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AVERAGE DENSITIES OF FREQUENCY BANDS OF WAVELETS[1]

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A function is called a wavelet if its integral translations and dyadic dilations form an orthonormal basis for $L^2(\mathbb{R})$. The support of the Fourier transform of a wavelet is called its frequency band. In this paper, we study the relation between diameters and measures of frequency bands of wavelets, precisely say, we study the ratio of the measure to the diameter. This reflects the average density of the frequency band of a wavelet. In particular, for multiresolution analysis (MRA) wavelets, we do further research. First, we discuss the relation between diameters and measures of frequency bands of scaling functions. Next, we discuss the relation between frequency bands of wavelets and the corresponding scaling functions. Finally, we give the precise estimate of the measure of frequency bands of wavelets. At the same time, we find that when the diameters of frequency bands tend to infinity, the average densities tend to zero.

Key words : Wavelet Analysis; frequency band; average density.

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1. INTRODUCTION

Wavelets are a tool of time-frequency analysis. There are many mathematical and physical situations in which one would like to control finely the frequency bands of wavelets. The geometry of the frequency bands of wavelets contains several aspects: structure, measure, diameter, density, and so on.

For the structure of the frequency bands, it is well known that for a band-limited wavelet whose Fourier transform is continuous, there exists a hole which contains the origin in its frequency band. Therefore, its frequency band Ω possesses a ring-like structure [4]

$$\Omega = S \setminus \widetilde{S} \ (0 \in \widetilde{S} \subset S).$$

For example, the frequency band of the Shannon wavelet is $\Omega = [-2\pi, 2\pi] \setminus (-\pi, \pi)$, the frequency band of the Meyer wavelet is $\Omega = \left[-\frac{8\pi}{3}, \frac{8\pi}{3}\right] \setminus \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$. If ψ is a minimally supported frequency (MSF) wavelet which is derived from an MRA, then the hole $\widetilde{S} = \frac{S}{2}$ [4].

For the measure of the frequency band, it is well known that the minimal measure of the frequency bands of wavelets is 2π [4]. In [7], we showed that for an MRA wavelet with the frequency band Ω, if $\mathrm{Diam}\,\Omega < 8\pi$ and the frequency band G of its corresponding scaling function satisfies $\partial G \bigcap \partial\left(\frac{G}{2}\right) = \emptyset$, then the measure $|\Omega| \leq 4\pi$, where the diameter of a set Ω is defined as $\mathrm{Diam}\,\Omega = \sup\{|x-y|, \ \ x, \ y \in \Omega\}$ and ∂E is the boundary of the set E. For any wavelet ψ, the minimal diameter is 4π and if $\mathrm{Diam}\,\Omega = 4\pi$, then the measure $|\Omega| = 2\pi$ [6].

In this paper, we discuss the relationship between diameters and measures of frequency bands Ω of wavelets. In Section 3, we use the characterization of wavelets to give a precise result for general wavelets as follows. If $\mathrm{Diam}\,\Omega < 4m_0\pi$, where m_0 is an odd number, then the measure

$$|\Omega| \leq 2m_0\pi.$$

We define the average density of the frequency band Ω as the ratio $|\Omega|/\mathrm{Diam}\,\Omega$ and denote it by $Ad(\Omega)$. If m_0 is the minimal odd number such that $\mathrm{Diam}\,\Omega < 4m_0\pi$, then the average density of the frequency band

$$Ad(\Omega) = \frac{|\Omega|}{\mathrm{Diam}\,\Omega} \leq \frac{1}{2} + \frac{4\pi}{\mathrm{Diam}\,\Omega - 8\pi}.$$

