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SOME RADII PROBLEMS RELATED TO CERTAIN P-VALENT MEROMORPHIC FUNCTIONS

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The purpose of the present paper is to introduce several new classes of p-valent meromorphic functions defined by Ruscheweyh derivative operator for meromorphic multivalent functions. Further, we investigate the radii problems of these analytic classes in the punctured unit disk.

Key words : p-valent functions; Ruscheweyh derivative; bounded boundary rotation; bounded radius rotation.

1. INTRODUCTION

Let Σ_p denote the class of meromorphic p-valent functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=p+n}^{\infty} a_k z^k, n \geq 1, \;\; p \in N = \{1, 2, ...\}, \tag{1.1}$$

which is analytic in $E = \{z : 0 < |z| < 1\}$.

Let $P(\gamma, n)$ be the class of functions p of the form

$$p(z) = 1 + \sum_{k=p+n}^{\infty} a_k z^k \tag{1.2}$$

which is analytic in E and satisfy $\mathrm{Re} p(z) > \gamma$ for $z \in E, 0 \leq \gamma$.

Using the concept of meromorphic p-valent functions, we define the following classes.

$$\mathcal{MS}^p(\gamma, n) = \left\{ f : f \in \Sigma_p, \text{ and } -\frac{zf'}{pf} \in P(\gamma, n) \right\}.$$

$$\mathcal{MC}^p(\gamma, n) = \left\{ f : f \in \Sigma_p, \text{ and } -\frac{(zf')'}{pf'} \in P(\gamma, n) \right\}.$$

It is clear that

$$f \in \mathcal{MC}^p(\gamma, n) \;\; \Leftrightarrow -\frac{zf'}{p} \in \mathcal{MS}^p(\gamma, n). \tag{1.3}$$

For $n = 1$ and $p = 1$ we have the well-known classes $\mathcal{MS}(\gamma)$ and $\mathcal{MC}(\gamma)$ of meromorphic starlike and meromorphic convex functions of order γ, see [4].

Similarly, we define the subclasses of Σ_p consisting of all meromorphic functions, which are respectively close -to-convex and quasi convex of order γ in E as follows.

$$\mathcal{MK}^p(\beta, \gamma) = \left\{ f : f \in \Sigma_p, \text{ and } -\frac{zf'}{pg} \in P(\gamma, n) \text{ for some } g \in S^*(\gamma, n) \right\}$$

$$\mathcal{MQ}^p(\beta, \gamma) = \left\{ f : f \in \Sigma_p, \text{ and} -\frac{(zf')'}{pg'} \in P(\gamma, n) \text{ for some } g \in C(\gamma, n) \right\}$$

We note that

$$f \in \mathcal{MQ}^p(\beta, \gamma) \;\; \Leftrightarrow \;\; -\frac{zf'}{p} \in \mathcal{MK}^p(\beta, \gamma), 0 \leq \beta < 1. \tag{1.4}$$

For $n = 1$, the above classes reduces to well-known classes $\mathcal{MK}^p(\beta, \gamma)$ and $\mathcal{MQ}^p(\beta, \gamma)$. We note that the class C^* of quasiconvex univalent functions, analytic in E, was first introduced and studied in [6]. For $k \geq 2$, we introduce the following analytic classes

$$P_k[\gamma, n] = \left\{ \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), p_1, p_2 \in P[\gamma, n] \right\}, \tag{1.5}$$

$$P'_k[\gamma, n] = \{p : p\prime \in P_k[\gamma, n]\}.$$

We note that $P_2[\gamma, n] = P(\gamma, n)$. For $\gamma = 0$,and $n = 1$, we have class P_k which was defined and introduced by Pinchuk [12].

The classes $\mathcal{MV}_k^p(\gamma, n)$ and $\mathcal{MR}_k^p(\gamma, n), k \geq 2$ generalize the concept of the classes $\mathcal{MC}^p(\gamma, n)$ and $MS^*(\gamma, n)$, defined as follows:

$$\mathcal{MV}_k^p(\gamma, n) = \left\{ f : f \in \Sigma_p, \text{ and } -\frac{(zf')'}{pf'} \in P_k(\gamma, n) \right\}$$

$$\mathcal{MR}_k^p(\gamma, n) = \left\{ f : f \in \Sigma_p, \text{ and} -\frac{zf'}{pf} \in P_k(\gamma, n) \right\}.$$

