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BESSEL FUNCTIONS FOR GL_2[1]

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Abstract. In classical analytic number theory there are several trace formulas or summation formulas for modular forms that involve integral transformations of test functions against classical Bessel functions. Two prominent such are the Kuznetsov trace formula and the Voronoi summation formula. With the paradigm shift from classical automorphic forms to automorphic representations, one is led to ask whether the Bessel functions that arise in the classical summation formulas have a representation theoretic interpretation. We introduce Bessel functions for representations of GL_2 over a finite field first to develop their formal properties and introduce the idea that the γ-factor that appears in local functional equations for L-functions should be the Mellin transform of a Bessel function. We then proceed to Bessel functions for representations of $GL_2(\mathbb{R})$ and explain their occurrence in the Voronoi summation formula from this point of view. We briefly discuss Bessel functions for GL_2 over a p-adic field and the relation between γ-factors and Bessel functions in that context. We conclude with a brief discussion of Bessel functions for other groups and their application to the question of stability of γ-factors under highly ramified twists.

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1. Introduction

In classical analytic number theory there are several trace formulas or summation formulas for modular forms that involve integral transformations of test functions against classical Bessel functions. Two prominent such are the Kuznetsov trace formula and the Voronoi summation formula. Not surprisingly, these formulas also involve related Kloosterman sums. In the 1960's the analytic paradigm for understanding modular forms shifted from a study of real or complex analytic functions on the upper half plane to the study of automorphic forms on GL_2 and automorphic representations of GL_2. This naturally leads one to ask whether the Bessel functions and Kloosterman sums that arise in the classical summation formulas have a representation theoretic interpretation.

Since the paradigm shift to automorphic representations had great impetus in the Soviet Union, it is not surprising that we find the first intimation of such a interpretation in the book of Gelfand, Graev, and Piatetski-Shapiro [15], albeit for GL_1. In Section 2.9 of Chapter 2 we find the Bessel function attached to a multiplicative character of a locally compact field K. The formula given there, formula (8) of that section defines the Bessel function of the multiplicative character π as a Mellin transform of the same expression one builds a Kloosterman sum from, that is, an additive character evaluated at $x + x^{-1}$. Moreover, in their expression (9) for the Bessel function it is presented as the convolution of a pair of Gauss sums, as appear in the functional equation of Tate's local functional equation.

Shifting focus to GL_2, from the GL_1 situation we expect that if we can attach Bessel functions to representations of $GL_2(K)$, where K is a local field, then it will be related to factors appearing in local functional equations. When asking questions about representations of local fields that occur as factors of automorphic representations or even questions about cuspidal automorphic representations, insight can often be gained by looking at the analogous questions over a finite field. One early mention of a Bessel function attached to a representation of GL_2 is over the finite field and can be found in Piatetski-Shapiro's book [20]. There he defines the Bessel function of a representation of $GL_2(K)$, where now K is a finite field. In doing so, he establishes the basic relation between the Bessel function of a representation π and the γ-factor $\gamma(\omega, \pi)$ that appears in the finite field analogue of the local functional equation of Jacquet and Langlands [17], namely that $\gamma(\omega, \pi)$ is the Mellin transform of the Bessel function. This fundamental relation over the local field has been used to great effect in recent years in proofs of Langlands' functoriality conjectures. We will review the theory of Bessel functions attached to representations of GL_2 over a finite field in Section 2 below.

Returning to the motivating question in the first paragraph, the question becomes whether there are Bessel functions associated to representations of $GL_2(\mathbb{R})$ and whether these play a role in the Kuznetsov and Voronoi formulas. The Bessel functions for representations of $PGL_2(\mathbb{R})$ were first defined in [9] following the lead from the finite field situation. The purpose of [9] was to then use these Bessel functions to establish a version of the Kuznetsov trace formula for arbitrary Fuchsian groups of the first kind $\Gamma \subset PGL_2(\mathbb{R})$. So indeed, in the case of the Kuznetsov formula, the Bessel functions that arise in the integral transforms do have a representation theoretic interpretation. The theory of Bessel functions of representations of $GL_2(\mathbb{R})$ is reviewed in Section 3 below. The question still remains for the integral transforms arising in the Voronoi formula. In Section 4 we give a derivation of the basic level 1 Voronoi summation formula for GL_2 in terms of the Bessel functions of representations. Along the same lines we show how to get the versions with additive twists and with square free level as well. Currently there are versions of Voronoi summation formula for GL_n, but none of them are in terms of Bessel functions of representations for $GL_n(\mathbb{R})$; in fact, we do not know of a definition of such Bessel functions at this time.

In Section 5 we turn to Bessel functions for GL_2 over a p-adic field. In this section we concentrate on the basic relation that the γ-factor that appears in the local functional equation should be the Mellin transform of a Bessel function.

What we find in this section is that the γ-factor/Bessel function relation satisfies one of the basic properties of integral transforms, namely that the transform of a product should be the convolution of the transforms. What would convolution mean in our context? What Soudry shows is that if one has two representations π_1 and π_2 of $GL_2(K)$ for K a p-adic field, then it is the *Rankin-Selberg convolution* factor $\gamma(s, \pi_1 \times \pi_2, \psi)$ which is obtained as the Mellin transform of the product of the Bessel functions $j_{\pi_1}(x) j_{\pi_2}(x)$. The Rankin-Selberg convolution L-functions or γ-functions are very subtle arithmetic invariants and to see the persistence of this basic property of integral transforms in this arithmetic context is something to ponder.

The fact that the local γ-factor appearing in the local functional equation can be written and analyzed in terms of a Bessel function associated to a representation has been used to great effect in recent work on functoriality and the fine analysis of local constants. In our final section we briefly discuss some of these relations and Bessel functions for groups other than GL_2.

One topic that we do not discuss is the use of Bessel functions in the relative trace formula of Jacquet. This is a pity, but it would take us too far afield. Jacquet's relative trace formula is a full representation theoretic version of the Kuznetsov trace formula. In the relative trace formula, Bessel functions of representations play an analogous role to the characters of representations in the Arthur-Selberg trace formula. Jacquet used the relative trace formula for GL_2 to reprove the result of Waldspurger on the central value of automorphic L-functions. Baruch and Mao continued Jacquet's program, and in particular undertook the local analysis of Jacquet's relative trace formula. In doing so they did extensive work on Bessel functions of representations of GL_2. I refer the reader to their papers [3, 4, 5] for more details.

