Indian J. Pure Appl. Math., **45**(5): 583-632, October 2014

ON THE GREATEST PRIME FACTOR OF $ab+1$

Étienne Fouvry[1]

Univ. Paris Sud, Laboratoire de Mathématiques d'Orsay, CNRS,
F-91405 *Orsay Cedex, France*
e-mail: Etienne.Fouvry@math.u-psud.fr

(*Received* 5 *November* 2013; *after final revision* 7 *August* 2014;
accepted 14 *August* 2014)

Abstract. We improve some results on the size of the greatest prime factor of the integers of the form $ab+1$ where a and b belong to some general given finite sequences $\mathcal{A}$ and $\mathcal{B}$ with rather large density.

Key words : Greatest prime factor; primes in arithmetic progressions

1. Introduction

Let N be an integer $\geqslant 1$ and let $\mathcal{A}$ and $\mathcal{B}$ be two sets of integers, both included in $[1,N]$. With these two finite sets, we build the set $\mathcal{C}=\mathcal{C}(\mathcal{A},\mathcal{B})$, defined by

$$\mathcal{C}=\mathcal{C}(\mathcal{A},\mathcal{B}):=\big\{ab+1\,;\ a\in\mathcal{A},\,b\in\mathcal{B}\big\}.$$

Let $P^+(n)$ be the greatest prime factor of the integer n if $n\geqslant 2$ and $P(1)=1$. The purpose of this paper is to give a lower bound for the integer

$$\Gamma^+(\mathcal{A},\mathcal{B},N):=\max_{c\in\mathcal{C}}P^+(c),$$

in terms of N and of the cardinalities $|\mathcal{A}|$ and $|\mathcal{B}|$. The interest of this question is that we make no assumption on the regularity of the sets $\mathcal{A}$ and $\mathcal{B}$, but we only impose some lower bound for $|\mathcal{A}|$ and $|\mathcal{B}|$. The purpose of the present paper is to improve the following

[1]The author benefited from the financial support of Institut Universitaire de France.

Theorem A. *(See* [27, Theorem 2]*) For any positive ϵ, there exist positive constants c_1, c_2 and c_3, depending at most on ϵ, in an effective way, such that, for any $N \geqslant c_1$, for any subsets $\mathcal{A}$ and $\mathcal{B}$ of $[1, N]$, satisfying the inequalities*

$$|\mathcal{A}|,\ |\mathcal{B}| \geqslant c_2 \frac{N}{((\log N)/\log\log N)^{\frac{1}{2}}},$$

we have the inequality

$$\Gamma^+(\mathcal{A}, \mathcal{B}, N) \geqslant \min\Big\{ N^{1+(1-\epsilon)(\min(|\mathcal{A}|, |\mathcal{B}|)/N)^2}, c_3(N/\log N)^{\frac{4}{3}} \Big\}. \tag{1}$$

It is important to note that, in the particular case where $\mathcal{A}$ and $\mathcal{B}$ have positive density, (which means that they satisfy $|\mathcal{A}|,\ |\mathcal{B}| \geqslant \delta N$, for some fixed positive δ and $N \to +\infty$), we then have $\Gamma^+(\mathcal{A}, \mathcal{B}, N) \gg_\delta N^{1+\delta_1}$, where δ_1 is a positive function of δ. However the relation (1) never produces a lower bound better than $N^{\frac{4}{3}}$.

Actually, much more is conjectured since in [24, Conj.1], the authors propose the following

Conjecture 1. *For every ϵ satisfying $0 < \epsilon < 1$, there exists $N(\epsilon)$ and $C(\epsilon) > 0$, such that, for every integer $N \geqslant N(\epsilon)$, for every $\mathcal{A}$ and $\mathcal{B} \subset [1, \dots, N]$ satisfying*

$$|\mathcal{A}|,\ |\mathcal{B}| > \epsilon N, \tag{2}$$

we have the inequality

$$\Gamma^+(\mathcal{A}, \mathcal{B}, N) \geqslant C(\epsilon) N^2. \tag{3}$$

Such a conjecture becomes false if we only impose the lower bounds $|\mathcal{A}|$ and $|\mathcal{B}| \geqslant \epsilon(N) \cdot N$, where $\epsilon(N)$ is a function of N tending to 0 as slowly as we want, when N tends to infinity. To see this, choose p a prime satisfying $(2\epsilon(N))^{-1} \leqslant p \leqslant \epsilon(N)^{-1}$ and consider $\mathcal{A} = \{a \leqslant N\,;\ a \equiv 1 \bmod p\}$ and $\mathcal{B} = \{b \leqslant N\,;\ b \equiv -1 \bmod p\}$. For such $\mathcal{A}$ and $\mathcal{B}$, we easily see that $\Gamma^+(\mathcal{A}, \mathcal{B}, N) \leqslant (N^2+1)/p = o(N^2)$.

Before stating our results, we first give some general considerations on the set $\mathcal{C}$.

1.1. The subset of Linnik–Vinogradov.

Let

$$\mathcal{LV}(N) := \{n\,; n \leqslant N^2,\ n = ab \text{ with } 1 \leqslant a,\, b \leqslant N\}.$$

Hence $\mathcal{LV}(N)$ is the set of (distinct) products of two integers $\leqslant N$. The study of the cardinality of this set is not an easy task at all, this a question due to Linnik and

Vinogradov. K.Ford [8, Corollary 3] has now solved this problem by proving

$$|\,\mathcal{LV}(N)\,| \asymp \frac{N^2}{(\log N)^{c_4}(\log\log N)^{\frac{3}{2}}}, \tag{4}$$

where c_4 has the value $c_4 = 1 - \frac{1+\log\log 2}{\log 2} = 0.086\,07\ldots$. (for a slightly weaker result see [16, Theorem 23]). The relation (4) shows that $\mathcal{LV}(N)$ is a sparse subset of $[1, N^2]$, but only by a tiny power of $\log N$.

Hence, for any $\mathcal{A}$ and $\mathcal{B}$, we have the trivial inclusion

$$\mathcal{C}(\mathcal{A},\mathcal{B}) \subset \mathcal{LV}(N) + \{1\},$$

which shows that $\mathcal{C}$ lives in a fairly sparse subset of $[1, N^2 + 1]$.

To complete the description of the scenery of our problem, we recall the basic properties of the classical function $\Psi(x, y)$, which counts the integers less than x with all their prime factors less than y. In other words, we define

$$S(x, y) := \{n \leqslant x\,;\ P^+(n) \leqslant y\}$$

and

$$\Psi(x, y) := |S(x, y)|.$$

We only appeal to the rather easy result

$$\Psi(x, y) = x\rho\Big(\frac{\log x}{\log y}\Big) + O\Big(\frac{x}{\log y}\Big),$$

uniformly for $x \geqslant y \geqslant 2$ (see [28, Théorème 6, p.371], for instance). Here ρ is the Dickman function (see [28, p.370]). This function quickly goes to zero, since it satisfies

$$\rho(u) \leqslant 1/\Gamma(u + 1),\ (u > 0).$$

Using the above formula, the Stirling formula, and the inclusion–exclusion principle, we see that

$$\Big(\mathcal{LV}(N) + \{1\}\Big) \cap \Big([1, \cdots, N^2 + 1] \setminus S(N^2 + 1, y)\Big) \neq \emptyset,$$

as soon as N is sufficiently large and y satisfies

$$y \geqslant \exp\Big(c_5 \frac{\log N\ \log\log\log N}{\log\log N}\Big),$$

where c_5 is some absolute positive constant. This means that, with a naive approach, we proved that the shifted Linnik–Vinogradov set $\mathcal{LV}(N)+\{1\}$, contains an element divisible by a prime

$$p > N^{c_5 \frac{\log\log\log N}{\log\log N}}. \tag{5}$$

We now state our results. They correspond to three different situations, which appear to be more and more difficult. We can already feel the depth of Conjecture 1 in the very particular case $\mathcal{A}=\mathcal{B}=[1,\dots,N]$ (this corresponds to the condition (2) with $\epsilon=1$, and $>$ replaced by $\geqslant$). This very particular situation will be the object of Theorem 1.

1.2. **The case $\mathcal{A}=\mathcal{B}=[1,\dots,N]$.** Our first step will be to prove

Theorem 1. *For every $A>0$, there exists $N_0=N_0(A)$, such that, for every $N\geqslant N_0$, the interval $[(1-(\log N)^{-A})N^2, N^2]$ contains a prime p of the form $p=ab+1$, where a and b are integers satisfying $1\leqslant a,b\leqslant N$.*

In particular, for $N\geqslant N_0(A)$ we have the inequality

$$\Gamma^+(\mathcal{A},\mathcal{B},N) \geqslant \Big(1-\frac{1}{(\log N)^A}\Big)N^2,$$

under the constraints $\mathcal{A}=\mathcal{B}=[1,\dots,N]$.