For $n = 1, p = 1$ we have $\mathcal{MV}_k(\gamma)$ and $\mathcal{MR}_k(\gamma)$ the well-known classes of meromorphic functions of bounded boundary rotation and bounded radius rotation of order γ, see [10] respectively. Let $f_j(z), j = 1, 2 \in \Sigma_p$ be given by

$$f_j(z) = \frac{1}{z^p} + \sum_{k=p+n}^{\infty} a_{k,j} z^k, \;\; p \in N = \{1, 2, ...\}.$$

Then the Hadamard product or convolution $f_1 * f_2$ of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} z^k, \;\; p \in N = \{1, 2, ...\} \tag{1.6}$$

By using the Hadamard product, the well known Ruscheweyh operator is defined as follows.

Denote by $D^{\alpha+p-1} : \Sigma_p \longrightarrow \Sigma_p$ the operator defined by

$$D^{\alpha+p-1} f(z) = \frac{1}{z^p (1-z)^{\alpha+p}} * f(z), \; \alpha > -p.$$

For the class of univalent functions, the above operator was first defined by Ruscheweyh, (see [13]).

For real number $\alpha > -1$, we have the well-known identity as follows:

$$z(D^{\alpha+p-1} f(z))' = (\alpha + p) D^{\alpha+p} f(z) - \alpha D^{\alpha+p-1} f(z). \tag{1.7}$$

Using the Ruscheweyh derivative, we define the following subclasses of meromorphic functions

$$\mathcal{MS}_\alpha^p(\gamma,n)=\left\{f: f\in\Sigma_p,\ \text{and } -D^{\alpha+p-1}f\in\mathcal{MS}^p(\gamma,n),\alpha>-1,z\in E\right\}$$

$$\mathcal{MC}_\alpha^p(\gamma,n)=\left\{f: f\in\Sigma_p,\ \text{and } -D^{\alpha+p-1}f\in\mathcal{MC}^p(\gamma,n),\alpha>-1,z\in E\right\}$$

$$\mathcal{MK}_\alpha^p(\beta,\gamma)=\{f: f\in\Sigma_p,\ \text{and } -D^{\alpha+p-1}f\in\mathcal{MK}^p(\beta,\gamma),\alpha>-1,z\in E\}$$

and

$$\mathcal{MQ}_\alpha^p(\beta,\gamma)=\left\{f: f\in\Sigma_p,\ \text{and } -D^{\alpha+p-1}f\in\mathcal{MQ}^p(\beta,\gamma),\alpha>-1,z\in E\right\}.$$

For univalent functions and $n=1$, the above classes have been discussed in [7]. See also [8] and [9]. In the present paper, we will discuss radius properties of various classes of meromorphic multivalent functions.

2. Preliminary Lemmas

In order to prove our results, we shall require the following lemmas

Lemma 1 — Let $h\in P(0,n)=P_n$ class of functions with positive real part for $z\in E$. Then

(1) $\left|\dfrac{h'(z)}{h(z)}\right|\leq\dfrac{2n|z|^{n-1}}{1-|z|^{2n}}$

(2) $|zh'(z)|\leq\dfrac{2n|z|^n\ Reh(z)}{1-|z|^{2n}}$

(3) $\dfrac{1-|z|^n}{1+|z|^n}\leq\ Reh(z)\leq|h(z)|\leq\dfrac{1+|z|^n}{1-|z|^n}$

For (1) we refer to [5], (2) will be found in [1] and for (3), see [14].

The following Lemma can easily be shown by using Lemma 1.

Lemma 2 — Let $p\in P_2(\gamma,n)$ for $z=re^{i\theta}\in E$. Then

$$\frac{1+(2\gamma-1)r^n}{1+r^n}\leq\ \text{Re}p(z)\leq|p(z)|\leq\frac{1-(2\gamma-1)r^n}{1-r^n}$$

and

$$|zp'(z)| \leq \frac{2n(1-\gamma)r^n \; Rep(z)}{(1-r^n)[1+(1-2\gamma)r^n]}.$$

For Lemma 2, we refer [2].

3. Main Results

***Theorem* 3.1** — *Let* $f,\ g \in \Sigma_p$ *and let* $\dfrac{f'}{g'} \in P_n$, *where* $g \in \mathcal{MV}_k^p(\gamma, n)$, *the class of p-valent meromorphic convex functions. Then* $f \in \mathcal{MV}_k^p(\gamma, n)$ *for* $|z| < r_0$, *where*

$$r_0^n = r_0^n(\gamma, n) = \frac{(n-\gamma+p)+\sqrt{n^2-2\gamma n+2np}}{p-\gamma}. \tag{3.1}$$

This result is sharp.

