 1$, a generalized derivation which acts as an n-Jordan homomorphism (S_n-homomorphism) on R must be a scalar multiplication by $\lambda \in C$ such that $\lambda^n = \lambda$.

If δ is actually a derivation, then we have

Corollary 4 — Let R be a prime ring. For $n > 1$, a derivation which acts as an n-Jordan homomorphism (S_n-homomorphism) on a nonzero right ideal must be zero.

§3. LIE IDEAL CASE

***Theorem* 2** — *Let R be a prime ring with a non-commutative Lie ideal L, δ a generalized derivation on R, and $f(Y) = \sum_{i=0}^{n} c_i Y^i \in C[Y]$ such that $\deg f(Y) = n > 1$ and $f(\xi) \neq c_1\xi + c_0$ for some $\xi \in C$. If $\delta \circ f(x) = f \circ \delta(x)$ for all $x \in L$, then there is a $\lambda \in C$ such that $c_0(\lambda - 1) = 0$ and $\delta(x) = \lambda x$ for all $x \in R$.*

PROOF : Let I be a nonzero ideal of R such that $[I, I] \subseteq L$ (see the proof of Lemma 1.3 in [9]). Let $g(Y) = f(Y) - c_0$. Suppose $\delta(x) = ax + \mu(x)$, where $a \in U$ and μ is an X-outer derivation on R. Then for any $x, y \in I$,

$$\delta(g([x,y])) = \sum_{i=1}^{n} \Big(ac_i[x,y]^i + \mu(c_i)[x,y]^i + c_i \sum_{j=0}^{i-1} [x,y]^j([\mu(x),y] + [x,\mu(y)])[x,y]^{i-1-j}\Big),$$

and

$$g(\delta([x,y])) = \sum_{i=1}^{n} c_i\big(a[x,y] + [\mu(x),y] + [x,\mu(y)]\big)^i.$$

By [9, Theorem 2] and [6, Theorem 1],

$$\sum_{i=1}^{n} \Big(ac_i[x,y]^i + \mu(c_i)[x,y]^i + c_i \sum_{j=0}^{i-1} [x,y]^j([z,y] + [x,w])[x,y]^{i-1-j}\Big) = \sum_{i=1}^{n} c_i(a[x,y] + [z,y] + [x,w])^i$$

for all $x, y, z, w \in R$. Taking $x = 0$ yields

$$c_n[z,y]^n + \cdots + c_2[z,y]^2 = 0 \tag{8}$$

for all $z, y \in R$. Obviously, (8) is a nontrivial GPI on R. Lemma 1 implies that there is an extension field F of C and a vector space V over F such that $End({}_F V)$

satisfies (8) and $\dim_F V > 1$ since R is non-commutative. Suppose $u, v \in V$ are C-independent. Choose $y_0, z_0 \in End(_F V)$ such that $vy_0 = 0$, $vz_0 = u$ and $uy_0 = \xi v$. Taking $z = z_0, y = y_0$ in (8) and then acting on v yield a contradiction

$$0 \neq (f(\xi) - c_1\xi - c_0)v = (c_n\xi^n + \cdots + c_2\xi^2)v = v(c_n[z_0, y_0]^n + \cdots + c_2[z_0, y_0]^2) = 0.$$

So δ must be X-inner on R. Let $\delta(x) = ax + xb$ on R, where $a, b \in U$. Then

$$\sum_{i=2}^{n} c_i\big((a[x,y] + [x,y]b)^i - (a[x,y]^i + [x,y]^i b)\big) = 0 \tag{9}$$

for all $x, y \in R$ (see [6, Theorem 1]). Taking $y = xy$ in (9) yields

$$\sum_{i=2}^{n} c_i\big((ax[x,y] + x[x,y]b)^i - (a(x[x,y])^i + (x[x,y])^i b)\big) = 0 \tag{10}$$

for all $x, y \in R$. Suppose $a \notin C$. Then (10) is a nontrivial GPI on R. By Lemma 1, there is an extension field F of C and a vector space V over F such that $End(_F V)$ satisfies (10) and $a \notin F$. Then Lemma 2 implies that there is a $v \in V$ such that va and v are independent. Choose $x_0, y_0 \in End(_F V)$ such that $vx_0 = 0$, $vax_0 = v$ and $vy_0 = -\xi va$. Taking $x = x_0$ in (8) and then acting on v yields a contradiction

$$\begin{aligned} 0 \neq& (f(\xi) - c_1\xi - c_0)v \\ =& v\sum_{i=2}^{n} c_i\big((ax_0[x_0,y_0] + x_0[x_0,y_0]b)^i - (a(x_0[x_0,y_0])^i + (x_0[x_0,y_0])^i b)\big) = 0. \end{aligned}$$

So $a \in C$. We will get $b \in C$ in a similar way. Setting $a + b = \lambda$ gives the conclusion.

Remark 1 : If the condition $|C| \geq n$ is added in Theorem 2, we can obtain an equivalent condition (see Corollary 5) similar to Theorem 1. But in this section, we will state the results in another style.

If $|C| = \infty$, then we get an equivalent condition.

Corollary 5 — Let R be a prime ring with infinite extended centroid C and a non-commutative Lie ideal L, let δ be a generalized derivation on R, and $f(Y) =$

$\sum_{i=0}^{n} c_i Y^i \in C[Y]$ such that $\deg f(Y) = n > 1$. Then $\delta \circ f(x) = f \circ \delta(x)$ for any $x \in L$ if and only if there is a $\lambda \in C$ such that $\delta(x) = \lambda x$ on R, $c_0(\lambda - 1) = 0$, and $c_i(\lambda^i - \lambda) = 0$ for all $2 \leq i \leq n$.