Such a result has to be compared with the weak result given in (5) and it is far from being trivial by the tools which will be involved. Theorem 1 implies that the inequality (3) of Conjecture 1 is true for any $C(\epsilon)<1$, in the particular case $\mathcal{A}=\mathcal{B}=[1,\dots,N]$. Its proof will be given in §5, one of its qualities is to give a first idea of the difficulty of the proof of Conjecture 1, if such a proof exists. In our proof, we shall appeal to the Siegel–Walfisz Theorem concerning primes in arithmetic progressions. This fact prevents to produce an effective value for $N_0(A)$, above. The same remark applies to Theorems 2 and 3 below.

A very delicate question is to find the asymptotic expansion of the cardinality of the set of primes belonging to $\mathcal{LV}(N)+\{1\}$. This question was treated in [17, Corollary 3], [8] and finally in [19, Corollary 1.1] which gives the asymptotic order of magnitude of this cardinality.

1.3. **The case $\mathcal{A}=[1,\dots,N]$ and $\mathcal{B}$ general.** This is the second step in our graduation of difficulty. In §6 we shall prove

Theorem 2. *Let δ satisfying $0 < \delta < 1$. There exist an absolute constant c_6, independent of δ, and a constant $c_7 = c_7(\delta)$, such that, for any $N \geqslant c_7$, for any subset $\mathcal{B}$ of $[1, \dots, N]$ satisfying the inequality*

$$\sum_{\substack{b \in \mathcal{B} \\ (1-\delta)N < b \leqslant N}} 1 \geqslant \frac{N}{\log^2 N} \cdot (\log\log N)^{c_6}, \tag{6}$$

there is a prime p in the interval $](1-2\delta)N^2, (1-\delta)N^2]$ of the form $p = ab+1$ with a and $b \leqslant N$ and $b \in \mathcal{B}$.

In particular, if $\mathcal{A} = [1, \dots, N]$ and if $\mathcal{B}$ satisfies (6), we have

$$\Gamma^+(\mathcal{A}, \mathcal{B}, N) \geqslant (1-2\delta) \cdot N^2,$$

for any sufficiently large N.

The condition (6) is not artificial at all for the following reason: the order of magnitude of the prime number p is almost N^2, hence, in the equality $p = ab + 1$, both a and b must be close to N. This implies that the set $\mathcal{B}$ must contain many elements in the neighborhood of N.

1.4. **The case $\mathcal{A}$ and $\mathcal{B}$ general.** In the more general situation, we shall prove

Theorem 3. *For every real number $0 < \delta < 1$, there exists a function $\varpi_\delta : \mathbb{N} \to \mathbb{R}_{\geq 0}$ tending to zero at infinity, such that, for every $N \geqslant 2$, for every subsets $\mathcal{A}$ and $\mathcal{B}$ of $[1, \dots, N]$, satisfying*

$$|\mathcal{A}| \geqslant |\mathcal{B}| \geqslant \frac{N}{(\log N)^\delta}, \tag{7}$$

the following inequality holds

$$\Gamma^+(\mathcal{A}, \mathcal{B}, N) \geqslant N^{1+(|\mathcal{A}|/N)(1-\varpi_\delta(N))}. \tag{8}$$

It is possible to describe the function ϖ_δ more precisely, but this description will depend on non explicit constants (see the comment after Theorem 1). Compared with (1), we see two advantages in Theorem 3. When $\mathcal{A}$ is more and more dense, the exponent of N in (8) tends to 2. This gives some consistency to Conjecture 1. In the other direction, if $\mathcal{A}$ and $\mathcal{B}$ satisfy $|\mathcal{A}| \sim |\mathcal{B}| \sim N/(\log N)^\delta$ with $0 < \delta < 1/2$, we obtain the lower bound

$$\Gamma^+(\mathcal{A}, \mathcal{B}, N) \geqslant N^{1+(1-\epsilon)(\log N)^{-\delta}},$$

for $N > N_0(\epsilon)$. By (1), we would have the same lower bound but with δ replaced by 2δ, thus Theorem 3 represents a valuable improvement for sparse sequences.

Actually, after a talk given by the author at the Congress *Activités Additives et Analytiques* (Lille, june 2009) where he exposed the results of the present paper, C. Elsholtz kindly turned our attention on a preprint of K. Matomäki on the same subject. This work is now published ([21]) from which we extract the following central result

Theorem B. *(*[21, Theorem 2]*) Let C_0 and c_1 be positive. Then for every c_2 satisfying*

$$c_2 < \frac{1 - 4c_1 - \frac{2}{C_0}}{4},$$

there exists $N_0(c_2)$ such that, for every $N \geqslant N_0(c_2)$, for every $\mathcal{A}$ and $\mathcal{B} \subset [1, \dots, N]$, satisfying

$$|\mathcal{A}| \geqslant C_0 \frac{N}{\log N} \quad \text{and } |\mathcal{B}| \geqslant \frac{|\mathcal{A}|}{N^{c_1 |\mathcal{A}|/N}},$$

we have

$$\Gamma^+(\mathcal{A}, \mathcal{B}, N) \geqslant N^{\sqrt{1 + c_2 |\mathcal{A}|/N}}.$$

When writing the proof of Theorem B, the author was unaware of [27], this is the reason why she only refers to the older and weaker result [26]. Hence it is worth comparing the strength of Theorems A & B. Theorem B really takes into account the situation where $\mathcal{B}$ is much thinner than $\mathcal{A}$ (a typical situation being $|\mathcal{A}| \asymp N$ and $|\mathcal{B}| \asymp |\mathcal{A}| \cdot N^{-\delta}$ with $\delta > 0$). In counterpart, Theorem B never produces a lower bound for $\Gamma^+(\mathcal{A}, \mathcal{B}, N)$ better than $\Gamma^+(\mathcal{A}, \mathcal{B}, N) \geqslant N^{\sqrt{1 + \frac{1}{4}} - \epsilon} = N^{1.118\cdots}$, instead of $\Gamma^+(\mathcal{A}, \mathcal{B}, N) \geqslant N^{1.333\cdots}$ by Theorem A.

It is also worth noticing that in [21, §4] the author already anticipated the potential impact of the work of Bombieri, Friedlander and Iwaniec [4] on that kind of results. The present paper confirms this intuition.

Acknowledgements. The author is grateful to C. Elsholtz for letting him know the existence of [21]. He also warmly thanks C. Stewart for his stimulating conversations on the subject and the referee for his numerous remarks.

2. Primes in arithmetic progressions

In the rest of this paper, we reserve the letter p to prime numbers. We shall also systematically write

$$\mathcal{L} = \log 2x,$$

where $x \geqslant 1$, is a real number we consider as tending to infinity.

The proofs of Theorems 1, 2 & 3 are based on deep properties of the classical function in prime number theory

$$\pi(x; q, a) = \sum_{\substack{p \leqslant x \\ p \equiv a \bmod q}} 1,$$

when a and q are coprime integers. In particular, its behavior has to be compared with the function $\pi(x)/\varphi(q)$, where $\pi(x)$ is the cardinality of the set of primes $\leqslant x$ and $\varphi(q)$ is the Euler function of the integer q. We recall some properties of this counting function on average in arithmetic progressions. The most classical one is the Bombieri–Vinogradov Theorem (see [1], [29], [2, Théorème 17], [18, Theorem 17.1], ... for instance)

Proposition 1. *For every A, there exists $B = B(A)$ such that*

$$\sum_{q \leqslant Q} \max_{y \leqslant x} \max_{(a,q)=1} \Big| \pi(y; q, a) - \frac{\pi(y)}{\varphi(q)} \Big| = O_A\big(x\, \mathcal{L}^{-A}\big), \tag{9}$$

uniformly for $Q \leqslant x^{\frac{1}{2}}\, \mathcal{L}^{-B}$ and $x \geqslant 1$.

In many applications, this proposition replaces the Riemann Hypothesis extended to Dirichlet's L–functions. The best constant for the moment is $B = A + 1$. But much more is conjectured: it is largely believed that (9) is true for $Q = x^{1-\epsilon}$, for any $\epsilon > 0$ (the O–constant depending now on A and ϵ). This is the content of the Elliott–Halberstam Conjecture (see [7]). The proof of the Bombieri–Vinogradov Theorem (see [18], for instance) is now presented as an elegant and deep consequence of the large sieve inequality for multiplicative characters and of the combinatorial structure of the characteristic function of the set of primes or of the van Mangoldt function $\Lambda(n)$ (see Lemma 7 below). It is a challenge to improve the value of Q in (9), even by modifying the way of summing the error terms $\pi(y; q, a) - \frac{\pi(y)}{\varphi(q)}$ or even by approaching the characteristic function of the set of primes by the characteristic function of another set of the same, but easier, combinatorial structure. The first breakthrough in that direction is due to Fouvry and Iwaniec [14] (see also [9]) and it was followed by several papers of Bombieri, Fouvry, Friedlander and Iwaniec ([10], [11], [15], [13], [3], [4], [5]...) Also see [2, §12 p.89–103] for an introduction to these techniques, based on Linnik's dispersion method and on several types of bounds for Kloosterman sums.