This paper grew out of a question of P. Michel during the 1999/2000 special year at the IAS in Princeton. In the process of writing [18], Kowalski, Michel, and Venderkam had discovered a relation among classical Bessel functions and Euler's Γ-function that amounted to the fact that the Mellin transform of a product of Bessel functions gave the Rankin-Selberg convolution γ-factor in the archimedean context. (See the calculations in Section 6.3 of [18].) Michel asked if there was a structural reason for this fact. In order to explain this I wrote a note explaining the philosophy of the relation between Bessel functions of representations and local γ-factors that appear in Sections 2, 3, and 5 below. Given the previous work with Bessel functions of representations and the Kuznetsov trace formula in [9] and the derivation of the Voronoi summation formula in [18], it was then natural to try to derive the Voronoi formula in these terms, and this second note became Section 4 here. I hope that making these two notes available to a broader public will be of value.

I would like to think Philippe Michel for asking the questions that prompted me to originally write these notes and Dinakar Ramakrishnan for encouraging me to make them available to a wider public. Finally, I would like to thank the referee for keeping me honest.

2. Bessel functions for $GL_2(K)$, with K a finite field

When asking questions about representations of local fields that occur as factors of automorphic representations or even questions about cuspidal automorphic representations, insight can often be gained by looking at the analogous questions over a finite field. This has the advantage that one doesn't have to worry about convergence issues, but one also loses the analytic tool of ignoring sets of measure 0, so one has to have a bit of care going from the finite field to local fields or global fields. Over the finite field there are two versions of Bessel functions attached to a representation, but we find here avatars of basic relations that are more subtle to prove in the local situations.

The basic reference for what follows is the book of Piatetski-Shapiro [20] and Section 3 of [9].

2.1. **Whittaker Models.** Let K be a finite field with q elements and ψ a non-trivial additive character of K. Let $G = GL_2(K)$ and $N = \left\{ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in K \right\}$ its maximal unipotent subgroup. ψ defines a character of N by $\psi(n) = \psi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \psi(x)$.

Consider the induced representation $\mathcal{W}(\psi) = ind_N^G(\psi) = \{f : G \to \mathbb{C} \mid f(ng) = \psi(n)f(g)\}$. It is not too hard to show the following facts.

(i) $\dim \mathcal{W}(\psi) = (q-1)^2(q+1)$

(ii) $\mathcal{W}(\psi)$ is the direct sum of all irreducible representations π of G with $\dim(\pi) > 1$, each with multiplicity one.

For any π of dimension greater than 1 we let $\mathcal{W}(\pi,\psi) \subset \mathcal{W}(\psi)$ denote the corresponding constituent of $\mathcal{W}(\psi)$. $\mathcal{W}(\pi,\psi)$ is called the Whittaker model of π. Then

$$\mathcal{W}(\psi) = \oplus \mathcal{W}(\pi,\psi).$$

If we have π realized on a space V_π then we denote the intertwining map $V_\pi \longrightarrow \mathcal{W}(\pi,\psi)$ by $v \mapsto W_v(g)$ and the intertwining property becomes $\pi(g')v \mapsto W_{\pi(g')v}(g) = W_v(gg')$.

2.2. **The Bessel function** $J_\pi(g)$**.** Every representation π of dimension greater than one has a unique (up to scalar multiples) Whittaker vector (or Bessel vector) which, given the character ψ, is a non-zero vector $v_0 \in V_\pi$ satisfying $\pi(n)v_0 = \psi(n)v_0$. If we fix a non-trivial G-invariant unitary form on V_π then we can normalize the Whittaker vector v_0 by $(v_0, v_0) = 1$. The map to the Whittaker model is then given by $v \mapsto (\pi(g)v, v_0) = W_v(g)$.

Definition 2.1. *The Bessel function $J_\pi(g)$ of π is the Whittaker function of the normalized Whittaker vector, i.e., $J_\pi(g) = (\pi(g)v_0, v_0) = W_{v_0}(g)$.*

Note that we could define the Bessel function using any choice of non-zero Whittaker vector v_0 by setting

$$J_\pi(g) = W_{v_0}(I)^{-1} W_{v_0}(g)$$

since in general $W_{v_0}(I) = (v_0, v_0) \neq 0$.

Some of the elementary properties of the Bessel function are:

(i) $J_\pi(n_1 g n_2) = \psi(n_1)\psi(n_2)J_\pi(g)$ for $n_1, n_2 \in N$.

(ii) $J_\pi(I) = 1$ and $J_\pi(aI) = \omega_\pi(a)$ where ω_π is the central character of π.

(iii) $J_\pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = 0$ if $a \neq 1$.

(iv) $\overline{J_\pi(g)} = J_\pi(g^{-1})$.

Some applications of the Bessel function are the following. For any finite group H and any function f on H we let

$$\int_H f(h)dh = \sum_{h\in H} f(h)$$

to ease the analogy with the local field situation. The following can be found in [9], or can be worked out fairly quickly.

- *A formula for J_π*: For any $W_v \in \mathcal{W}(\pi,\psi)$ we have

$$\frac{1}{|N|}\int_N W_v(gn)\psi^{-1}(n)\ dn = W_v(I)J_\pi(g).$$

- *Reproducing kernel*: If $W_v \in \mathcal{W}(\pi,\psi)$ then

$$W_v(h) = \frac{\dim V_\pi}{|G|}\int_G W_v(g)J_\pi(hg^{-1})\ dg.$$

- *Spectral projector*: According to the decomposition

$$\mathcal{W}(\psi) = \oplus_\pi \mathcal{W}(\pi,\psi).$$

  for any $f \in \mathcal{W}(\psi)$ we have the decomposition

$$f(g) = \sum_\pi F_\pi(f)(g)$$

  with $F_\pi(f) \in \mathcal{W}(\pi,\psi)$. Then

$$F_\pi(f)(h) = \frac{\dim V_\pi}{|G|}\int_G f(g)J_\pi(hg^{-1})\ dg.$$

One can formulate a version of the Kuznetsov formula over the finite field as in [9]. For any $\gamma \in G$ we can define a Kloosterman distribution on $\mathcal{W}(\psi)$ by

$$f \in \mathcal{W}(\psi) \mapsto K(f,\gamma) = \frac{1}{|N|}\int_N f(\gamma n)\psi^{-1}(n)\ dn$$

Then this distribution has a spectral decomposition

$$K(f,\gamma) = \sum_\pi J_\pi(\gamma)\frac{\dim V_\pi}{|G|}\int_G f(g)J_\pi(g^{-1})\ dg.$$

This is a finite field version of the Petersson/Kuznetsov formula. We see that the Bessel function of representations play the role of the kernel function for the integral transform that appears.