PROOF : We can write

$$f'(z) = g'(z)h(z), \text{ where } g \in MV_k(\gamma, n),\ h \in P_n, z \in E.$$

Logarithmic differentiation yields

$$\begin{aligned} -\frac{(zf'(z))'}{f'(z)} &= -\frac{(zg'(z))'}{g'(z)} - \frac{zh'(z)}{h(z)}, h \in P_k(\gamma, n) \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\{(p-\gamma)P_1(z)+\gamma\} - \left(\frac{k}{4}-\frac{1}{2}\right)\{(p-\gamma)P_2(z)+\gamma\} - \frac{zh'(z)}{h(z)}, \end{aligned}$$

where $P_i \in P_n,\ i = 1, 2.$ This gives us

$$\frac{1}{p-\gamma}\left\{-\frac{(zf'(z))'}{f'(z)} - \gamma\right\} = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{P_1(z) - \frac{1}{p-\gamma}\frac{zh'(z)}{h(z)}\right\}$$

$$+\left(\frac{k}{4}-\frac{1}{2}\right)\{P_2(z) - \frac{1}{p-\gamma}\frac{zh'(z)}{h(z)}\}.$$

Now for $i = 1, 2$, we use Lemma 1 with $|z| < r_o$, to get

$$\begin{aligned} \operatorname{Re}\left\{P_i(z) - \frac{1}{p-\gamma}\frac{zh'(z)}{h(z)}\right\} &\geq \operatorname{Re}P_i(z) - \frac{1}{p-\gamma}\left|\frac{zh'(z)}{h(z)}\right| \\ &\geq \frac{1-r^n}{1+r^n} - \frac{1}{p-\gamma}\frac{2nr^n}{1-r^{2n}} \\ &= \frac{(p-\gamma)(1-r^n)^2 - 2nr^n}{(p-\gamma)(1-r^{2n})}. \end{aligned}$$

Therefore for $|z| < r_0$, where r_0 is given in 3.1,

$$\operatorname{Re}\left\{P_i(z) - \frac{1}{p-\gamma}\frac{zh'(z)}{h(z)}\right\} > 0.$$

This implies that

$$\frac{1}{p-\gamma}\left\{-\frac{(zf'(z))'}{f'(z)} - \gamma\right\} \in P_k[0,n] \ \text{ for } |z| < r_0.$$

Sharpness can be found by taking

$$P_i(z) = h(z) = \frac{1-z^n}{1+z^n} \in P_n.$$

***Theorem* 3.2** — *Let $f, g \in \Sigma_p$ and $g \in \Sigma_p$ and $g \in P_2'(\gamma, n) = P'(\gamma, n)$ in E. If $\frac{f'}{g'} \in P(\gamma, n)$, then $f \in \mathcal{MV}_2(0, n)$ for $|z| < r_1$ where r_1 is given by*

$$r_1^n = \frac{p}{2n(\gamma-1) + \sqrt{4n^2(\gamma-1)^2 + p^2}} \tag{3.2}$$

This result is sharp.

PROOF : Let

$$\frac{f'(z)}{g'(z)} = h(z), \ \text{ where } h \in P(\gamma, n)$$

After a simple computation, we get

$$\frac{zf''(z)}{pf'(z)} = \frac{zh'(z)}{ph(z)} + \frac{zg''(z)}{pg'(z)}.$$

That is

$$-\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) = \frac{-zh'(z)}{ph(z)}-\frac{zP'(z)}{pP(z)}-\frac{1}{p},$$
$$-\operatorname{Re}\frac{(zf'(z))'}{pf'(z)} \geq -\left|\frac{zh'(z)}{ph(z)}\right|-\left|\frac{zP'(z)}{pP(z)}\right|+\frac{1}{p},$$

where $g' = P \in P(\gamma, n)$ in E.

Now by using Lemma 2, we have

$$-Re\frac{(zf'(z))'}{pf'(z)} \geq \frac{1}{p}-\frac{4n(1-\gamma)r^n}{p(1-r^n)(1+r^n)} = \frac{(1-r^{2n})-4n(1-\gamma)r^n}{p(1-r^{2n})}.$$

Hence $f \in \mathcal{MV}_2(0,n)$ for $|z| < r_1$ where r_1 is given by 3.2.