For the problem we are studying in the present paper, we shall restrict to two points of view. The first one is

Proposition 2. *(See* [12, Corollaire 1] *&* [3, Theorem 9]*) For every non zero integer a, for every A, we have the equality*

$$\sum_{\substack{q\leqslant Q\\(q,a)=1}}\Big(\pi(x;q,a)-\frac{\pi(x)}{\varphi(q)}\Big)=O_{a,A}\big(x\,\mathcal{L}^{-A}\big), \tag{10}$$

uniformly for $Q\leqslant x\,\mathcal{L}^{-200A-200}$ and $x\geqslant 1$.

Note that in (10), we are summing the error terms, without absolute value, on consecutive moduli q. Hence we benefit from oscillations of the signs of this error term. In the proof, this oscillation is exploited by *kloostermania i.e* by the study of sums of Kloosterman sums with consecutive denominators. This is the heart of the work of Deshouillers and Iwaniec [6].

To continue the presentation of our tools we recall the classical functions in prime number theory $\theta(x)$, $\psi(x)$ and $\psi(x;q,a)$ given by

$$\theta(x)=\sum_{p\leqslant x}\log p,\ \ \psi(x)=\sum_{n\leqslant x}\Lambda(n)\ \text{and}\ \psi(x;q,a)=\sum_{\substack{n\leqslant x\\ n\equiv a\bmod q}}\Lambda(n).$$

We also introduce the notation

$$q\sim Q \tag{11}$$

to mean that q satisfies the inequalities $Q\leqslant q<2Q$.

In the second variation in the thema of Proposition 1 we sum the error terms with absolute values giving

Proposition 3. *(see* [4, Main Theorem]*) There exists an absolute constant B_1 with the following property :*

For every integer $a\neq 0$, for every x, y, and Q satisfying $x\geqslant y\geqslant 3$, $Q^2\leqslant xy$, we have the inequality

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\Big|\psi(x;q,a)-\frac{\psi(x)}{\varphi(q)}\Big|\ \ll x\Big(\frac{\log y}{\log x}\Big)^2\cdot(\log\log x)^{B_1},$$

where the constant implied in $\ll$ depends on a at most.

The trivial upper bound for the quantity studied in Proposition 3 is $O(x)$. The same is also true for the quantities which are bounded in Propositions 1 and 2. Hence, when $Q=x^{\frac{1}{2}}$, the upper bound given in Proposition 3 is non trivial only by a factor $(\log x)^{-2}(\log\log x)^{B_1}$. It will be sufficient for our proof, however.

Proposition 3 is also interesting for $Q = x^{\frac{1}{2}+\epsilon(x)}$, with $\epsilon(x) \to 0$ as $x \to \infty$, giving an asymptotic expansion of $\psi(x; q, a) \sim \frac{\psi(x)}{\varphi(q)}$, for almost all $q \sim Q$, satisfying $(q, a) = 1$. The technique of proof of Proposition 3 was followed up in [5], leading to the relation $\psi(x; q, a) \asymp_\delta \frac{\psi(x)}{\varphi(q)}$, for almost all $q \sim Q$, satisfying $(q, a) = 1$, with $Q \leqslant x^{\frac{1}{2}+\delta}$ and where δ is a tiny positive constant.

Actually, we shall use Proposition 3 in the following form

$$\sum_{\substack{q\sim Q\\(a,q)=1}} \Big|\pi(x; q, a) - \frac{\pi(x)}{\varphi(q)}\Big| \ll \frac{x}{\log x} \cdot \Big(\frac{\log y}{\log x}\Big)^2 \cdot (\log\log x)^{B_1}. \tag{12}$$

This is a standard consequence of the inequality

$$0 \leqslant \psi(x) - \theta(x) \ll x^{\frac{1}{2}}\, \mathcal{L},$$

and of the Abel summation formula written under the form

$$\pi(x) = \int_{\frac{3}{2}}^{x} \frac{1}{\log t}\ [\mathrm{d}\,\theta(t)],$$

and a similar formula for $\pi(x; q, a)$. The upper bound (12) is well suited to the proof of Theorem 2 but is not sufficient for Theorem 3. In §4, we shall adapt the original proof of Proposition 3 to show

Theorem 4. *There exists an absolute constant B_2 with the following property :*

For every integer $a \neq 0$, for every x, y, P_1, P_2 and Q satisfying $3 \leqslant y \leqslant x$, $Q^2 \leqslant xy$ and $1 \leqslant P_1 \leqslant P_2$, we have the inequality

$$\sum_{\substack{q\sim Q\\(q,a)=1}} \Big| \sum_{\substack{P_1<p\leqslant P_2,\\ pm\equiv a \bmod q}} \sum_{pm\leqslant x} \log p - \frac{1}{\varphi(q)} \sum_{\substack{P_1<p\leqslant P_2,\\(pm,q)=1}} \sum_{pm\leqslant x} \log p\Big| \ll x \cdot \frac{(\log y)^2}{\log x} \cdot (\log\log x)^{B_2},$$

where the constant implied in $\ll$ depends on a at most.

Note that the trivial bound for the quantity now studied is

$$\begin{aligned}
&\sum_{q\sim Q}\Big(\sum_{\substack{n\equiv a \bmod q\\ n\leqslant x}} \sum_{p\mid n} \log p + \frac{1}{\varphi(q)} \sum_{\substack{(n,q)=1\\ n\leqslant x}} \sum_{p\mid n} \log p\Big)\\
&\ll \sum_{q\sim Q}\Big(\sum_{\substack{n\leqslant x\\ n\equiv a \bmod q}} \log n + \frac{1}{\varphi(q)} \sum_{\substack{n\leqslant x\\(n,q)=1}} \log n\Big)\\
&\ll x\,(\log 2x). \qquad (13)
\end{aligned}$$

The proof we shall give is highly based on the work [4]. We shall use the same tools and, as far as possible the same notations, but our proof is more than a paraphrase of the original proof of [4]: the intrusion of the integer variable m creates a new combinatorial situation that we cannot ignore. In the same order of ideas, the variable m brings unsuspected difficulty linked with coprimality conditions (we shall work a lot to circumvent the condition $(A_3(x))$ below).

We now have at our disposal the striking result of Zhang [30] on the average behaviour of the function $\pi(x;q,a)$ for q a smooth integer $\sim Q$, with $Q = x^{\frac{1}{2}+\delta}$, where δ is a (small) fixed positive constant. This breakthrough has a celebrated application to bounded gaps between primes via the Goldston–Pintz–Yıldırım method. However it seems difficult to insert Zhang's result in our method to improve Theorem 3 since our approach is essentially based on the Tchebychev–Hooley method (see (108) below) and an interpretation of the equality $ab+1 = pm$ ($a \in \mathcal{A}$ and $b \in \mathcal{B}$) as a congruence $pm \equiv 1 \bmod b$, where b, by definition of $\mathcal{B}$, essentially satisfies the inequality $b \ll (pm)^{\frac{1}{2}}$ (see (117) below). In other words, our method is surprisingly independent of information concerning the function $\pi(x;q,a)$ for $q \sim x^{\frac{1}{2}+\delta}$, for some fixed positive δ.

3. Analytic preparation

We first recall some results concerning the average behavior of the divisor functions. The following subsection contains the results of Lemmas 11–15 of [4].

3.1. **Lemmas on divisor functions.** Let $\ell \geqslant 0$ be an integer, and let $n \geqslant 1$ be an integer, then we define

$$\tau_\ell(n) := \sum_{n=n_1\cdots n_\ell} 1,$$

(this is the generalized divisor function of order ℓ), then

$$\tau(n) = \tau_2(n),$$

is the classical divisor function. Of course $\tau_0(n) = 1$ if and only if $n = 1$, otherwise, its value is 0. Again some notations:

- $\mathbf{1}_{\mathcal{E}}$ is the characteristic function of the given subset of integers $\mathcal{E}$,
- $\mathfrak{z}(n)$ is the characteristic function of the set of integers n divisible by no prime factor $< z$, where $z \geqslant 2$ is a given number.

We first make a list of several upper bounds for sums of $\tau_\ell^k(n)$ and of $\mathfrak{z}(n)\tau_\ell(n)$.