2.3. **The Bessel function $j_\pi(y)$.** For some applications, a variant of the Bessel function as a function on $K^\times$ is preferable. By the Bruhat decomposition $G = B \cup NwB$ where $B = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the transformation properties of J_π given above we see that J_π is completely determined by its values $J_\pi \begin{pmatrix} 0 & 1 \\ y & 0 \end{pmatrix}$. So we set

$$j_\pi(y) = J_\pi \begin{pmatrix} 0 & 1 \\ y^{-1} & 0 \end{pmatrix}.$$

Two applications of this avatar of the Bessel function of a representation are:

• *The action of w in the Kirillov model*: Often it is useful to restrict the Whittaker functions to the diagonal: $W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. This gives the Kirillov model $\mathcal{K}(\pi, \psi)$ of π. For $v \in V_\pi$ set $\varphi_v(x) = W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. Then the action of w in this model is given by

$$\varphi_{\pi(w)v}(x) = \omega_\pi(x) \int_{K^\times} \varphi_v(y) j_\pi(xy)\ dy$$

that is, the Bessel function j_π gives an integral kernel for the action of the Weyl element w in the Kirillov model.

•*Relation with the γ-factor*: The γ-factor of a representation is defined by the relation

$$\gamma(\omega, \pi) \int_{K^\times} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \omega(x)\ dx = \int_{K^\times} W_v \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \omega(x)\ dx$$

where ω is a multiplicative character of $K^\times$. This is the analogue of the "local functional equation" of Jacquet and Langlands [17] and it holds for all $v \in V_\pi$.

In particular, if we take v to be our Whittaker vector we get the Bessel function and the above becomes

$$\gamma(\omega,\pi) = \int_{K^\times} j_\pi(x)\omega^{-1}(x)\ dx$$

that is, the γ-factor is the Mellin transform of the Bessel function.

3. Bessel functions for $GL_2(\mathbb{R})$

Now take $K = \mathbb{R}$ and (π, V_π) an irreducible infinite dimensional unitary generic representation of $G = GL_2(\mathbb{R})$. Let V_π^∞ be the space of smooth vectors for π. The reference for this section is [9].

3.1. **The Whittaker and Kirillov models.** Fix a non-trivial additive character of $\mathbb{R}$. π will have a Whittaker model $\mathcal{W}(\pi,\psi)$, that is, a realization as a subspace of the space of functions $\mathcal{W}(\psi) = ind_N^G(\psi) \subset \{f : GL_2(\mathbb{R}) \to \mathbb{C} \mid f(ng) = \psi(n)f(g)\}$ as before. Again we let the map $V_\pi \to \mathcal{W}(\pi,\psi)$ be given by $v \mapsto W_v(g)$. For $v \in V_\pi^\infty$ the functions $W_v(g)$ are smooth.

These representations of $GL_2(\mathbb{R})$ also have a Kirillov model realization on $L^2(\mathbb{R}^\times)$, denoted by $\mathcal{K}(\pi,\psi)$. If we denote the map from $V_\pi \to \mathcal{K}(\pi,\psi)$ by $v \mapsto \varphi_v(x)$ then the Whittaker and Kirillov model are related by

$$\varphi_v(x) = W_v\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

In the Kirillov model, the center Z of G acts by the central character ω_π, and the group of matrices of the form $\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right\}$ acts by

$$\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} v \mapsto \varphi_{\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} v}(x) = \psi(bx)\varphi_v(ax).$$

So this action is the same for all representations π and we see that the different Kirillov models $\mathcal{K}(\pi,\psi)$ are distinguished by the action of the center (through the central character) and the Weyl group element $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In the Kirillov model, the action of w can be given by an integral kernel $k_\pi(x,y)$ so that

$$\varphi_{\pi(w)v}(x) = \int_{\mathbb{R}^\times} k_\pi(x,y)\varphi_v(y)\ d^\times y.$$

3.2. **The Bessel function $j_\pi(x)$.** Using the kernel $k_\pi(x,y)$, we define the one variable Bessel function $j_\pi(x)$ by analogy with one of the properties of the one variable Bessel function over the finite field, namely that it should give the action of w in the Kirillov model.

Definition 3.1. *If π is an infinite dimensional unitary representation of $GL_2(\mathbb{R})$ we define $j_\pi(x) = \omega_\pi^{-1}(x)k_\pi(x,1)$.*

Then one checks that in fact for $\varphi_v \in \mathcal{K}(\pi,\psi)$ one has

$$\varphi_{\pi(w)v}(x) = \omega_\pi(x)\int_{\mathbb{R}^\times} j_\pi(xy)\varphi_v(y)\ d^\times y.$$

Note: This definition differs slightly from that taken in [9] by the inclusion of the central character in the definition of j_π. The advantage is that it makes the formulas for the action of w in the Kirillov model for $\mathbb{R}$ and the finite field agree. This should be compared with the corresponding formula over the finite field. It does not effect the results of [9] since we quickly assumed trivial central character there.

From now on, we will restrict to representations having trivial central character, i.e., representations of $PGL_2(\mathbb{R})$.

These Bessel functions can be explicitly computed. They are as follows. We take $\psi(x) = e^{2\pi i n x}$.

Let $\pi = \sigma(d)$ be holomorphic discrete series, $d = 1, 2, 3, \dots$. $\sigma(d)$ corresponds to holomorphic forms of weight $2d$, in the sense that if one takes a holomorphic modular form of weight $2d$ which is a Hecke eigenform and lifts it to an automorphic representation of GL_2, as in say [7], then the archimedean component of that representation will be isomorphic to $\sigma(d)$. Then

$$j_{\sigma(d)}(x) = (-1)^d 2\pi|n|\sqrt{x}J_{2d-1}(4\pi|n|\sqrt{x})$$

if $x > 0$ and $j_{\sigma(d)}(x) = 0$ for $x < 0$.

Let $\pi = \pi(ir)$ with $r \in \mathbb{R}$. These correspond to Maass forms. Then

$$j_{\pi(ir)}(x) = \frac{-\pi|n|\sqrt{x}}{\sin(\pi ir)}\{J_{2ir}(4\pi|n|\sqrt{x}) - J_{-2ir}(4\pi|n|\sqrt{x})\}$$

for $x > 0$ and a similar expression in I_{2ir} and I_{-2ir} for $x < 0$.

Let$\pi = \pi(r)$ with $-\frac{1}{2} < r < \frac{1}{2}$. These will correspond to exceptional eigenvalues. Then

$$j_{\pi(r)}(x) = \frac{-\pi|n|\sqrt{x}}{\sin(\pi r)}\{J_{2r}(4\pi|n|\sqrt{x}) - J_{-2r}(4\pi|n|\sqrt{x})\}$$

for $x > 0$ and a similar expression in I_{2r} and I_{-2r} for $x < 0$.