From this, we deduce that when the diameter tends to infinity, the upper limit

$$\limsup\, Ad(\Omega) \leq \frac{1}{2}.$$

In Section 4, we use the characterization of frequency bands of scaling functions to give precise results for MRA wavelets. We first discuss the relationship between diameters and measures of frequency bands of scaling functions. Next, we discuss the relationship between frequency bands of wavelets and that of the corresponding scaling functions. Finally, we give the precise estimate of the measure of the frequency bands Ω of wavelets. If l is the minimal integer such that supp $\widehat{\psi} \subset [-2^l\pi,\ 2^l\pi]$, then the measure of the frequency band

$$|\Omega| \leq \frac{2^{l+2}\pi - 4\pi}{l+1}$$

and the average density of the frequency band

$$Ad(\Omega) \leq \frac{8}{\log_2\left(\dfrac{\mathrm{Diam}\,\Omega}{\pi}\right)}\left(1 - \frac{2\pi}{\mathrm{Diam}\,\Omega}\right).$$

This implies that when the diameter of the frequency band tends to infinity, the average density tends to zero.

The density of the frequency band Ω of a wavelet at a point ω_0 is defined by

$$d(\Omega,\, \omega_0) = \lim_{\epsilon \longrightarrow 0} |\Omega \bigcap (\omega_0 - \epsilon,\ \omega_0 + \epsilon)|/(2\epsilon).$$

The paper [2] showed that for any wavelet, its frequency band Ω satisfies $d(\Omega,\, 0) = 0$ even if there is no hole containing the origin in Ω. On the other hand, from Theorem 1.6 in [2], we know that for any $\omega_0 \neq 0,\ 0 \leq \alpha \leq 1$, there is a wavelet ψ with $\Omega = \mathrm{supp}\,\widehat{\psi}$ such that $d(\Omega,\, \omega_0) = \alpha$. Here the average of density discussed by us should be regarded as average of densities of all points in a frequency band. It reflects the sparsity of frequency bands of band-limited wavelets.

2. Preliminaries

At first we recall some notations, definitions, and known lemmas.

2.1. Some notations

For any set $E \subset \mathbb{R}$, χ_E is the characteristic function of E, $|E|$ is the measure of E, and

$$E + 2\pi\mathbb{Z} = \bigcup_{\nu \in \mathbb{Z}} (E + 2\nu\pi).$$

Denote the Fourier transform of $f \in L^2(\mathbb{R})$ by $\widehat{f}$:

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-it\omega}\,\mathrm{d}t.$$

For convenience, the support of the Fourier transform of f is called the frequency band of f, i.e.,

$$\operatorname{supp} \widehat{f} = \{\omega \in \mathbb{R}; \quad \widehat{f}(\omega) \neq 0\}.$$

If the frequency band of a function is bounded, then we say it is a band-limited function. For a set E, define the diameter of E by

$$\operatorname{Diam} E = \sup\{|x-y|, \quad x,\ y \in E\}$$

and define the average density of E by

$$Ad(E) = \frac{|E|}{\operatorname{Diam} E}.$$

2.2. Some definitions

A function $\psi \in L^2(\mathbb{R})$ is a wavelet if $\{\psi_{j,k} :\ j,\ k \in \mathbb{Z}\}$ is an orthonomal basis for $L^2(\mathbb{R})$, where $\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j \cdot - k)$ [3]-[6]. Wavelets can be characterized by using its Fourier transform and wavelets can be constructed by using multiresolution analyses (MRAs). A multiresolution analysis [3] consists of a sequence of subspaces $\{V_m\}_m$ of $L^2(\mathbb{R})$ satisfying

(i) $V_m \subset V_{m+1}, \quad \overline{\bigcup_m V_m} = L^2(\mathbb{R}), \quad \bigcap_m V_m = \{0\};$

(ii) $f \in V_m \quad \Longleftrightarrow \quad f(2\cdot) \in V_{m+1}, \;\; m \in \mathbb{Z}$;

(iii) there exists a $\varphi \in V_0$ such that $\{\varphi(\cdot - n)\}_n$ is an orthonormal basis of V_0, where φ is called a scaling function.