Lemma 1. *Let $k \geqslant 0$ and $\ell \geqslant 1$ be integers and let $\epsilon > 0$. We then have the inequality*

$$\sum_{x-y<n\leqslant x} \tau_\ell^k(n) \ll_\epsilon y\,(\log 2x)^{\ell^k-1},$$

uniformly for $x \geqslant y \geqslant x^\epsilon$ and $x \geqslant 1$.

Proof. See [20, Lemma 1.1.5] for instance or deduce this lemma from the classical result of Shiu [25]. □

It is well known that the main part of the divisor function $\tau_\ell(n)$ comes from the small divisors of n. Hence the summatory functions of $\mathfrak{z}(n)\tau_\ell(n)$ and $\tau_\ell(n)$ have different behaviors when z becomes larger and larger. This is the object of the next lemma.

Lemma 2. *Let $j \geqslant 0$ be an integer. We have the six relations*

$$(i) \qquad \sum_{n\leqslant x} \mathfrak{z}(n)\tau_j(n) \ll \frac{x}{\log 2x}\cdot\Big(\frac{\log 2xz}{\log 2z}\Big)^j,$$

$$(i') \qquad \sum_{n\leqslant x} \mathfrak{z}(n)\tau_j(n)n^{-1} \ll \Big(\frac{\log 2xz}{\log 2z}\Big)^j,$$

$$(i'') \qquad \sum_{w<n\leqslant x} \mathfrak{z}(n)\tau_j(n)n^{-1}(\log 2n)^{-1} \ll \frac{1}{\log 2w}\cdot\Big(\frac{\log 2xz}{\log 2z}\Big)^j,$$

$$(ii) \qquad \sum_{x<n\leqslant xy} \mathfrak{z}(n)\tau_j(n)n^{-1} \ll \Big(\frac{\log 2y}{\log 2x}\Big)\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^j,$$

$$(iii) \qquad \sum_{nt\leqslant x} \mathfrak{z}(n)\tau_j(n) \ll x\cdot(\log\log 3x)\cdot\Big(\frac{\log 2xz}{\log 2z}\Big)^j,$$

and

$$(iv) \qquad \sum_{x<nt\leqslant xy} \mathfrak{z}(n)\tau_j(n)(nt)^{-1} \ll (\log 2y)\cdot(\log\log 3xy)\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^j,$$

uniformly for real w, x, y, $z \geqslant 1$.

Proof. For $j = 0$, all the results are trivial. For $j \geqslant 1$, the items (i) and (ii) are exactly [4, Lemma 13]. Note that (i) can also be seen as a direct consequence of Shiu's result ([25, Theorem 1]) concerning sums of multiplicative functions, with the adequate remarks concerning the uniformity of this result (see [22, p.258] or [23, p.119]).

The item (i') is a direct consequence of Mertens formula. The inequality (i'') is a trivial consequence of (i').

In the items (iii) and (iv), we impose no sifting condition on the variable t. This explains the change in the asymptotic order. We pass from (i) to (iii) by writing

$$\sum_{nt\leqslant x}\mathfrak{z}(n)\tau_j(n)=\sum_{t\leqslant x}\sum_{n\leqslant x/t}\mathfrak{z}(n)\tau_j(n)\ll\Big(\frac{\log 2xz}{\log 2z}\Big)^j\sum_{t\leqslant x}\frac{x/t}{\log(2x/t)},$$

and summing over t.

Finally for (iv), we decompose

$$\begin{aligned}\sum_{x<nt\leqslant xy}&\mathfrak{z}(n)\tau_j(n)(nt)^{-1}\\&=\Big\{\sum_{t\leqslant x}t^{-1}\sum_{x/t<n\leqslant xy/t}+\sum_{x<t\leqslant xy}t^{-1}\sum_{1<n\leqslant xy/t}\Big\}\mathfrak{z}(n)\tau_j(n)n^{-1}\\&:=\Sigma_1+\Sigma_2,\end{aligned}\tag{14}$$

say. For Σ_1 we use (ii) to write the relations

$$\Sigma_1\ll(\log 2y)\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^j\sum_{t\leqslant x}\frac{1}{t\log(2x/t)}\ll(\log 2y)\cdot(\log\log 3x)\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^j,$$

which is acceptable in view of (iv). Finally, for Σ_2, we use (i') to write

$$\sum_{1\leqslant n\leqslant xy/t}\mathfrak{z}(n)\tau_j(n)n^{-1}\ll\Big(\frac{\log 2xyz}{\log 2z}\Big)^j,$$

for $t\geqslant x$. Inserting this bound into (14), we complete the proof of (iv). □

We continue our investigations for more intricate sums.

Lemma 3. *Let j_1, j_2, j_3 and j_4 be integers $\geqslant 0$. We then have*

$$\begin{aligned}\sum\sum\sum\sum_{\substack{n_1n_2n_3n_4\leqslant x\\ w\leqslant n_4\leqslant n_3\leqslant n_2\leqslant n_1\\ n_3\leqslant yn_4,\,n_1\leqslant yn_2}}&\mathfrak{z}(n_1n_2n_3n_4)\tau_{j_1}(n_1)\tau_{j_2}(n_2)\tau_{j_3}(n_3)\tau_{j_4}(n_4)\\&\ll\frac{x}{\log 2w}\Big(\frac{\log 2y}{\log 2x}\Big)^2\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_1+j_2+j_3+j_4},\end{aligned}\tag{15}$$

uniformly for x, y, z, $w \geqslant 1$. The constant implied in $\ll$ only depends on j_1, j_2, j_3 and j_4. Similarly we have

$$\sum\sum\sum\sum\sum_{\substack{tn_1n_2n_3n_4\leqslant x\\ w\leqslant n_4\leqslant n_3\leqslant n_2\leqslant tn_1\\ n_3\leqslant yn_4,\, tn_1\leqslant yn_2}} \mathfrak{z}(n_1n_2n_3n_4)\,\tau_{j_1}(n_1)\tau_{j_2}(n_2)\tau_{j_3}(n_3)\tau_{j_4}(n_4)$$

$$\ll (\log\log 3xyz)\cdot\frac{x}{\log 2w}\cdot\frac{(\log 2y)^2}{\log 2x}\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_1+j_2+j_3+j_4}. \tag{16}$$

Finally, the relation (16) remains true if the summation is replaced by each of the three following ones

$$\sum\sum\sum\sum\sum_{\substack{tn_1n_2n_3n_4\leqslant x\\ w\leqslant n_4\leqslant n_3\leqslant tn_2\leqslant n_1\\ n_3\leqslant yn_4,\, n_1\leqslant tyn_2}},\quad \sum\sum\sum\sum\sum_{\substack{tn_1n_2n_3n_4\leqslant x\\ w\leqslant n_4\leqslant tn_3\leqslant n_2\leqslant n_1\\ tn_3\leqslant yn_4,\, n_1\leqslant yn_2}} \text{ or } \sum\sum\sum\sum\sum_{\substack{tn_1n_2n_3n_4\leqslant x\\ w\leqslant tn_4\leqslant n_3\leqslant n_2\leqslant n_1\\ n_3\leqslant tyn_4,\, n_1\leqslant yn_2}}. \tag{17}$$

Proof. The upper bound (15) is exactly [4, Lemma 14]. Remark that in (16) & (17), we are dealing with sums in dimension five since we have replaced the variable n_i in (15) by tn_i. In that case, we say that the variable t *is glued to* n_i. This extra variable t, without sifting conditions, explains why the upperbound in (16) is larger than the corresponding one in (15) by a $\log 2x$–factor. In our application the value of the exponent of the $\log\log$–factor has no importance. It remains to adapt the proof of [4, Lemma 14] to obtain (16) by appealing to Lemma 2 and the upper bound (15) of Lemma 3.

We now give all the details for the proof of (16), which corresponds to the case where t is glued to n_1. By dyadic subdivision, we restrict the summation to $x/2 < tn_1n_2n_3n_4 \leqslant x$. Playing with the conditions of summation in the left part of (16), we deduce that the variables n_2, n_3 and n_4 satisfy

$$n_2n_3n_4 \leqslant x^{\frac{3}{4}},\ n_3n_4 \leqslant x^{\frac{1}{2}},\ w \leqslant n_4 \leqslant x^{\frac{1}{4}} \text{ and } x/2y < n_2^2n_3n_4 \leqslant x. \tag{18}$$

We first assume that

$$y \leqslant x^{\frac{1}{3}}. \tag{19}$$

We first sum on t and n_1, by using Lemma 2 (iii)

$$\sum_{\substack{t,\ n_1\\ tn_1\leqslant x/(n_2n_3n_4)}} \mathfrak{z}(n_1)\,\tau_{j_1}(n_1) \ll \frac{x}{n_2n_3n_4}\cdot(\log\log 3xyz)\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_1}. \tag{20}$$

Then, by Lemma 2 (ii), by (18) and the restriction (19), we have

$$\sum_{\sqrt{x/2yn_3n_4}<n_2\leqslant\sqrt{x/n_3n_4}}\mathfrak{z}(n_2)\tau_{j_2}(n_2)n_2^{-1}\ll\frac{\log 2y}{\log 2x}\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_2},\tag{21}$$

$$\sum_{n_4\leqslant n_3\leqslant yn_4}\mathfrak{z}(n_3)\tau_{j_3}(n_3)n_3^{-1}\ll\frac{\log 2y}{\log 2n_4}\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_3},\tag{22}$$

and finally

$$\sum_{w\leqslant n_4\leqslant x^{\frac14}}\mathfrak{z}(n_4)\tau_{j_4}(n_4)n_4^{-1}(\log 2n_4)^{-1}\ll\frac{1}{\log 2w}\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_4},\tag{23}$$

by Lemma 2 (i''). Putting together (20), (21), (22) and (23), we obtain (16) in the case of (19).