These can all be put in a single formula. If we set

$$\kappa = \begin{cases} d - \frac{1}{2} & \pi = \sigma(d) \\ ir & \pi = \pi(ir) \\ r & \pi = \pi(r) \end{cases}$$

then we have

$$j_\pi(x) = \begin{cases} \frac{-\pi|n|\sqrt{x}}{\sin(\pi\kappa)}\{J_{2\kappa}(4\pi|n|\sqrt{x}) - J_{-2\kappa}(4\pi|n|\sqrt{x})\} & x > 0 \\ \frac{-\pi|n|\sqrt{x}}{\sin(\pi\kappa)}\{I_{2\kappa}(4\pi|n|\sqrt{x}) - I_{-2\kappa}(4\pi|n|\sqrt{x})\} & x < 0 \end{cases}.$$

While these formulas are to be found in [9], there derivation is not. One can find these formulas derived in the archimedean paper of Baruch and Mao [4].

One can derive the Kloosterman-Spectral formula for modular forms for a discrete subgroup $\Gamma \subset PGL_2(\mathbb{R})$ in terms of these j_π. This is the content of the first half of [9]. It leads to a variant of the Petersson/Kuznetsov formula.

These Bessel functions are also related to the γ–factors coming from the integral representations. In terms of the Kirillov models, the γ-factors satisfy

$$\gamma(\omega, \pi, \psi) \int_{\mathbb{R}^\times} \varphi_v(x)\omega(x)\; d^\times x = \int_{\mathbb{R}^\times} \varphi_{\pi(w)v}(x)\omega^{-1}(x)\; d^\times x$$

for all $\varphi_v \in \mathcal{K}(\pi, \psi)$. In terms of the local L-function and ϵ-factors (i.e., the usual "Γ"–factors in the functional equation) we have, for $\omega(x) = |x|^{s-\frac{1}{2}}$,

$$\gamma(s, \pi, \psi) = \frac{\epsilon(s, \pi, \psi)L(1 - s, \tilde{\pi})}{L(s, \pi)}.$$

In accordance with what happens in the finite field, we once again get that γ is the Mellin transform of Bessel, i.e.,

$$\gamma(\omega,\pi,\psi)=\int_{\mathbb{R}^\times} j_\pi(x)\omega^{-1}(x)\ d^\times x.$$

4. Voronoi Summation

The Voronoi summation formula for GL_2 has a natural formulation in terms of the Bessel functions of representations of $\mathbb{R}$, through the local functional equation at the archimedean place and the relation with the γ-factor and Mellin inversion. This is more or less what the derivation of Kowalski, Michel, and Vanderkam does. (See Appendix B of [18] for their derivation of this formula. Note that their J_g is essentially j_π for $x>0$ and their K_g is essentially j_π for $x<0$.) In this section we would like to give a conceptually simple proof based on the fact that the Bessel function gives the action of w in the Kirillov model.

We start with a cuspidal automorphic representation of $GL_2(\mathbb{A})$, where for simplicity $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. Take $\varphi\in V_\pi$ which is decomposable, so under the isomorphism $V_\pi\to\otimes V_{\pi_v}$ we have $\varphi=\otimes\varphi_v$ with $\varphi_v(g_v)$ the normalized K_v-fixed vector for almost all $v<\infty$. φ_∞ will be arbitrary. We let $W(g)$ be the associated Whittaker function

$$W(g)=\int_{\mathbb{Q}\backslash\mathbb{A}}\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\psi^{-1}(x)\ dx$$

where ψ is the standard additive character for $\mathbb{A}$ trivial on $\mathbb{Q}$ as in Tate's thesis. If φ is decomposable as above then this will factor as $W(g)=\prod_v W_v(g)$ where $W_v(g)\in\mathcal{W}(\pi_v,\psi_v)$ is in the Whittaker model of the corresponding local representation. For more details on this one can consult [6] or [7]. Set $W_f=\prod_{v<\infty}W_v$ the finite part of the Whittaker function.

4.1. **Simple Voronoi summation – Level 1.**

In the level one case, here is an elementary derivation. We assume trivial central character for now. We take a cuspidal representation π such that for all $v<\infty$ we have π_v is unramified. Take $\varphi\in V_\pi$ with $\varphi=\otimes\varphi_v$ with φ_v the normalized unramified vector at all finite v.

The classical Fourier expansion of $\varphi(g_\infty)$ is then

$$\varphi(g_\infty) = \sum_n W_f \begin{pmatrix} n & \\ & 1 \end{pmatrix} W_\infty \left(\begin{pmatrix} n & \\ & 1 \end{pmatrix} g_\infty \right)$$

so $\varphi(I) = \sum_n a(n)F(n)$ where

$$a(n) = W_f \begin{pmatrix} n & \\ & 1 \end{pmatrix} \qquad \text{and} \qquad F(x) = W_\infty \begin{pmatrix} x & \\ & 1 \end{pmatrix} \in \mathcal{K}(\pi_\infty, \psi).$$

Now, by invariance of φ under the rational points we (essentially) have

$$\begin{aligned} \varphi(I) = \varphi(w) &= \sum_n W_f \left(\begin{pmatrix} n & \\ & 1 \end{pmatrix} w \right) W_\infty \left(\begin{pmatrix} n & \\ & 1 \end{pmatrix} w \right) \\ &= \sum_n W_f \begin{pmatrix} n & \\ & 1 \end{pmatrix} W_\infty \left(\begin{pmatrix} n & \\ & 1 \end{pmatrix} w \right) \\ &= \sum_n a(n) \int_{\mathbb{R}^\times} j_{\pi_\infty}(ny) F(y) \; d^\times y \end{aligned}$$

since convolution with j_{π_∞} gives the action of w in the Kirillov model of π_∞.

Hence we have the Voronoi summation formula in this simple case:

$$\sum_n a(n)F(n) = \sum_n a(n) \int_{\mathbb{R}^\times} j_{\pi_\infty}(ny) F(y) \; d^\times y.$$

The different representations are distinguished by the action of w, that it, their Bessel functions.

The proof in the more complicated situation of [18] is now simply the question of (1) level and (2) using a W-operator in place of simply w.

4.2. **Voronoi summation with additive twists – Level 1.** We start with a cuspidal automorphic representation π of $GL_2(\mathbb{A})$ with trivial central character as in the previous section, $\varphi \in V_\pi$ factorizable, $W(g)$ its Whittaker function.