If $|C| < \infty$, then the conditions in Corollary 5 are not sufficient. For example

Example 2 : Let $R = \mathrm{M}_2(\mathbb{Z}_2)$, $\delta(x) = 2x = \lambda x$, $L = \{A \in R \mid \mathrm{tr}A = 0\}$, and $f(x) = 2x^6 + 2x^3 + x^2$. Then $\delta \circ f(x) = f \circ \delta(x)$ for all $x \in L$, but $c_6(\lambda^6 - \lambda) \neq 0$ and $c_2(\lambda^2 - \lambda) \neq 0$.

If the generalized derivation in Theorem 2 is actually a derivation, then we will get the equivalent condition without any restriction on C.

Corollary 6 — Let R be a prime ring, L a non-commutative Lie ideal of R, μ a derivation of R, $f(Y) = \sum_{i=0}^{n} c_i Y^i \in C[Y]$ such that $\deg f(Y) = n > 1$ and $f(\xi) \neq c_1\xi + c_0$ for some $\xi \in C$. Then $\mu \circ f(x) = f \circ \mu(x)$ for all $x \in L$ if and only if $\mu = 0$ and $c_0 = 0$.

In particular, if $f(Y) = Y^n$ $(n > 1)$, then we get the following corollaries.

Corollary 7 — Let R be a prime ring. For $n > 1$, a generalized derivation which acts as an n-Jordan homomorphism (S_n-homomorphism) on a non-commutative Lie ideal of R must be a scalar multiplication by $\lambda \in C$ such that $\lambda^n = \lambda$.

In particular, for derivations we have

Corollary 8 — Let R be a prime ring. For $n > 1$, a derivation acting as an n-Jordan homomorphism (S_n-homomorphism) on a non-commutative Lie ideal of R must be zero.

The condition "$f(\xi) \neq c_1\xi + c_0$ for some $\xi \in C$" in the last two sections is necessary. It is not only needed technically.

Example 3 : Let $R = \mathrm{M}_2(\mathbb{Z}_2)$, $\delta(x) = [e_{11}, x]$ and $f(x) = x^4 + x^2$. Then $\delta \neq 0$, but still $\delta \circ f = f \circ \delta$ on R.

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REFERNECES

1. A. Asma and K. Deepak, Derivation which acts as a homomorphism or as an anti-homomorphism in a prime ring, *Int. Math. Forum*, **2** (2007), 1105-1110.

2. A. Asma and K. Deepak, Generalized derivations as homomorphisms or as anti-homomorphisms in a prime ring, *Hacet. J. Math. Stat.*, **38** (2009), 17-20.

3. A. Asma, N. Rehman and A. Shakir, On Lie ideals with derivations as homomorphisms and anti-homomorphisms, *Acta Math. Hungar.*, **101** (2003), 79-82.

4. K. I. Beidar, W. S. Martindale and A. V. Mikhalev, *Rings with generalized identities*, Marcel Dekker, New York (1996).

5. H. E. Bell and L. C. Kappe, Rings in which derivations satisfy certain algebraic conditions, *Acta Math. Hungar.*, **253** (1989), 339-346.

6. C. L. Chuang, Differential identities with automorphisms and antiautomorphisms II, *J. Algebra*, **160** (1993), 130-171.

7. V. De Filippis, Generalized derivations as Jordan homomorphisms on Lie ideals and right ideals, *Acta Math. Sin. (Engl. Ser.)*, **25** (2009), 1965-1974.

8. T. S. Erickson, W. S. Martindale and J. M. Osborn, Prime nonassociative algebras, *Pacific J. Math.*, **60** (1975), 49-63.

9. I. N. Herstein, *Topics in ring theory*, Univ. Chicago Press (1969).

10. V. K. Kharchenko, Differential identities of semiprime rings, *Algebra i Logika*, **18** (1979), 86-119.

11. C. Lanski, An Engel condition with derivation for left ideal, *Proc. Amer. Math. Soc.*, **125** (1997), 339-345.

12. P. H. Lee and T. K. Lee, Derivations with Engel conditions on multilinear polynomials, *Proc. Amer. Math. Soc.*, **124** (1996), 2625-2629.

13. T. K. Lee, Generalized derivations of left faithful rings, *Comm. Algebra*, **27** (1999), 4057-4073.

14. W. S. Martindale 3rd, Prime rings satisfying a generalized polynomial identity, *J. Algebra*, **12** (1969), 576-584.

15. L. Oukhtite, S. Salhi and L. Taoufiq, σ-Lie ideals with derivations as homomorphisms and anti-homomorphisms, *Int. J. Algebra*, **1** (2007), 235-239.

16. N. Rehman, On generalized derivations which acts as homomorphisms and anti-homomorphisms, *Glas. Mat.*, **39** (2004), 27-30.

17. Y. Wang and H. You, Derivations as homomorphisms or anti-homomorphisms, *Acta Math. Sin. (Engl. Ser.)*, **23** (2007), 1149-1152.

18. X. W. Xu, J. Ma and F. W. Niu, Annihilators of power central values of generalized derivations, *Chinese Ann. Math. Ser. A*, **28** (2007), 131-140.

19. X. W. Xu, J. Ma and F. W. Niu, Generalized derivations having the same power values with left multiplications, *Algebra Colloq.*, **20** (2013), 369-382.