We now suppose

$$y>x^{\frac13}.\tag{24}$$

Since y is large, we lose almost nothing in forgetting the conditions $n_3\leqslant yn_4$ and $tn_1\leqslant yn_2$. The sum that we are studying is less or equal to

$$\sum_{n_4\leqslant x^{\frac14}}\mathfrak{z}(n_4)\tau_{j_4}(n_4)\sum_{n_3\leqslant x^{\frac13}}\mathfrak{z}(n_3)\tau_{j_3}(n_3)\sum_{n_2\leqslant x^{\frac12}}\mathfrak{z}(n_2)\tau_{j_2}(n_2)\sum_{tn_1\leqslant x/(n_2n_3n_4)}\sum\mathfrak{z}(n_1)\tau_{j_1}(n_1).\tag{25}$$

Applying Lemma 2 (iii) and (i') three times, we see that the above quantity is

$$\ll x\cdot(\log\log 3xyz)\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_1+j_2+j_3+j_4},\tag{26}$$

which is less than the bound claimed in (16), because of (24). This completes the proof of (16) in all the cases.

The first case of (17) concerns the situation where t is glued to n_2. The inequalities (18) are changed into

$$tn_2n_3n_4\leqslant x^{\frac34},\ n_3n_4\leqslant x^{\frac12},\ w\leqslant n_4\leqslant x^{\frac14}\text{ and }x/2y<t^2n_2^2n_3n_4\leqslant x,\tag{27}$$

and we suppose that (19) is satisfied. By using respectively the items (i) and (iv) of Lemma 2 we can write

$$\sum_{\substack{n_1\\ n_1\leqslant x/(tn_2n_3n_4)}}\mathfrak{z}(n_1)\,\tau_{j_1}(n_1)\ll\frac{x/(tn_2n_3n_4)}{\log 2x}\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_1},\tag{28}$$

$$\sum_{\substack{t,\ n_2\\ \sqrt{x/2yn_3n_4}<tn_2\leqslant\sqrt{x/n_3n_4}}}\sum \mathfrak{z}(n_2)\tau_{j_2}(n_2)(tn_2)^{-1}\ll(\log\log 3xyz)\cdot(\log 2y)\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_2},$$

and we use (22) and (23) again. Putting together the four above results, we obtain (16), for the first sum appearing in (17) in the case where (19) is satisfied.

By similar techniques we also prove that the second and third sums of (17) satisfy (16) under the restriction (19), which means y small.

When (24) is satisfied (y large) each of the three sums listed in (17) is less than the sum studied in (25). By (26), we see that these three sums also satisfy (16). □

Our last lemma concerning divisors is

Lemma 4. *Let $x,\ y,\ z,\ w$ be real numbers $\geqslant 1$, let $s=5$ or 6 and let $j_1,\dots,j_s$ be integers $\geqslant 0$. We then have*

$$\sum_{\substack{n_1\cdots n_s\leqslant x\\ w\leqslant n_s\leqslant\cdots\leqslant n_1\\ n_{s-2}\leqslant yn_s}}\cdots\sum\mathfrak{z}(n_1\cdots n_s)\,\tau_{j_1}(n_1)\cdots\tau_{j_s}(n_s)\ll\frac{x}{\log 2x}\Big(\frac{\log 2y}{\log 2w}\Big)^2\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_1+\cdots+j_s},\tag{29}$$

where the constant implied in $\ll$ depends at most on $j_1,\dots,j_s$. Similarly, we have for $s=5$ or 6 and $1\leqslant\nu\leqslant s$, the inequality

$$\begin{aligned}\sum_{(t,n_1,\dots,n_s)\in\mathcal{E}(s,\nu)}\cdots\sum\mathfrak{z}(n_1\cdots n_s)\,\tau_{j_1}(n_1)\cdots\tau_{j_s}(n_s)&\\ \ll x\cdot(\log\log 3xyz)^s\cdot\Big(\frac{\log 2y}{\log 2w}\Big)^2\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_1+\cdots+j_s},&\end{aligned}\tag{30}$$

where $\mathcal{E}(s,\nu)$ denotes the set of $s+1$–uples $(t,n_1,\dots,n_s)$ satisfying the inequalities

$$\begin{cases}n_1\cdots(tn_\nu)\cdots n_s\leqslant x&\\ w\leqslant n_s\cdots\leqslant(tn_\nu)\leqslant\cdots\leqslant n_1,&\\ n_{s-2}\leqslant yn_s, & \text{if } \nu\neq s \text{ and } s-2,\\ tn_{s-2}\leqslant yn_s, & \text{if } \nu=s-2,\\ n_{s-2}\leqslant ytn_s, & \text{if } \nu=s.\end{cases}\tag{31}$$

Proof. Actually this lemma is also true for $s=4$, but we shall only use it for $s=5$ or $s=6$ (see the end of §4.5). The upper bound (29) is exactly [4, Lemma 15]. The bound (30) is a consequence of Lemma 2. Note that we pass from the conditions of summation of (29) to $\mathcal{E}(s,\nu)$, by gluing (as we defined after Lemma 3) the variable t to the variable n_ν.

We now give the proof of (30) in the particular case $s=\nu=5$ (in other words, this is the case where t is glued to n_5) since the other ten cases are similar. We write the inequality

$$\begin{aligned}(32)\qquad \sum_{(t,n_1,\dots,n_5)\in\mathcal{E}(5,5)}\cdots\sum \leqslant & \sum_{w\leqslant tn_5\leqslant x^{\frac{1}{5}}}\mathfrak{z}(n_5)\tau_{j_5}(n_5)\sum_{tn_5\leqslant n_4\leqslant ytn_5}\mathfrak{z}(n_4)\tau_{j_4}(n_4)\\ &\times\sum_{tn_5\leqslant n_3\leqslant ytn_5}\mathfrak{z}(n_3)\tau_{j_3}(n_3)\sum_{n_3\leqslant n_2\leqslant x^{\frac{4}{5}}/(n_3n_4tn_5)}\mathfrak{z}(n_2)\tau_{j_2}(n_2)\\ &\times\sum_{n_1\leqslant x/(n_2n_3n_4tn_5)}\mathfrak{z}(n_1)\tau_{j_1}(n_1).\end{aligned}$$

By Lemma 2 (i) we have

$$(33)\qquad \sum_{n_1\leqslant x/(n_2n_3n_4tn_5)}\mathfrak{z}(n_1)\tau_{j_1}(n_1)\ll\frac{x/(n_2n_3n_4tn_5)}{\log 2x}\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_1}.$$

By Lemma 2 (i'), we get

$$(34)\qquad \sum_{n_3\leqslant n_2\leqslant x^{\frac{4}{5}}/(n_3n_4tn_5)}\mathfrak{z}(n_2)\tau_{j_2}(n_2)n_2^{-1}\ll\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_2}.$$

By Lemma 2 (ii), we have, for $i=3$ or 4, the inequality

$$(35)\qquad \sum_{tn_5\leqslant n_i\leqslant ytn_5}\mathfrak{z}(n_i)\tau_{j_i}(n_i)\,n_i^{-1}\ll\frac{\log 2y}{\log 2tn_5}\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_i},$$

and finally

$$(36)\qquad \sum_{w\leqslant tn_5\leqslant x^{\frac{1}{5}}}\mathfrak{z}(n_5)\tau_{j_5}(n_5)(tn_5)^{-1}(\log 2tn_5)^{-2}\ll\frac{\log x}{(\log 2w)^2}\cdot(\log\log 3xyz)\cdot\Big(\frac{\log 2xyz}{\log 2z}\Big)^{j_5},$$

by Lemma 2 (iv) and the lower bound $\log(2tn_5)\gg\log 2w$. Gathering (32),...,(36), we deduce (30) in the particular case $(s,\nu)=(5,5)$.