First note that

$$\varphi(1) = \varphi(1_\infty, 1_f) = \sum_{\gamma \in \mathbb{Q}^\times} W \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} = \sum_{\gamma \in \mathbb{Q}^\times} W_f \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} W_\infty \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}.$$

Now $W_f\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} = 0$ unless $\gamma \in \mathbb{Z}$, so that

$$\varphi(1) = \sum_{n\in\mathbb{Z}}{}' W_f\begin{pmatrix}n & \\ & 1\end{pmatrix} W_\infty\begin{pmatrix}n & \\ & 1\end{pmatrix} = \sum_{n\in\mathbb{Z}}{}' a(n)F(n)$$

where $a(n) = W_f\begin{pmatrix}n & \\ & 1\end{pmatrix}$ is the Fourier coeffecient and $F(x) = W_\infty\begin{pmatrix}x & \\ & 1\end{pmatrix}$ is in the Kirillov model $\mathcal{K}(\pi_\infty, \psi_\infty)$ of π_∞ and $\psi_\infty(x) = e(x) = e^{2\pi i x}$.

Let $a, c \in \mathbb{Z}$ with $c \neq 0$ and $(a, c) = 1$. Complete this to a 2×2 matrix

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$$

and let

$$\delta = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & \\ & c^2\end{pmatrix} w^{-1} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix} & 1\\ -c^2 & \end{pmatrix} = \begin{pmatrix}-bc^2 & a\\ -dc^2 & c\end{pmatrix} \in GL_2(\mathbb{Q}).$$

Now consider

$$\varphi(1_\infty, \delta_f) = \sum_{\gamma\in\mathbb{Q}^\times} W_f\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\delta\right) W_\infty\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}.$$

By a simple matrix calculation (via Bruhat) we have

$$\delta = \begin{pmatrix}1 & \frac{a}{c}\\ & 1\end{pmatrix}\begin{pmatrix}c & \\ & c\end{pmatrix}\alpha$$

where

$$\alpha = \begin{pmatrix}-1 & 0\\ -dc^2 & 1\end{pmatrix} \in SL_2(\mathbb{Z}).$$

Hence

$$W_f\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\delta\right) = \psi_f\left(\gamma\left(\frac{a}{c}\right)\right) W_f\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}$$

and as before for this to be non-zero we must have $\gamma = n \in \mathbb{Z}$. Then $\psi_f(\gamma(a/c)) = \psi_f(na/c) = \psi_\infty(-na/c)$ the last equality following from the fact that $\psi = \psi_\infty\psi_f$

is trivial on $\mathbb{Q}$. Hence we have

$$\varphi(1_\infty, \delta_f) = {\sum_{n\in\mathbb{Z}}}' e(-an/c) W_f \begin{pmatrix} n & \\ & 1 \end{pmatrix} W_\infty \begin{pmatrix} n & \\ & 1 \end{pmatrix} = {\sum_{n\in\mathbb{Z}}}' e(-an/c) a(n) F(n)$$

and we have achieved our additive twists.

For the other side, we use the fact that $\varphi(g)$ is left invariant under $PGL_2(\mathbb{Q})$, and in particular under

$$w \begin{pmatrix} 1 & \\ & c^{-2} \end{pmatrix} = \begin{pmatrix} 0 & -c^{-2} \\ 1 & 0 \end{pmatrix}$$

so that

$$\begin{aligned} \varphi(1_\infty, \delta_f) &= \varphi\left(w \begin{pmatrix} 1 & \\ & c^{-2} \end{pmatrix}, w \begin{pmatrix} 1 & \\ & c^{-2} \end{pmatrix} \delta_f \right) \\ &= \sum_{\gamma\in\mathbb{Q}^\times} W_f \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & \\ & c^{-2} \end{pmatrix} \delta \right) W_\infty \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & \\ & c^{-2} \end{pmatrix} \right). \end{aligned}$$

Now another simple matrix calculation gives

$$w \begin{pmatrix} 1 & \\ & c^{-2} \end{pmatrix} \delta = \begin{pmatrix} 0 & -c^{-2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -c^2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\bar{a}/c \\ & 1 \end{pmatrix} \eta$$

where $\bar{a}$ is an inverse to $a \mod c$ and

$$\eta = \begin{pmatrix} d - \bar{a}bc & \frac{1}{c}(a\bar{a} - 1) \\ -bc^2 & a \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Thus

$$\begin{aligned} W_f \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & \\ & c^{-2} \end{pmatrix} \delta \right) &= W_f \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -\bar{a}/c \\ & 1 \end{pmatrix} \eta \right) \\ &= \psi_f(-\gamma\bar{a}/c) W_f \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \end{aligned}$$

and again we have $\gamma = n \in \mathbb{Z}$ and $\psi_f(-n\bar{a}/c) = e(n\bar{a}/c)$. Hence

$$\begin{aligned}
\varphi(1_\infty, \delta_f) &= \varphi\left(w\begin{pmatrix}1 & \\ & c^{-2}\end{pmatrix}, w\begin{pmatrix}1 & \\ & c^{-2}\end{pmatrix}\delta_f\right) \\
&= \sum_{n\in\mathbb{Z}}{}' e(n\bar{a}/c) W_f\begin{pmatrix}n & \\ & 1\end{pmatrix} W_\infty\left(\begin{pmatrix}n & \\ & 1\end{pmatrix} w \begin{pmatrix}1 & \\ & c^{-2}\end{pmatrix}\right) \\
&= \sum_{n\in\mathbb{Z}}{}' e(n\bar{a}/c) a(n) W_\infty\left(\begin{pmatrix}nc^{-2} & \\ & 1\end{pmatrix} w\right)
\end{aligned}$$

and since the action of w in the Kirillov model is given by convolution with the Bessel function j_{π_∞} we have

$$\varphi(1_\infty, \delta_f) = \sum_{n\in\mathbb{Z}}{}' a(n)e(n\bar{a}/c)\int_{\mathbb{R}^\times} F(x) j_{\pi_\infty}\left(\frac{nx}{c^2}\right)\, d^\times x.$$

So, equating our two expressions for $\varphi(1_\infty, \delta_f)$ we have the Voronoi summation formula with additive twists

$$\sum_{n\in\mathbb{Z}}{}' a(n)e(-an/c)F(n) = \sum_{n\in\mathbb{Z}}{}' a(n)e(n\bar{a}/c)\int_{\mathbb{R}^\times} F(x) j_{\pi_\infty}\left(\frac{nx}{c^2}\right)\, d^\times x$$

for all $F \in \mathcal{K}(\pi_\infty, \psi_\infty)$.