If a wavelet ψ belongs to the space W_0, where $V_0 \bigoplus W_0 = V_1$, then ψ is called an MRA wavelet. Many authors [4] characterized wavelets whose frequency bands have the minimal measure, which are called minimally supported frequency (MSF) wavelets. A wavelet ψ is a MSF wavelet if and only if $|\widehat{\psi}| = \chi_W$ and $|W| = 2\pi$ [4].

2.3. Some known lemmas

The following two lemmas give the characterization of wavelets and the characterization of frequency bands of scaling functions, respectively.

Lemma 2.1 [4] — A function $\psi \in L^2(\mathbb{R})$, with $\| \psi \|_2 = 1$, is a wavelet if and only if

(i) $\quad \sum\limits_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 = 1,$

(ii) $\quad \sum\limits_{j=0}^{\infty} \widehat{\psi}(2^j\omega)\overline{\widehat{\psi}}(2^j(\omega + 2m\pi)) = 0 \;$ for $\; m \in 2\mathbb{Z}+1.$

Lemma 2.2 [8] — Let G be a bounded closed set in $\mathbb{R}$. Then there is a scaling function φ with $G = \operatorname{supp}\widehat{\varphi}$ if and only if

(i) $G \subset 2G$;

(ii) $\overline{\bigcup\limits_{m\in\mathbb{Z}} 2^m G} = \mathbb{R}$;

(iii) $G + 2\pi\mathbb{Z} = \mathbb{R}$;

(iv) $\left(G \setminus \frac{G}{2}\right) \bigcap \left(\frac{G}{2} + 2\pi\nu\right) = \emptyset \quad (\nu \in \mathbb{Z}).$

These two lemmas are our main tool to discuss the relationship between diameters and measures of frequency bands of wavelets. The following two lemmas show that the diameters of frequency bands of band-limited wavelets have a lower bound 4π, but have no upper bound.

Lemma 2.3 [4] — Let ψ be a wavelet with $\Omega = \operatorname{supp}\widehat{\psi}$. Then $\operatorname{Diam}\Omega \geq 4\pi$. If $\operatorname{Diam}\Omega$

$= 4\pi$, then the measure $|\Omega| = 2\pi$ and

$$\Omega = [-2\alpha,\ -\alpha] \bigcup [2\pi - \alpha,\ 4\pi - 2\alpha] \qquad \text{for some } \ 0 < \alpha < 2\pi.$$

Lemma 2.4 [8] — For each $n \in \mathbb{Z}_+$, there exists an MSF wavelet ψ_n with $B_n = \operatorname{supp} \widehat{\psi}_n$ such that

$$\lim_{n\to\infty} \text{Diam}\, B_n = \infty.$$

We also need the following lemmas which are used to explain the relationship between frequency bands of wavelets and that of the corresponding scaling functions.

Lemma 2.5 [4] — Let ψ be an MRA wavelet derived by a scaling function φ. Then

$$|\widehat{\varphi}(\omega)|^2 = \sum_{m=1}^{\infty} |\widehat{\psi}(2^m\omega)|^2 \qquad (\omega \in \mathbb{R}).$$

Lemma 2.6 [7] — Let $\varphi \in L^2(\mathbb{R})$ be a band-limited scaling function with $G = \operatorname{supp} \widehat{\varphi}$. If the frequency band G satisfies

(i) $\partial G \bigcap \partial \left(\frac{G}{2}\right) = \emptyset$,

(ii) $\omega + 4\pi\nu \notin \partial G$ for any $\omega \in \partial G$ and $\nu \in \mathbb{Z} \setminus \{0\}$,

then the frequency band Ω of the corresponding wavelet satisfies

$$\Omega = \operatorname{supp} \widehat{\psi} \subset 2G \setminus G.$$

3. General Wavelets

First, based on the characterization of wavelets, we give estimates of the average densities of the frequency bands of general wavelets. Suppose that ψ is a band-limited wavelet with $\Omega = \operatorname{supp} \widehat{\psi}$. By Lemma 2.3, we know that Diam $\Omega \geq 4\pi$ and if Diam $\Omega = 4\pi$, then $|\Omega| = 2\pi$, so $Ad(\Omega) = \frac{1}{2}$. Therefore, we only need to discuss the case Diam $\Omega > 4\pi$.