The other cases are treated similarly. □

3.2. **Convolution of two sequences in arithmetic progressions.** We continue to follow the notations of [4], in order to quote the necessary results from this paper. Let f an arithmetic function with finite support. We define

$$\|f\|:=\Big(\sum_n|f(n)|^2\Big)^{\frac{1}{2}}.$$

For a and q coprime integers, we introduce

$$\Delta(f;q,a) := \sum_{n\equiv a \bmod q} f(n) - \frac{1}{\varphi(q)} \sum_{(n,q)=1} f(n).$$

Hence $\Delta(f;q,a)$ measures the distribution of the sequence $f(n)$ in the arithmetic progression $n \equiv a \bmod q$. We shall be mainly concerned by the situation where f is the arithmetic convolution product $f = \boldsymbol{\alpha} * \boldsymbol{\beta}$, of two complex sequences $\boldsymbol{\alpha} = (\alpha_m)_{m\sim M}$ and $\boldsymbol{\beta} = (\beta_n)_{n\sim N}$, with $MN = x$ say and M, $N > x^{\epsilon}$. (See (11) for the meaning of $\sim$). We shall also study the convolution of three sequences.

The following assumption is crucial in the context of dispersion technique. Let $B > 0$ be a real number and $\kappa : \mathbb{R} \to \mathbb{R}$ a real function. Now consider the condition $(A_1(B,\kappa))$ concerning $\boldsymbol{\beta} = (\beta_n)_{n\sim N}$

$$(A_1(B,\kappa)) \quad \begin{cases} \text{For any } A > 0, \text{ for any integers } d,\ k \geqslant 1, \ell \neq 0, (k,\ell) = 1 \text{ we have} \\ \\ \Big| \sum\limits_{\substack{n\equiv \ell \bmod k \\ (n,d)=1}} \beta_n - \dfrac{1}{\varphi(k)} \sum\limits_{(n,dk)=1} \beta_n \Big| \leqslant \kappa(A) \|\boldsymbol{\beta}\|\, \tau^B(d) N^{1/2} (\log 2N)^{-A}. \end{cases}$$

Of course any $(\beta_n)_{n\sim N}$ satisfies $(A_1(B,\kappa))$ by chosing for κ a huge function of N and A (for intance $\kappa(A) = (\log 2N)^A$)). This is an uninteresting case. The situation is quite different when we deal with sequences $(\beta_n)_{n\geqslant 1}$, which satisfy Siegel–Walfisz type theorem (for instance the characteristic function of the set of primes). If, in that case, we consider the truncated sequence $\boldsymbol{\beta} = (\beta_n)_{n\sim N}$, then, the condition $A_1(B,\kappa))$ is satisfied by $\boldsymbol{\beta} = (\beta_n)_{n\sim N}$, but with a function $A \mapsto \kappa(A)$ independent of N. Then we are in an interesting situation, on letting N tend to infinity, and choosing A very large, but fixed.

We shall also frequently suppose that, on average, the sequences are less than a power of $\log 2n$ by introducing, for $B > 0$, the assumption

$$(A_2(B)) \qquad\qquad |\beta_n| \leqslant B\, \tau^B(n) \text{ for all } n \sim N.$$

Sometimes it will be asked that $\beta_n = 0$ when n has a small prime divisor in the following sense: let $x \geqslant 3$ be a real number and let $(A_3(x))$ be the hypothesis

$(A_3(x))$ $\qquad \beta_n \neq 0 \Rightarrow \big\{p \mid n \Rightarrow p > \exp(\log x/(\log\log x)^2)\big\}.$

In other words, we ask the support of $\boldsymbol{\beta}$ to be included in the set of quasi primes. We shall also sometimes work with very particular $\boldsymbol{\lambda} = (\lambda_\ell)_{\ell\sim L}$ satisfying

$(A_4(z))$ $\qquad$ There exists an interval $\mathfrak{L} \subset [L, 2L[$ and $z \geqslant 2$ such that $\boldsymbol{\lambda} = \mathfrak{z}\mathbf{1}_{\mathfrak{L}}$.

First recall a classical consequence of the large sieve inequality, which, after combinatorial preparations, leads to Proposition 1 (Bombieri–Vinogradov Theorem).

Proposition 4. *Let ϵ, x, B, M and N be real numbers such that $\epsilon > 0$, $B > 0$, $x = MN$ and $M,\ N \geqslant \max(2, x^\epsilon)$. Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a real function. Let $\boldsymbol{\alpha} = (\alpha_m)_{m\sim M}$, $\boldsymbol{\beta} = (\beta_n)_{n\sim N}$ be two complex sequences such that $\boldsymbol{\beta}$ satisfies $(A_1(B, \kappa))$. Then, for every $C > 0$, there exists A_0, depending only on B and C such that the following inequality holds*

$$\sum_{q\leqslant x^{\frac12}\mathcal{L}^{-A_0}} \max_{(a,q)=1} \big|\, \Delta(\boldsymbol{\alpha} * \boldsymbol{\beta}; q, a)\,\big| \ll \|\boldsymbol{\alpha}\|\, \|\boldsymbol{\beta}\|\, x^{\frac12}\, \mathcal{L}^{-C},$$

where the constant implied in the $\ll$–symbol depends at most on ϵ, κ, B and C.

However, Proposition 4 says nothing when $Q \asymp x^{\frac12}$. We now recall several situations, when Q (*level of distribution*) can be taken greater than $x^{\frac12}$. The relative sizes of the factors of the convolution are crucial to allow to go beyond $x^{\frac12}\mathcal{L}^{-A_0}$, which is the natural limit of the large sieve.

The first situation is

Proposition 5. *Let $a \neq 0$ be an integer. Let ϵ, x, B, M and N be real numbers such that $\epsilon > 0$, $B > 0$, $x = MN$ and $M,\ N \geqslant \max(2, x^\epsilon)$. Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a real function. Let $\boldsymbol{\alpha} = (\alpha_m)_{m\sim M}$, $\boldsymbol{\beta} = (\beta_n)_{n\sim N}$ be two complex sequences such that $\boldsymbol{\beta}$ satisfies $(A_1(B, \kappa))$, $(A_2(B))$ and $(A_3(x))$.*

Then for every $C > 0$, we have

$$\sum_{\substack{q\sim Q\\(q,a)=1}} |\Delta(\boldsymbol{\alpha} * \boldsymbol{\beta}; q, a)| \ll \|\boldsymbol{\alpha}\|\, \|\boldsymbol{\beta}\|\, x^{\frac12}\, \mathcal{L}^{-C},$$

uniformly for

$$x^{\epsilon-1}Q^2 < N < x^{\frac56-\epsilon}Q^{-\frac43},$$

where the constant implied in the $\ll$–symbol depends at most on ϵ, κ, a, B and C.

The first and stronger version of Proposition 5 can be found in [11, Théorème 1] (without the restriction $(A_3(x))$). A new proof is given in [3, Theorem 3] and it appears again as [4, Theorem 1]. It is obvious that we can take $Q \asymp x^{\frac{1}{2}}$ as soon as N satisfies $x^\epsilon < N < x^{\frac{1}{6}-\epsilon}$. This result is quite convenient for applications.

We shall also use the following result which is one of the key ingredient in the proof of Proposition 2.

Proposition 6. *([4, Theorem 2]) Let $a \neq 0$ be an integer. Let ϵ, x, y_1, y_2, B, C, N and Q be real numbers such that $\epsilon > 0$, $B > 0$, $C > 0$, $y_2 > y_1 > 0$, $x \geqslant 1$, $x^\epsilon \leqslant N \leqslant x^{\frac{1}{3}-\epsilon}$ and $1 \leqslant Q \leqslant x^{1-\epsilon}$. Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a real function. Let $\boldsymbol{\beta} = (\beta_n)_{n\sim N}$ be a complex sequence such that $\boldsymbol{\beta}$ satisfies $(A_1(B,\kappa))$ and $(A_3(x))$.*
Then, for every double sequence $\boldsymbol{\xi} = \xi(\ell, m)$ of complex numbers, we have the inequality

$$\sum_{\substack{q\sim Q\\(q,a)=1}} \Big(\sum_{\substack{\ell mn\sim x,\ n\sim N\\ y_1<m/n<y_2\\ \ell mn\equiv a \bmod q}}\sum\sum \xi(\ell,m)\beta_n - \frac{1}{\varphi(q)} \sum_{\substack{\ell mn\sim x,\ n\sim N\\ y_1<m/n<y_2\\ (\ell mn,q)=1}}\sum\sum \xi(\ell,m)\beta_n \Big) \ll \|\boldsymbol{\xi}\|\ \|\boldsymbol{\beta}\|\ x^{\frac{1}{2}}\,\mathcal{L}^{-C},$$

where the constant implied in $\ll$ depends at most on ϵ, κ, a, B and C, and where

$$\|\boldsymbol{\xi}\| = \Big(\sum_{x/2N\leqslant \ell m\leqslant 2x/N} \sum |\xi(\ell,m)|^2 \Big)^{\frac{1}{2}}.$$

Note that we are summing the error terms without absolute values, this is why the level of distribution Q can be taken so large. If we fix $y_1 = 0$ and $y_2 = \infty$ and define $\boldsymbol{\alpha}$ by the formula $\alpha_k = \sum_{\ell m=k} \xi(\ell, m)$, Proposition 6 deals with the convolution $\boldsymbol{\alpha} * \boldsymbol{\beta}$.