The sharpness can be seen by considering

$$g(z) = \int_0^z \frac{1-(2\gamma-1)t^n}{1-t^n}dt, h(z) = \frac{1-(2\gamma-1)z^n}{1-z^n} \quad \text{and} \quad f(z) = \int_0^z \left(\frac{(1-(2\gamma-1)t^n}{1-t^n}\right)^2 dt.$$

***Theorem* 3.3** — *Let $f \in \mathcal{MS}^p_{\alpha-1}(\gamma,n), \alpha \geq -2$. Then $f \in \mathcal{MS}^p_{\alpha}(\gamma,n)$ for $|z| < r_0(\alpha,\gamma)$ given by*

$$r_o^n(\alpha,\gamma) = \frac{\alpha-p}{\gamma-p+n+\sqrt{(\gamma-p+n)^2-(\alpha-p)(2\gamma-p-\alpha)}} \tag{3.3}$$

PROOF : Since $f \in \mathcal{MS}^p_{\alpha-1}(\gamma,n)$, we can write it as

$$\frac{-z(D^{\alpha+p-1}f(z))'}{D^{\alpha+p-1}f(z)} = H(z) = (p-\gamma)h(z)+\gamma,$$

where $H \in P(\gamma, n)$ and so $h \in P(0,n) = P_n$. Now using 1.7, we have

$$-\frac{z(D^{\alpha+p}f(z))'}{D^{\alpha+p}f(z)} = (p-\gamma)h(z)+\gamma+\frac{zh'(z)}{-(p-\gamma)h(z)-\gamma+\alpha}$$

or

$$\frac{1}{(p-\gamma)}\left[-\frac{(z(D^{\alpha+p}f(z))'}{D^{\alpha+p}f(z)}-\gamma\right] = h(z)+\frac{zh'(z)}{-(p-\gamma)h(z)-\gamma+\alpha}.$$

Therefore

$$\mathrm{Re}\frac{1}{(p-\gamma)}\left[-\frac{(z(D^{\alpha+p}f(z))'}{D^{\alpha+p}f(z)}-\gamma\right]\geq \mathrm{Re}h(z)\left\{1-\frac{\frac{2nr^n}{1-r^{2n}}}{(\gamma-p)\left(\frac{1-r^n}{1+r^n}\right)-\gamma+\alpha}\right\}$$

$$=\mathrm{Re}h(z)\left\{\frac{r^{2n}(-p+2\gamma-\alpha)-2(-p+\gamma+n)r^n+(\alpha-p)}{(\gamma-p)(1-r^n)^2+(\alpha-\gamma)(1-r^{2n})}\right\}$$

The right hand side of the above inequality is positive if $r^n < r_o^n$ and so is given by 3.3.

***Theorem* 3.4** — *Let for $\alpha \geq 0$, $f \in \mathcal{MC}^p_{\alpha-1}(\gamma,n)$ in E. Then $f \in \mathcal{MC}^p_{\alpha}(\gamma,n)$ in E. Then $f \in \mathcal{MC}^p_{\alpha+1}(\gamma,n)$ for $|z| < r_0$, where r_0 is given by 3.3.*

PROOF : By the definition of $\mathcal{MC}^p_{\alpha+p-1}(\gamma,n)$,we have

$$\begin{array}{rcl} f & \in & \mathcal{MC}^p_{\alpha+p-1}(\gamma,n) \Leftrightarrow -D^{\alpha+p-1}f \in C(\gamma,n) \\ & \Leftrightarrow & -z(D^{\alpha+p-1}f)' \in \mathcal{MS}^p(\gamma,n) \text{ in } E. \\ & \Leftrightarrow & D^{\alpha+p-1}(-zf') \in \mathcal{MS}^p(\gamma,n) \text{ in } E. \\ & \Leftrightarrow & -zf' \in \mathcal{MS}^p_\alpha(\gamma,n) \text{ in } |z|<r_0. \\ & \Leftrightarrow & D^{\alpha+p}(-zf') \in \mathcal{MS}^p(\gamma,n) \text{ in } |z|<r_0 \\ & \Leftrightarrow & -z(D^{\alpha+p}f)' \in \mathcal{MS}^p(\gamma,n) \text{ in } |z|<r_0. \\ & \Leftrightarrow & -D^{\alpha+p}f \in \mathcal{MC}^p(\gamma,n) \text{ in } |z|<r_0. \\ & \Leftrightarrow & f \in \mathcal{MC}^p_{\alpha+p}(\gamma,n) \text{ in } |z|<r_0. \end{array}$$

which is the required result.