***Theorem* 3.1** — *Let $\psi \in L^2(\mathbb{R})$ be a band-limited wavelet with $\Omega =$ supp $\widehat{\psi}$ and Diam $\Omega > 4\pi$.*

(i) If m_0 is the minimal odd number such that Diam $\Omega < 4m_0\pi$, then $|\Omega| \leq 2m_0\pi$;

(ii) $Ad(\Omega) \leq \frac{1}{2} + \frac{4\pi}{\text{Diam}\,\Omega - 8\pi}$;

PROOF : By the assumption, we can deduce that for any odd number m satisfying $|m| \geq m_0$,

$$\widehat{\psi}(2^j\omega)\overline{\widehat{\psi}(2^j(\omega+2m\pi))} = 0 \quad a.e. \quad \text{for } \ j \geq 1. \tag{3.1}$$

In fact, if $\omega \notin 2^{-j}\Omega$, then $2^j\omega \notin \Omega$, hence $\widehat{\psi}(2^j\omega) = 0$. If $\omega \in 2^{-j}\Omega$, then $2^j\omega \in \Omega$. Since

$$|2^j(\omega+2m\pi) - 2^j\omega| = 2^j \cdot 2|m|\pi \geq 4m_0\pi \quad (j \geq 1, \ |m| \geq m_0)$$

and $\operatorname{Diam}\Omega < 4m_0\pi$, it follows that for $\omega \in 2^{-j}\Omega$,

$$2^j(\omega+2m\pi) \notin \Omega \qquad (j \geq 1, \ \ |m| \geq m_0).$$

So $\widehat{\psi}(2^j(\omega+2m\pi)) = 0$. Hence (3.1) holds.

By Lemma 2.1 (ii), we have

$$\sum_{j=0}^{\infty} \widehat{\psi}(2^j\omega)\,\overline{\widehat{\psi}(2^j(\omega+2m\pi))} = 0 \quad a.e. \quad m \in 2\mathbb{Z}+1.$$

From this and (3.1), we deduce that for any odd number m satisfying $|m| \geq m_0$,

$$\widehat{\psi}(\omega)\,\overline{\widehat{\psi}(\omega+2m\pi)} = 0 \quad a.e.$$

So $|\operatorname{supp}(\widehat{\psi}(\omega)\,\overline{\widehat{\psi}(\omega+2m\pi)})| = 0$. Again, since

$$\operatorname{supp}\widehat{\psi}(\omega)\,\overline{\widehat{\psi}(\omega+2m\pi)} = \operatorname{supp}\widehat{\psi}(\omega)\bigcap \operatorname{supp}\widehat{\psi}(\omega+2m\pi) = \Omega\bigcap(\Omega-2m\pi),$$

we have

$$|\Omega\bigcap(\Omega-2m\pi)| = 0 \quad \text{for} \quad m \in 2\mathbb{Z}+1, \qquad |m| \geq m_0. \tag{3.2}$$

By $\operatorname{Diam}\Omega < 4m_0\pi$, we may take

$$a \leq \inf\{\omega \in \Omega\}, \qquad b \geq \sup\{\omega \in \Omega\}$$

such that $\Omega \subset [a,b], \ b-a = 4m_0\pi$. Moreover,

$$|\Omega| = \left|\Omega\bigcap\left[a, \ \frac{a+b}{2}\right]\right| + \left|\Omega\bigcap\left[\frac{a+b}{2}, \ b\right]\right| =: |I_1| + |I_2|.$$