4.3. **Voronoi summation with additive twists – square free Level D.** Let D be a square free integer, $D > 1$. We begin with a cuspidal automorphic representation $\pi = \otimes'\pi_v$ of $GL_2(\mathbb{A})$ corresponding to forms of level D with Nebentypus character of finite order mod D. Therefore, if we let

$$K_0(p) = \left\{k = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in GL_2(\mathbb{Z}_p) \mid c \equiv 0 \mod p\right\}$$

then for each $p|D$ we have that there is a unique (normalized) vector φ_p° such that for each $k \in K_0(p)$ as above we have $\pi_p(k)\varphi_p^\circ = \omega_p(d)\varphi_p^\circ$ where ω_p is the central character of π_p. At those finite p which do not divide D we have a unique (normalized) K_p fixed vector φ_p°. The component π_∞ will have central character ω_∞ either trivial or the sign character.

We take $\varphi \in V_\pi$ a cuspidal automorphic form and assume that $\varphi = \otimes\varphi_v$ is decomposable and for each $p < \infty$ we have $\varphi_p = \varphi_p^\circ$. As always, φ_∞ is arbitrary. If we let $W(g)$ be the Whittaker function associated to φ as before, we have $W = W_f W_\infty$ and now W_f is satisfies $W_f(gk) = \omega_f(d)W_f(g)$ for $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(D) = \prod_{p|D} K_0(p) \prod_{p\nmid D} K_p$. Note that $GL_2(\mathbb{Q}) \cap K_0(D) = \Gamma_0(D)$.

As before we have

$$\varphi(1) = \varphi(1_\infty, 1_f) = \sum_{\gamma\in\mathbb{Q}^\times} W\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} = \sum_{\gamma\in\mathbb{Q}^\times} W_f\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} W_\infty\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}$$
$$= \sum_{n\in\mathbb{Z}}{}' W_f\begin{pmatrix} n & \\ & 1 \end{pmatrix} W_\infty\begin{pmatrix} n & \\ & 1 \end{pmatrix} = \sum_{n\in\mathbb{Z}}{}' a(n)F(n).$$

Fix integers a, c such that $(a, c) = 1$. Let $D_1 = (c, D)$ and write $D = D_1D_2$ and $c = D_1c_2$. Since D is square free we have $(D_2, c) = 1$. Then we also have $(aD_2, c) = 1$. We extend the relatively prime pair aD_2, c to a 2×2 matrix $\begin{pmatrix} aD_2 & b \\ c & d \end{pmatrix}$. Note that

$$\begin{pmatrix} aD_2 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{aD_2}{c} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -dc & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & \\ & c \end{pmatrix} w.$$

Hence if we set

$$\delta = \begin{pmatrix} aD_2 & b \\ c & d \end{pmatrix} w^{-1} \begin{pmatrix} c & \\ & c^{-1} \end{pmatrix} \begin{pmatrix} D_2 & \\ & 1 \end{pmatrix}$$

we have

$$\delta = \begin{pmatrix} 1 & \frac{aD_2}{c} \\ & 1 \end{pmatrix} \begin{pmatrix} D_2 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -dcD_2 & 1 \end{pmatrix}$$

and that

$$\begin{pmatrix} 1 & 0 \\ -dcD_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -dc_2D & 1 \end{pmatrix} \in \Gamma_0(D).$$

Write $g \in GL_2(\mathbb{A})$ as $g = (g_\infty, g_f)$ with $g_\infty \in GL_2(\mathbb{R})$ and $g_f \in GL_2(\mathbb{A}_f)$.

Now consider

$$\begin{aligned}
\varphi\left(\begin{pmatrix} D_2 & \\ & 1\end{pmatrix}_\infty, \delta_f\right) &= \sum_{\gamma\in\mathbb{Q}^\times} W_f\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\delta\right) W_\infty\begin{pmatrix}\gamma D_2 & \\ & 1\end{pmatrix}\\
&= \sum_{\gamma\in\mathbb{Q}^\times} W_f\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \frac{aD_2}{c} \\ & 1\end{pmatrix}\begin{pmatrix}D_2 & \\ & 1\end{pmatrix}\right) W_\infty\begin{pmatrix}\gamma D_2 & \\ & 1\end{pmatrix}\\
&= \sum_{\gamma\in\mathbb{Q}^\times} \psi_f(\gamma a D_2/c) W_f\begin{pmatrix}\gamma D_2 & \\ & 1\end{pmatrix} W_\infty\begin{pmatrix}\gamma D_2 & \\ & 1\end{pmatrix}.
\end{aligned}$$

As before, this forces $\gamma D_2 = n \in \mathbb{Z}$. So we write $\gamma = n/D_2$ we get

$$\begin{aligned}
\varphi\left(\begin{pmatrix} D_2 & \\ & 1\end{pmatrix}_\infty, \delta_f\right) &= {\sum_{n\in\mathbb{Z}}}' \psi_f\left(\frac{n}{D_2}\frac{aD_2}{c}\right) W_f\begin{pmatrix} n & \\ & 1\end{pmatrix} W_\infty\begin{pmatrix} n & \\ & 1\end{pmatrix}\\
&= {\sum_{n\in\mathbb{Z}}}' e\left(-\frac{na}{c}\right) a(n)F(n).
\end{aligned}$$

This is the left hand side of our summation formula.

To get the right hand side, we use that φ is left invariant under $GL_2(\mathbb{Q})$ and left translate by the element

$$\begin{pmatrix} c^{-1} & \\ & c\end{pmatrix} w = \begin{pmatrix} 0 & -c^{-1} \\ c & 0\end{pmatrix}.$$

Then

$$\begin{aligned}
\varphi\left(\begin{pmatrix} D_2 & \\ & 1\end{pmatrix}_\infty, \delta_f\right) &= \varphi\left(\begin{pmatrix} 0 & -c^{-1} \\ c & 0\end{pmatrix}\begin{pmatrix} D_2 & \\ & 1\end{pmatrix}_\infty, \begin{pmatrix} 0 & -c^{-1} \\ c & 0\end{pmatrix}\delta_f\right)\\
&= \sum_{\gamma\in\mathbb{Q}^\times} W_f\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix} 0 & -c^{-1} \\ c & 0\end{pmatrix}\delta\right)\\
&\qquad W_\infty\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix} 0 & -c^{-1} \\ c & 0\end{pmatrix}\begin{pmatrix} D_2 & \\ & 1\end{pmatrix}\right).
\end{aligned}$$