***Theorem* 3.5** — *Let for $\alpha \geq 0$, $f \in \mathcal{MK}^p_{\alpha-1}(\beta,\gamma)$ in E. Then $f \in \mathcal{MK}^p_{\alpha}(\beta,\gamma)$ for $|z| < r_0 = r_0(\alpha,\gamma)$.*

PROOF : Since $f \in \mathcal{MK}^p_{\alpha-1}(\beta,\gamma)$, there exist $g \in \mathcal{MS}^p_{\alpha-1}(\gamma,n)$ such that

$$-\frac{(zD^{\alpha+p-1}f(z))'}{D^{\alpha+p-1}f(z)} = H(z) = (p-\beta)h(z)+\beta, \tag{3.4}$$

Also since $g \in \mathcal{MS}^p_{\alpha-1}(\gamma, n)$, we can write

$$-\frac{z(D^{\alpha+p-1}g(z))'}{D^{\alpha+p-1}g(z)} = H(z) = (p-\gamma)H(z) + \gamma,\ H \in P(0,n) \tag{3.5}$$

Using 1.7, we have

$$-\frac{z(D^{\alpha+p}f(z))'}{D^{\alpha+p}g(z)} = -\frac{z(D^{\alpha+p}(zf'(z))}{D^{\alpha+p}g(z)} = \frac{\frac{-1}{(p+\alpha)}z(D^{\alpha+p-1}(zf'(z))' + \frac{-\alpha}{\alpha+p}D^{\alpha+p-1}(zf'(z))}{\frac{1}{(p+\alpha)}\frac{z(D^{\alpha+p-1}(g(z))'}{z(D^{\alpha+p-1}(g(z))} + \alpha}$$

By using 3.4 and 3.5, we have

$$-\frac{z(D^{\alpha+p}f(z))'}{D^{\alpha+p}g(z)} = \frac{\frac{-z(D^{\alpha+p-1}(zf'(z))'}{D^{\alpha+p-1}g(z)} + \alpha\{-((p-\beta)h(z)-\beta\}}{-(p-\gamma)H(z)-\gamma+\alpha} \tag{3.6}$$

From 3.4, we have

$$z(D^{\alpha+p-1}f(z))' = (D^{\alpha+p-1}g(z))\{\{((p-\beta)h(z)+\beta\}.$$

Differentiating both sides, we have

$$(z(D^{\alpha+p-1}f(z))')' = -(p-\beta)zh'(z)(D^{\alpha+p}g(z)) + (D^{\alpha+p}g(z)'\{-\{((p-\beta)h(z)-\beta\}.$$

That is

$$\frac{(z(D^{\alpha+p-1}(zf(z))')'}{D^{\alpha+p-1}g(z)} = -(p-\beta)zh'(z)+[-(p-\gamma)H(z)-\gamma\}\{-\{((p-\beta)h(z)-\beta\} \tag{3.7}$$

Using 3.7 in 3.6, we obtain

$$\left\{\frac{1}{p-\beta}\left(-\frac{z(D^{\alpha+p}f(z))'}{D^{\alpha+p}g(z)} - \beta\right)\right\} = \left\{h(z) + \frac{zh'(z)}{-(p-\gamma)H(z)-\gamma+\alpha}\right\}$$

Now

$$\text{Re}\left\{\frac{1}{p-\beta}\left(-\frac{z(D^{\alpha+p}f(z))'}{D^{\alpha+p}g(z)} - \beta\right)\right\} = \text{Re}h(z)\left\{1 - \frac{\frac{2nr^n}{1-r^n}}{-(p-\gamma)\frac{1-r^n}{1+r^n} - \gamma + \alpha}\right\}.$$

We note that right hand side is positive for $|z| < r_0$, given by 3.3 and also $g \in \mathcal{MS}^p_{\alpha-1}(\gamma, n)$ for $|z| < r_o$.

Using the same method as in Theorem 3.4 with the relation 1.4, we can easily prove the following.

***Theorem* 3.6** — *Let $f \in \mathcal{MC}^p_{\alpha-1}(\beta,\gamma)$, for $\alpha \geq 0$, $z \in E$. Then $f \in \mathcal{MC}^p_{\alpha}(\beta,\gamma)$ for $|z| < r_0$, where r_0 is given by 3.3.*

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