Since $\left[\frac{a+b}{2}, b\right] - 2m_0\pi = \left[\frac{a+b}{2},\ \ b\right] - \frac{1}{2}(b-a) = \left[a,\ \ \frac{a+b}{2}\right]$, we have

$$I_2 - 2m_0\pi = \left(\ \Omega \bigcap \left[\frac{a+b}{2},\ \ b\right]\ \right) - 2m_0\pi = (\Omega - 2m_0\pi) \bigcap \left[a,\ \ \frac{a+b}{2}\right]. \tag{3.3}$$

Noticing that

$$I_1 = \Omega \bigcap [a,\ \ \frac{a+b}{2}]. \tag{3.4}$$

By (3.2), we have $|(\Omega - 2m_0\pi) \bigcap \Omega| = 0$. From this and (3.3), and (3.4), it follows that

$$|(I_2 - 2m_0\pi) \bigcap I_1| = 0 \quad \text{and} \quad I_1,\ I_2 - 2m_0\pi \subset \left[a,\ \ \frac{a+b}{2}\right].$$

So

$$|\Omega| = |I_1| + |I_2| = |I_1| + |I_2 - 2m_0\pi| \leq \left(\frac{a+b}{2} - a\right) = \frac{b-a}{2} = 2m_0\pi.$$

We get (i).

Now we prove (ii). Since ψ is band-limited and $\operatorname{Diam}\Omega > 4\pi$, there exists an odd number $m_0 > 1$ such that

$$4(m_0 - 2)\pi \leq \operatorname{Diam}\Omega < 4m_0\pi.$$

By (i), we have $|\Omega| \leq 2m_0\pi$, This implies that

$$Ad(\Omega) = \frac{|\Omega|}{\operatorname{Diam}\Omega} \leq \frac{2m_0\pi}{4(m_0-2)\pi} = \frac{1}{2} + \frac{4\pi}{4m_0\pi - 8\pi} \leq \frac{1}{2} + \frac{4\pi}{\operatorname{Diam}\Omega - 8\pi}.$$

We get (ii). $\square$

From this, we deduce that for any sequence $\{\psi_n\}$ of wavelets, if their diameters tend to infinity, we have

$$\limsup\ Ad(\operatorname{supp}\widehat{\psi}_n) \leq \frac{1}{2}.$$

On the other hand, by Lemma 2.4, there exists a sequence of MSF wavelets $\{\psi_n\}$ such that diameters of its frequency bands tend to infinity. Again, since the measure of each MSF wavelet ψ_n is 2π, we get

$$\lim_{n\to\infty} Ad(\operatorname{supp}\widehat{\psi}_n) = 0.$$

From this, we see that when the diameter is sufficiently large, the average density τ of the frequency band of a wavelet satisfies $0 < \tau < \frac{1}{2} + \epsilon$ $(\epsilon > 0)$.

4. MRA Wavelets

For MRA wavelets, we first consider a corresponding result on scaling functions by using a characterization of frequency bands of band-limited scaling functions.

***Theorem* 4.1** — *Let φ be a scaling function satisfying $G = \text{supp}\, \widehat{\varphi} \subset [-2^k\pi,\ \ 2^k\pi]$, where k is a nonnegative integer. Then*

$$|G| \le \frac{2^{k+2}\pi}{k+2}. \tag{4.1}$$

When φ is the Shannon scaling function or the Meyer scaling function, the equality holds.

PROOF : Since φ is a scaling function, by Lemma 2.2 (i) and (iv), we have

$$\frac{G}{2} \subset G, \qquad \left(G \setminus \frac{G}{2}\right) \bigcap \left(\frac{G}{2} + 2\pi\mathbb{Z}\right) = \emptyset. \tag{4.2}$$

Let $E_j = \dfrac{G}{2^{j-1}} \setminus \dfrac{G}{2^j}$ $(j = 1, ..., k)$ and $E_{k+1} = \dfrac{G}{2^k}$. We give a partition of the set G as follows:

$$G = \bigcup_{j=1}^{k+1} E_j \qquad \text{(a disjoint union)}. \tag{4.3}$$

By (4.2), we know that for $j,\ l = 1, ..., k+1,\ \ j \neq l$,

$$E_j \bigcap (E_l + 2\pi\mathbb{Z}) = \emptyset. \tag{4.4}$$

For any set E, we define an associated set $\text{mod}_{2\pi}E \subset E$ which satisfies $\text{mod}_{2\pi}E + 2\pi\mathbb{Z}$ $= E + 2\pi\mathbb{Z}$ and $\{\text{mod}_{2\pi}E + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are mutually disjoint.