Now one computes that

$$\begin{pmatrix} 0 & -c^{-1} \\ c & 0\end{pmatrix}\delta = \begin{pmatrix} 1 & -\frac{\overline{aD_2}}{c} \\ & 1\end{pmatrix} W_{D_2}$$

where $\overline{aD_2}$ is an inverse to $aD_2 \mod c$ and W_{D_2} is an Atkin-Lehner involution as in [18]:

$$W_{D_2} = \begin{pmatrix} -D_2(b + b\overline{aD_2}) & \frac{1}{c}(\overline{aD_2}aD_2 - 1) \\ bcc_2D & aD_2 \end{pmatrix}$$

and that

$$\begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix} \begin{pmatrix} D_2 & \\ & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{c^2 D_2} & \\ & 1 \end{pmatrix} \begin{pmatrix} cD_2 & \\ & cD_2 \end{pmatrix} w.$$

Thus

$$W_f \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix} \delta \right) = e(\gamma\overline{aD_2}/c) W_f \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} W_{D_2} \right)$$

and

$$\begin{aligned} W_\infty \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix} \begin{pmatrix} D_2 & \\ & 1 \end{pmatrix} \right) &= W_\infty \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{c^2 D_2} & \\ & 1 \end{pmatrix} \begin{pmatrix} cD_2 & \\ & cD_2 \end{pmatrix} w \right) \\ &= \omega_\infty(cD_2) W_\infty \left(\begin{pmatrix} \frac{\gamma}{c^2 D_2} & \\ & 1 \end{pmatrix} w \right) \\ &= \omega_\infty(cD_2)\omega_\infty(\gamma/c^2 D_2) \int_{\mathbb{R}^\times} F(x) j_{\pi_\infty}(\gamma x / c^2 D_2) \, d^\times x \\ &= \omega_\infty(\gamma/c) \int_{\mathbb{R}^\times} F(x) j_{\pi_\infty}(\gamma x / c^2 D_2) \, d^\times x. \end{aligned}$$

So we have

$$\begin{aligned} \varphi \left(\begin{pmatrix} D_2 & \\ & 1 \end{pmatrix}_\infty, \delta_f \right) = \sum_{\gamma \in \mathbb{Q}^\times} e(\gamma\overline{aD_2}/c) W_f \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} W_{D_2} \right) \omega_\infty(\gamma/c) \times \\ \times \int_{\mathbb{R}^\times} F(x) j_{\pi_\infty}(\gamma x / c^2 D_2) \, d^\times x. \end{aligned}$$

Now, an elementary matrix calculation shows that if $m \in \mathbb{Z}$ then

$$\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} W_{D_2} = W_{D_2} \eta_m$$

with $\eta_m \in \Gamma_0(D)$. Hence we have for $m \in \mathbb{Z}$

$$\begin{aligned} W_f\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} W_{D_2}\right) &= W_f\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} W_{D_2}\eta_m\right) \\ &= W_f\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix}1 & m \\ & 1\end{pmatrix} W_{D_2}\right) \\ &= \psi_f(\gamma m) W_f\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} W_{D_2}\right)\end{aligned}$$

and hence this is 0 unless $\psi_f(\gamma m) = e(-\gamma m) = 1$ for all $m \in \mathbb{Z}$. So we must have $\gamma = n \in \mathbb{Z}$. So the above becomes

$$\varphi\left(\begin{pmatrix}D_2 & \\ & 1\end{pmatrix}_\infty, \delta_f\right) = \sum_{n\in\mathbb{Z}}{}' e(n\overline{aD_2}/c) W_f\left(\begin{pmatrix}n & \\ & 1\end{pmatrix} W_{D_2}\right)\omega_\infty(n/c) \times \\ \times \int_{\mathbb{R}^\times} F(x) j_{\pi_\infty}(nx/c^2D_2)\, d^\times x.$$

Now, let $\varphi'(g) = \varphi(gW_{D_2})$. Then this will have a Fourier expansion

$$\varphi'(g_\infty, 1_f) = \sum_{n\in ZZ}{}' a'(n) W'_\infty\left(\begin{pmatrix}n & \\ & 1\end{pmatrix} g_\infty\right)$$

with

$$a'(n) = W'_f\begin{pmatrix}n & \\ & 1\end{pmatrix} = W_f\left(\begin{pmatrix}n & \\ & 1\end{pmatrix} W_{D_2}\right).$$

With this notation the above becomes

$$\varphi\left(\begin{pmatrix}D_2 & \\ & 1\end{pmatrix}_\infty, \delta_f\right) = \sum_{n\in\mathbb{Z}}{}' e(n\overline{aD_2}/c) a'(n)\omega_\infty(n/c)\int_{\mathbb{R}^\times} F(x) j_{\pi_\infty}(nx/c^2D_2)\, d^\times x.$$

This is our left hand side.

Equating our two expressions for $\varphi\left(\begin{pmatrix}D_2 & \\ & 1\end{pmatrix}_\infty, \delta_f\right)$ we get our summation formula

$$\sum_{n\in\mathbb{Z}}{}' e(-an/c)a(n)F(n) = \sum_{n\in\mathbb{Z}}{}' e(n\overline{aD_2}/c) a'(n)\omega_\infty(n/c)\int_{\mathbb{R}^\times} F(x) j_{\pi_\infty}(nx/c^2D_2)\, d^\times x$$

where the $a(n)$ are the Fourier coefficients of $\varphi(g)$, the $a'(n)$ are the Fourier coefficients of $\varphi'(g) = \varphi(gW_{D_2})$, and $F \in \mathcal{K}(\pi_\infty, \psi_\infty)$.

Note that Kowalski, Michel, and Vanderkam [18] have a slightly different form and use a slightly different W_{D_2}. Also recall that either $\omega_\infty(y) \equiv 1$ or $\omega_\infty(y) = sgn(y)$.

4.4. **Voronoi summation for GL_n.** Recently Ichino and Templier have developed an adelic version of the Voronoi summation formula for GL_n [16]. This is based on the functional equation for the Rankin-Selberg convolution for $GL_n \times GL_1$. The transforms that arise there are based on the local functional equations, which define the local γ-factors $\gamma(s, \pi_v \times \chi_v, \psi_v)$. The philosophy presented here is that these local factors, and the transforms that occur in the Voronoi formula, should come from Bessel functions of representations. The theory of Bessel functions for $GL_n(\mathbb{R})$, other than the case $n = 2$ treated above, is not well developed. Zhi Qi is working on the theory of archimedean Bessel functions for GL_n in his OSU thesis.