3.3. **Convolution of three sequences in arithmetic progressions.** The second type of results concerns the convolution of three sequences

$$\begin{cases} \boldsymbol{\eta} = (\eta_k)_{k\sim K},\ \boldsymbol{\lambda} = (\lambda_\ell)_{\ell\sim L},\ \boldsymbol{\alpha} = (\alpha_m)_{m\sim M}, \\ \\ x = KLM,\ \mathcal{L} = \log 2x,\ \text{with } K,\ L,\ M \geqslant 1. \end{cases} \tag{37}$$

We have

Proposition 7. *Let ϵ, B and C be given positive real numbers. Let $a \neq 0$ be an integer. Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a real function. Let x, K, L, M be real numbers and $\boldsymbol{\eta}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ be three sequences as in (37). Furthermore, suppose that the following conditions are satisfied*

- K, L, $M \geqslant x^{\epsilon}$,
- $\boldsymbol{\eta}$ *satisfies* $(A_2(B))$ *and* $(A_3(x))$,
- $\boldsymbol{\lambda}$ *satisfies* $(A_1(B,\kappa))$, $(A_2(B))$ *and* $(A_3(x))$,
- $\boldsymbol{\alpha}$ *satisfies* $(A_2(B))$.

Then, there exists A_0, depending only on B and C, such that the following inequality

$$\sum_{\substack{q\sim Q\\(q,a)=1}} \big|\Delta(\boldsymbol{\eta}*\boldsymbol{\lambda}*\boldsymbol{\alpha};q,a)\big| \ll x\mathcal{L}^{-C}. \tag{38}$$

holds as soon as one of two sets of inequalities is verified

$$Q\mathcal{L}^{A_0} < KL,\ K^2L^3 < Qx\mathcal{L}^{-A_0} \text{ and } K^4L^2(K+L) < x^{2-\epsilon}, \tag{$S1$}$$

or

$$Q\mathcal{L}^{A_0} < KL,\ KL^2Q^2 < x^2\mathcal{L}^{-A_0} \text{ and } K^2x^{\epsilon} < Q. \tag{$S2$}$$

The constant implied in the $\ll$–symbol of (38) depends at most on ϵ, κ, a, B and C.

The conditions $(S1)$ correspond to [4, Theorem 3], and the set $(S2)$ to [4, Theorem 4]. Note that in the original statement of [4, Theorems 3 & 4], the sequence $\boldsymbol{\alpha}$ is supposed to satisfy $(A_3(x))$. Actually, this restriction is unnecessary, since the proof of [4, Formula (4.3)], based on Cauchy–Schwarz inequality does not require such a condition.

Note that if in $(S1)$ or $(S2)$ the factor $\mathcal{L}^{A_0}$ was replaced by the larger factor x^{ϵ}, Proposition 7 would be too weak for the proof of Proposition 3 and Theorem 4. This is the reason why we cannot appeal to [3, Theorem 4], which also deals with the convolution of three sequences.

3.4. **Other types of results on the convolution of three sequences.** The condition $(A_3(x))$ that must satisfy $\boldsymbol{\lambda}$ in Proposition 7 is rather annoying in the application that we have in mind. It could certainly be removed by writing with great care the original proof of Theorems 3 & 4 of [4]. We prefer to modify the proof of the following result of Fouvry [10, Théorème 2].

Proposition 8. *Let a be an integer. Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a real function. Let ϵ, x, C, L, M, N be real numbers such that : ϵ and $C > 0$, L, M and $N \geqslant 1$, $x = LMN$,*

$0 < |a| \leqslant x$ *and such that*

$$L^2N \leqslant M^{2-\epsilon},\ L^3N^4 \leqslant M^{4-\epsilon} \text{ and } \log N \geqslant \epsilon \log M. \tag{S3}$$

Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ *and* $\boldsymbol{\lambda}$ *be the characteristic functions of three sets of integers respectively included in* $[M, 2M[$, $[N, 2N[$ *and* $[L, 2L[$. *Suppose that* $\boldsymbol{\beta}$ *satisfies*

$$\Big| \sum_{n \equiv b \bmod q} \beta_n - \frac{1}{\varphi(q)} \sum_{(n,q)=1} \beta_n \Big| \leqslant \kappa(A) \Big(\sum_n |\, \beta_n \,| \Big) \, (\log 2N)^{-A},$$

for every real A *and for every integers* b *and* q *such that* $(b, q) = 1$. *Then we have the inequality*

$$\sum_{\substack{q \leqslant (LN)^{1-\epsilon} \\ (q,a)=1}} \Big| \, \Delta(\boldsymbol{\alpha} * \boldsymbol{\beta} * \boldsymbol{\lambda}; q, a) \, \Big| \ll x \, \mathcal{L}^{-C},$$

where the constant implied in the $\ll$*–symbol depends at most on* ϵ, κ *and* C.

It is worth to notice the large uniformity over a compared with the results contained in Propositions 5–7. This is due to the use of Weil's classical bound for Kloosterman sums instead of kloostermania. However we shall not use this uniformity here. Nevertheless the condition $(A_3(x))$ is now absent from the hypothesis, but the range of summation for q is too short for our application. As in Proposition 7, we would like to go up to $q \leqslant (LN)\, \mathcal{L}^{-A_0}$.

We now give the improvement of Proposition 8 necessary for our application.

Proposition 9. *Let* a *be an integer. Let* $\kappa : \mathbb{R} \to \mathbb{R}$ *be a real function. Let* ϵ, x, B, C, L, M, N *be real numbers such that :* ϵ, B *and* $C > 0$, L, M *and* $N \geqslant x^{\epsilon}$, $x = LMN$, *such that* (S_3) *is satisfied. Let* a *be an integer such that* $0 < |a| \leqslant x$. *Let* $\boldsymbol{\alpha} = (\alpha_m)_{m \sim M}$, $\boldsymbol{\beta} = (\beta_n)_{n \sim N}$ *and* $\boldsymbol{\lambda} = (\lambda_\ell)_{\ell \sim L}$ *be three sequences such that*

- $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, *and* $\boldsymbol{\lambda}$ *satisfy* $(A_2(B))$,
- $\boldsymbol{\beta}$ *satisfies* $(A_1(B, \kappa))$.

Then there exists A_0 *depending only on* B *and* C *such that we have*

$$\sum_{\substack{q \sim Q \\ (q,a)=1}} \Big| \, \Delta(\boldsymbol{\alpha} * \boldsymbol{\beta} * \boldsymbol{\lambda}; q, a) \, \Big| \ll x \mathcal{L}^{-C},$$

for $Q \leqslant (LN)\, \mathcal{L}^{-A_0}$. *The constant implied in the* $\ll$*–symbol depends at most on* ϵ, κ, B *and* C.

Proof. When $Q \leqslant (LN)^{1-\epsilon}$, the extension from Proposition 8 to Proposition 9 is straightforward by following the proof of [10, Théorème 2].

Hence we are left with the case

$$(LN)^{1-\epsilon} < Q \leqslant (LN)\mathcal{L}^{-A_0}. \tag{39}$$

We shall follow the notations of [10] as most as possible even if they differ sometimes from [4]. Let

$$\boldsymbol{\gamma} = \boldsymbol{\beta} * \boldsymbol{\lambda}.$$

Hence $\boldsymbol{\gamma} = (\gamma_k)_k$ has its support included in $[K, 4K[$, with $K := LN$. Note that

$$|\gamma_k| \leqslant B^2 \tau^{2B+1}(k), \tag{40}$$

by $(A_2(B))$. In the following proof, we shall denote by B^* a constant depending only on the constant B appearing in the assumptions $(A_1(B,\kappa))$ and $(A_2(B))$. The value of B^* may change at each time it appears.