By (4.2)-(4.4), we get

$$\text{mod}_{2\pi}G = \bigcup_{j=1}^{k+1} \text{mod}_{2\pi}E_j \qquad \text{(a disjoint union)}. \tag{4.5}$$

Since $E_j = \frac{G}{2^{j-1}} \setminus \frac{G}{2^j}$ and $G \subset [-2^k\pi,\ 2^k\pi]$, we have $E_j \subset [-2^{k-j+1}\pi,\ 2^{k-j+1}\pi]$. For $a, e,\ \omega \in \mathrm{mod}_{2\pi} E_j$, the points $\{\omega + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ have at most 2^{k-j+1} values lie in E_j, so we have

$$2^{k-j+1}|\mathrm{mod}_{2\pi} E_j| \geq |E_j|. \tag{4.6}$$

By Lemma 2.2 (iii), we have $G + 2\pi\mathbb{Z} = \mathbb{R}$ and so

$$\mathrm{mod}_{2\pi} G + 2\pi\mathbb{Z} = G + 2\pi\mathbb{Z} = \mathbb{R}.$$

Again, since $\{\mathrm{mod}_{2\pi} G + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are mutually disjoint, we have $|\mathrm{mod}_{2\pi} G| = 2\pi$. By (4.5) and (4.6), we get

$$2\pi = |\mathrm{mod}_{2\pi} G| = \sum_{j=1}^{k+1} |\mathrm{mod}_{2\pi} E_j| \geq \sum_{j=1}^{k+1} \frac{|E_j|}{2^{k-j+1}}.$$

Denote $\alpha = |E_{k+1}| = \left|\frac{G}{2^k}\right|$. Then, since $G \subset 2G$, we have

$$|E_j| = \left|\frac{G}{2^{j-1}}\right| - \left|\frac{G}{2^j}\right| = 2^{k-j+1}\alpha - 2^{k-j}\alpha = 2^{k-j}\alpha \qquad (j = 1, ..., k),$$

and so

$$2\pi \geq \sum_{j=1}^{k} \frac{2^{k-j}\alpha}{2^{k-j+1}} + \alpha = \frac{k\alpha}{2} + \alpha.$$

This implies that $\alpha \leq \frac{4\pi}{k+2}$, and so

$$|G| = 2^k\alpha \leq \frac{2^{k+2}\pi}{k+2}. \qquad \square$$

Now we discuss the relationship between the frequency band of a scaling function and that of the corresponding wavelet.

Lemma 4.2 — Let ψ be a band-limited scaling function with $G = \mathrm{supp}\,\widehat{\varphi}$ and ψ be the corresponding wavelet with $\Omega = \mathrm{supp}\,\widehat{\psi}$. Then

(i) $G = 2G \setminus E$, where $E = \{\omega \in G,\ \left|\widehat{\varphi}\left(\frac{\omega}{2}\right)\right| = |\widehat{\varphi}(\omega)|\}$,

(ii) Diam $\Omega = 2\,$Diam G,

(iii) $|G| \le |\Omega| \le 2|G|$,

(iv) $\frac{1}{2}\,Ad(G) \le Ad(\Omega) \le Ad(G)$.