5. Bessel functions for $GL_2(K)$ with K a p-adic field

The main reference for this section is the paper of Soudry [22]. It illustrates an interesting phenomenon in the relation between Bessel functions and γ-factors. We will follow Soudry's normalizations, which are a bit different from the above. Let q be the order of the residue field of K, i.e., $q = |\mathfrak{o}/\mathfrak{p}|$.

5.1. **Whittaker models and Kirillov models.** Again take π to be an infinite dimensional irreducible generic representation of $GL_2(K)$. Fix a non-trivial character ψ of K. Such π will have a Whittaker model $\mathcal{W}(\pi, \psi)$ and Kirillov model $\mathcal{K}(\pi, \psi)$ as in the real case, again related by

$$\varphi_v(x) = W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \text{ for } v \in V_\pi.$$

5.2. **The Bessel function $j_\pi(x)$.** In this context Soudry defines the Bessel function in a way related to the formula for the Bessel function J_π over the finite field. If $W_v \in \mathcal{W}(\pi, \psi)$ then the integral

$$\ell(v, x) = \int_N W_v \left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right) \psi^{-1}(n)\, dn$$

converges in the sense of stabilizing as one integrates over an exhaustive system of compact open subgroups of N. For fixed x, this defines a Whittaker functional on V_π, as does the functional $v \mapsto W_v(I)$. Whittaker functionals are unique, so these functionals differ by a constant.

Definition 5.1. *The Bessel function $j_\pi(x)$ is the constant of proportionality (a function of x as $\ell(v,x)$ varies), that is,*

$$\int_N W_v\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right)\psi^{-1}(n)\ dn = j_\pi(x)W_v(I).$$

This does in fact give the action of w in the Kirillov model as above, in the sense that if v is such that $W_v\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \varphi_v(x) \in \mathcal{S}(K^\times) \subset \mathcal{K}(\pi,\psi)$ ($\mathcal{S}(K^\times)$ the Schwartz-Bruhat space of locally constant compactly supported functions) then

$$W_v\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix} = \int_{K^\times} \omega_\pi^{-1}(y) j_\pi(xy) W_v\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\ d^\times y.$$

We have the local γ-factor as before, which we will now write as

$$\gamma(s,\pi,\psi) = \frac{\epsilon(s,\pi,\psi)L(1-s,\tilde{\pi})}{L(s,\pi)}.$$

Unfortunately, Soudry never shows that $\gamma(s,\pi,\psi)$ is the Mellin transform of $j_\pi(x)$, but I believe this is not difficult. Instead he does something more intriguing – he computes the Mellin transform of the product of two Bessel functions. In particular, he proves the following (his Lemma 4.5)

$$\gamma(s,\pi_1\times\pi_2,\psi) = \int_{|c|\le q^l} \psi^{-1}(c)\omega^{-1}(c)|c|^{2(1-s)} \int_{|cx^2|\le q^j} j_{\pi_1}(x) j_{\pi_2}(x)\omega^{-1}(x)|x|^{-s}\ d^\times x\ d^\times c$$

for l and j sufficiently large and $\omega = \omega_{\pi_1}\omega_{\pi_2}$. So, the Mellin transform of the product of the Bessel functions is the *Rankin–Selberg convolution* γ-factor.

The Rankin-Selberg convolution L-functions or γ-factors are very subtle arithmetic invariants and to see the persistence of this basic property of integral transforms, that the transform of a product is given by a convolution, in this arithmetic context is intriguing. It is a property we expect to persist in other situations, but it is only in the GL_2 situation that we know of proofs. The archimedean version of this relation is essentially what is calculated in Section 6.3 of [18].

6. Bessel functions for $G(K)$ with K a local field

The paradigm of writing local γ-factors as Mellin transforms of appropriate Bessel functions for groups G other than GL_2 has been used with remarkable success recently. However, there are differences.

Once the group G is of higher rank, we find that Bessel functions aren't just attached to representations π but rather to pairs consisting of π and a Weyl group element w that supports a Bessel function. Not all Weyl group elements support Bessel functions; to do so, they must be able to be written as $w = w_\ell \cdot w_\ell^M$ where w_ℓ is the long Weyl element for G itself and w_ℓ^M is the long Weyl element for a proper Levi subgroup M of G. The rationale for this can be found in [10] or [11]. The Bessel function $j_{\pi,w}$ associated to π and the Weyl element w is essentially supported on the Bruhat cell $C(w)$ associated to w. As we move from one Bruhat cell $C(w)$ to one on its boundary, say $C(w')$, we expect the Bessel function $j_{\pi,w}$ to approach $j_{\pi,w'}$ asymptotically [1]. In the case of GL_2 discussed above, there are a dearth of Weyl elements and what we have referred to as "the" Bessel function of a representation is the one attached to the long Weyl element. (The Bessel function attached to the trivial Weyl element is essentially the central character of the representation). Bessel functions associated to relevant cells take various forms depending on one's starting point. Examples for K a p-adic field can be found in the work of Baruch for GL_n [2], Lapid and Mao for split groups [19], and in our work [10, 11].

As the rank increases, the relation between Bessel functions and γ-factors also becomes more complicated. To analyze a given γ-factor, one has to find the appropriate Weyl element w and often, for analytic reasons, one has to deal with only partial Bessel functions (again, see [10] and [11]). The γ-factors and Bessel functions discussed above all have their origins in the theory of integral representations, but there are also γ-factors that arise in the Langlands-Shahidi method. As it turns out, these can also be written as Mellin transforms of appropriate (partial) Bessel functions [21].

One of the primary uses of this relation in the p-adic case is to show that γ-factors become stable under highly ramified twists [10, 12]. Roughly this takes the following form. Suppose π_1 and π_2 are generic representations of $G(K)$ with the same central character. Then for all sufficiently highly ramified characters χ of $K^\times$ we have

$$\gamma(s, \pi_1 \times \chi, \psi) = \gamma(s, \pi_2 \times \chi, \psi).$$

Relations of this sort are established by writing $\gamma(s, \pi, \psi)$ as the Mellin transform of an appropriate Bessel function and then analyzing the asymptotics of the associated Bessel function. Such stability results were crucial in establishing functoriality from the classical groups to GL_n via the method of L-functions [8, 13], where they were used to finesse the lack of the Local Langlands Conjecture for the classical groups. Currently we are using such stabilities to analyze the exterior and symmetric square ε-factors for $GL_n(K)$ [14].

We know of no instances of Bessel functions of representations for $G(\mathbb{R})$ when G is not GL_2. In light of the connections with Kuznetsov and Voronoi mentioned above, such Bessel functions would be quite interesting. It seems that this gap may be partially filled in the forthcoming thesis of Zhi Qi at OSU.

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