PROOF : Denote

$$a = \inf\{\omega,\ \ \omega \in G\}, \qquad b = \sup\{\omega,\ \ \omega \in G\}. \tag{4.7}$$

First, we easily prove that $a < 0 < b$. If it is not true, then $a > 0$ or $b < 0$. This is contrary to $\overline{\bigcup_{m\in\mathbb{Z}} 2^m\, G} = \mathbb{R}$ (Lemma 2.2). By Lemma 2.5, we have

$$|\widehat{\psi}(\omega)|^2 = \left|\widehat{\varphi}\left(\frac{\omega}{2}\right)\right|^2 - |\widehat{\varphi}(\omega)|^2 \qquad (\omega \in \mathbb{R}). \tag{4.8}$$

Since

$$\text{supp}\,|\widehat{\varphi}(\frac{\cdot}{2})| = 2G, \qquad \text{supp}\,|\widehat{\varphi}|^2 = G, \qquad \text{supp}\,|\widehat{\psi}|^2 = \Omega,$$

and $G \subset 2G$ (Lemma 2.2), by (4.8), we get

$$\Omega = (2G \bigcup G) \setminus E = 2G \setminus E, \tag{4.9}$$

where $E = \{\omega \in G,\ \ |\widehat{\varphi}(\frac{\omega}{2})| = |\widehat{\varphi}(\omega)|\}$, i.e., (i) holds. Since $E \subset G \subset [a,\, b]$ and $a < 0 < b$, by (4.9), we have

$$2a = \inf\{\omega,\ \ \omega \in \Omega\}, \qquad 2b = \sup\{\omega,\ \ \omega \in \Omega\}.$$

From this and (4.7), we get (ii). By (4.9) and $E \subset G$, we get $2G \setminus G \subset \Omega \subset 2G$, and so

$$|G| = 2|G| - |G| \le |\Omega| \le 2|G|,$$

i.e., (iii) holds. Finally, from (ii) and (iii), we get (iv). $\square$

From Theorem 4.1 and Lemma 4.2, we deduce that under conditions of Lemma 4.2, we easily obtain the following estimates of frequency bands for band-limited MRA wavelets: If l is the minimal integer such that $\Omega \subset [-2^l\pi,\ 2^l\pi]$, then

(i) $|\Omega| \le \dfrac{2^{l+2}\pi}{l+1}$,

(ii) $Ad(\Omega) \le \dfrac{8}{\log_2\left(\dfrac{\mathrm{Diam}\,\Omega}{\pi}\right)}$.

Now we further improve the above results.

***Theorem* 4.3** — *Let ψ be a band-limited MRA wavelet. If l is the minimal integer such that the frequency band Ω satisfies $\Omega \subseteq [-2^l\pi,\ 2^l\pi]$, then*

(i) $|\Omega| \le \frac{2^{l+2}\pi-4\pi}{l+1}$. For the Shannon wavelet and Meyer wavelets, the equality holds.

(ii) $\lim\limits_{l\to\infty} Ad(\Omega) = 0$ and $Ad(\Omega) \le \dfrac{8}{\log_2\left(\dfrac{Diam\,\Omega}{\pi}\right)}\left(1-\dfrac{2\pi}{Diam\,\Omega}\right)$.

PROOF : By assumption and $\mathrm{Diam}\,G = \frac{1}{2}\,\mathrm{Diam}\,\Omega$ (Lemma 4.2 (ii)), we know that l is the minimal integer such that

$$G \subset [-2^{l-1}\pi,\ 2^{l-1}\pi]. \tag{4.10}$$

Letting $k = l-1$, by (4.3), we know that

$$G = \bigcup_{j=1}^{l} E_j \tag{4.11}$$

is a partition of G, where $E_j = \frac{G}{2^{j-1}} \setminus \frac{G}{2^j}$ $(j = 1, ..., l-1)$ and $E_l = \frac{G}{2^{l-1}}$. By (4.4) and $E_l \subset [-\pi,\ \pi]$, we have that for $\nu \ne 0$,

$$(E_l + 2\nu\pi)\bigcap E_j = \emptyset \qquad \text{for} \quad j = 1, ..., l. \tag{4.12}$$

Since φ is a scaling function, we have [3]

$$\sum_{\nu\in\mathbb{Z}} |\widehat{\varphi}(\omega+2\nu\pi)|^2 = 1 \qquad (\omega\in\mathbb{R}). \tag{4.13}$$

By (4.12), we know that for $a.e.$ $\omega \in E_l$ and $\nu \ne 0$,

$$\omega + 2\nu\pi \notin E_j \qquad (j = 1, ..., l).$$

Again, by (4.11), we deduce that $\omega + 2\nu\pi \notin G$ for $\omega \in E_l$ $(\nu \notin 0)$. From this and (4.13), it follows that

$$|\widehat{\varphi}(\omega)| = 1 \qquad (\omega \in E_l = \frac{G}{2^{l-1}}).$$

Since $\frac{G}{2} \subset G$, we have $\left|\widehat{\varphi}\left(\frac{\omega}{2}\right)\right| = 1 \ \ (\omega \in \frac{G}{2^{l-1}})$. Finally, by (4.8), we have

$$\widehat{\psi}(\omega) = 0 \qquad (a.e.\ \omega \in \frac{G}{2^{l-1}}). \tag{4.14}$$

By Lemma 4.2 (i), we have $\Omega \subset 2G$. Again, by $\frac{G}{2^{l-1}} \subset G \subset 2G$, and (4.14), we get

$$\Omega = 2G \setminus \frac{G}{2^{l-1}} \qquad \text{and} \qquad |\Omega| \leq \left(2 - \frac{1}{2^{l-1}}\right)|G|. \tag{4.15}$$

By (4.10) and Theorem 4.1, we have $|G| \leq \frac{2^{l+1}\pi}{l+1}$. From this and (4.15), we get (i). By the assumption, we know that $\Omega \subset [-2^l\pi,\ 2^l\pi]$, but Ω is not contained in $[-2^{l-1}\pi,\ 2^{l-1}\pi]$. Noticing that $0 \in G$, we have

$$2^{l-1}\pi \leq \text{Diam}\,\Omega \leq 2^{l+1}\pi.$$

By (i), we get

$$Ad(\Omega) = \frac{|\Omega|}{\text{Diam}\,\Omega} \leq \frac{8}{l+1}\left(1 - \frac{1}{2^l}\right).$$

Since $\text{Diam}\,\Omega \leq 2^{l+1}\pi$, we have

$$l + 1 \geq \log_2\left(\frac{\text{Diam}\,\Omega}{\pi}\right), \qquad 2^l \geq \frac{\text{Diam}\,\Omega}{2\pi}.$$

We get (ii). □

By Lemma 4.2 and (4.14), we get the following:

Corollary 4.4 — If ψ is an MRA wavelet with $\Omega = \text{supp}\,\widehat{\varphi} \subset [-2^l\pi,\ 2^l\pi]$ for some $l \in \mathbb{Z}_+$, then

$$\widehat{\psi}(\omega) = 0, \qquad \omega \in \frac{\Omega}{2^l}.$$

For a kind of MRA wavelets, we have a more precise result.

Suppose that ψ is a band-limited MRA wavelet with $\Omega = \text{supp}\,\widehat{\psi}$. Its corresponding scaling function φ with $G = \text{supp}\,\widehat{\varphi}$. If the set G satisfies the conditions (i) and (ii) in Lemma 2.6, then $\Omega \subset 2G \setminus (\frac{G}{2})$. So

$$|\Omega| \leq 2|G| - \frac{1}{2}|G| = \frac{3}{2}|G|.$$

Let $\Omega \subset [-2^l\pi,\ 2^l\pi]$. By the argument of Theorem 4.3, we have $G \subset [-2^{l-1}\pi,\ 2^{l-1}\pi]$. Using Theorem 4.1, we get

$$|\Omega| \leq \frac{3}{2}|G| \leq \frac{3\pi \cdot 2^l}{l+1}.$$

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