Let $E(Q)$ be the sum

$$E(Q) := \sum_{\substack{q\sim Q\\(q,a)=1}} \sum_{(m,q)=1} |\alpha_m| \Big| \sum_{k\equiv a\overline{m} \bmod q} \gamma_k - \frac{1}{\varphi(q)} \sum_{(k,q)=1} \gamma_k \Big|.$$

(here $\overline{m}$ is the multiplicative inverse of $m \bmod q$.) Obviously, $E(Q)$ satisfies the inequality

$$\sum_{\substack{q\sim Q\\(q,a)=1}} \Big| \Delta(\boldsymbol{\alpha} * \boldsymbol{\beta} * \boldsymbol{\lambda}; q, a) \Big| \leqslant E(Q).$$

By the Cauchy–Schwarz inequality, by the assumption $(A_2(B))$ for $\boldsymbol{\alpha}$ and by inversion of summation, we get the inequality (see [10, p.365])

$$E^2(Q) \leqslant \|\boldsymbol{\alpha}\|\ Q\ D(Q) \ll M\, Q\, \mathcal{L}^{B^*}\, D(Q), \tag{41}$$

where the dispersion $D(Q)$ is

$$D(Q) := W(Q) - 2V(Q) + U(Q), \tag{42}$$

with

$$U(Q) := \sum_{\substack{q\sim Q\\(q,a)=1}} \sum_{\substack{m\sim M\\(m,q)=1}} \Big(\frac{1}{\varphi(q)} \sum_{(k,q)=1} \gamma_k\Big)^2,$$

$$V(Q) := \sum_{\substack{q\sim Q\\(q,a)=1}} \sum_{\substack{m\sim M\\(m,q)=1}} \Big(\sum_{k_1\equiv a\overline{m} \bmod q} \gamma_{k_1}\Big)\Big(\frac{1}{\varphi(q)} \sum_{(k_2,q)=1} \gamma_{k_2}\Big),$$

and

$$W(Q) := \sum_{\substack{q\sim Q\\(q,a)=1}} \sum_{\substack{m\sim M\\(m,q)=1}} \Big(\sum_{k\equiv a\overline{m} \bmod q} \gamma_k \Big)^2.$$

Let also

$$A(Q) := \sum_{\substack{q\sim Q\\(q,a)=1}} \frac{1}{q\,\varphi(q)} \Big(\sum_{(k,q)=1} \gamma_k \Big)^2.$$

Following the proof of [10, Form.(6)] and appealing to Lemma 1, we prove the equality

$$U(Q) = MA(Q) + O\big(K^2\,Q^{-1}\,\mathcal{L}^{B^*}\big). \tag{43}$$

By the proof of [10, Form.(12)], we also have

$$V(Q) = MA(Q) + O_\epsilon\big(KQ^{-1}x^{1-\epsilon} + K^{\frac{5}{2}}Q^{-1}x^{7\epsilon}\big), \tag{44}$$

where ϵ appears in (39).

The study of $W(Q)$ is more delicate. Firstly we take some care to get rid of the common divisors. Let

$$\Delta := 3KQ^{-1}, \tag{45}$$

and, by (39), we can suppose that

$$3\mathcal{L}^{A_0} \leqslant \Delta < 3K^{\epsilon}. \tag{46}$$

Then we notice that if k_1 and k_2 are two distinct integers of the interval $[K, 4K[$, satisfying $(k_1k_2, q) = 1$ and $k_1 - k_2 = q_0q$, for some $q \sim Q$ and some positive integer q_0 we then have

$$(k_1, k_2) = (k_1, k_1 - k_2) = (k_1, q_0q) = (k_1, q_0) \leqslant q_0 \leqslant \Delta. \tag{47}$$

Following [10, §VI], we write the equality

$$W(Q) = \sum_{\substack{q\sim Q\\(q,a)=1}} \sum_{\substack{k_1\equiv k_2 \bmod q\\(k_1k_2,q)=1}} \gamma_{k_1}\gamma_{k_2} \sum_{\substack{m\sim M\\m\equiv a\overline{k}_1 \bmod q}} 1. \tag{48}$$

We first notice that the contribution, say $W^{=}(Q)$, to $W(Q)$ of the (k_1, k_2) with $k_1 = k_2$ satisfies

$$\big|W^{=}(Q)\big| \ll \sum_{k\leqslant 4K} \tau^{4B+2}(k) \sum_{\substack{m\sim M\\km\neq a}} \tau(|km-a|) + Qx^{\epsilon}.$$

Writing $t = km$, we deduce that

$$\big|W^{=}(Q)\big| \ll \sum_{\substack{t\leqslant 8x\\ t\neq a}} \tau^{4B+3}(t)\tau(|t-a|) + Qx^{\epsilon} \ll x\mathcal{L}^{B^*}, \tag{49}$$

by the Cauchy–Schwarz inequality and by Lemma 1. We will see that the bound (49) is acceptable in view of (39) & (41) by choosing A_0 sufficiently large.

Let $W^{\neq}(Q)$ be the contribution to $W(Q)$ of the pairs (k_1, k_2) with $k_1 \neq k_2$ (see (48)). By (47), we know that $d := (k_1, k_2)$ is less than Δ. Decomposing $W^{\neq}(Q)$ according to the the value of d and writing $k_i = dk_i'$ ($i = 1, 2$) we have the equality (compare with [10, Form.(13)])

$$W^{\neq}(Q) = \sum_{\substack{q\sim Q\\ (q,a)=1}} \sum_{\substack{d\leqslant\Delta\\ (d,q)=1}} \sum_{\substack{k_1'\equiv k_2' \bmod q,\, k_1'\neq k_2'\\ (k_1',k_2')=(k_1'k_2',q)=1}} \gamma_{dk_1'}\gamma_{dk_2'} \sum_{\substack{m\sim M\\ m\equiv a\overline{dk_1'} \bmod q}} 1. \tag{50}$$

We continue to prepare the variable by extracting from k_1' all the prime factors appearing also in d. So we write $k_1' = d_1k_1''$ with $d_1 \mid d^{\infty}$ and $(k_1'', d) = 1$ and we use the following crude estimate

Lemma 5. *Uniformly for d integer $\geqslant 1$ and $y \geqslant 1$, we have the inequality*

$$\sum_{\substack{d_1\mid d^{\infty}\\ d_1\geqslant y}} \frac{1}{d_1} \ll \frac{\tau(d)}{y^{\frac{1}{2}}}.$$

Proof. We may restrict to the case where $d = p_1 \cdots p_r$ is squarefree. Following Rankin's method, we write, for every $\kappa \in]0, 1[$, the inequality

$$\sum_{\substack{d_1\mid d^{\infty}\\ d_1\geqslant y}} \frac{1}{d_1} \leqslant \sum_{d_1\mid d^{\infty}} \frac{1}{d_1}\cdot\Big(\frac{d_1}{y}\Big)^{\kappa} = \frac{1}{y^{\kappa}}\prod_{i=1}^{r}\Big(1-p_i^{\kappa-1}\Big)^{-1} \ll \frac{1}{y^{\kappa}}\exp\Big(\sum_{i=1}^{r} p_i^{\kappa-1}\Big).$$

Fixing $\kappa = 1/2$, we get the desired upper bound. □

Inspired by [10, p.368], we see that the contribution to the right part of (50) of $d_1 > y$ is

$$= \sum_{\substack{q\sim Q\\ (q,a)=1)}} \sum_{\substack{d\leqslant\Delta\\ (d,a)=1}} \sum_{\substack{d_1\mid d^{\infty}\\ d_1>y}} \sum_{\substack{d_1k_1''\equiv k_2' \bmod q\\ d_1k_1''\neq k_2'\\ (k_1'',dk_2')=(k_1''k_2',d_1q)=1}} \gamma_{dd_1k_1''}\gamma_{dk_2'} \sum_{\substack{m\sim M\\ m\equiv a\overline{dd_1k_1''} \bmod q}} 1.$$

Using (40) and the inequality $\tau(n) \ll X^{\frac{\epsilon}{20(2B+1)}}$ ($0 < n \leqslant X$) several times, and separating the cases $dd_1k_1''m - a \neq 0$ from the case $dd_1k_1''m - a = 0$, we see, by (45),

that the above contribution is

$$\begin{aligned}&\ll Mx^{\frac{\epsilon}{5}}\sum_{\substack{d\leqslant\Delta\\(d,a)=1}}\sum_{\substack{d_1|d^\infty\\d_1>y}}\sum_{k_1''\leqslant 4K/(dd_1)}\frac{K}{dQ}+Kx^{\frac{\varepsilon}{10}}\\&\ll K^2MQ^{-1}x^{\frac{\epsilon}{3}}y^{-\frac12}+Kx^{\frac{\varepsilon}{10}}\\&\ll K^2MQ^{-1}x^{-\frac{\epsilon}{6}},\end{aligned}\tag{51}$$

by choosing

$$y=x^{\epsilon},\tag{52}$$

and applying Lemma 5. Note that (51) is acceptable in view of (39) & (41). Gathering (48), (49), (50) & (51) and changing notations, we obtain the equality

$$\begin{aligned}W(Q)=&\sum_{\substack{q\sim Q\\(q,a)=1}}\sum_{\substack{d\leqslant\Delta\\(d,q)=1}}\sum_{\substack{d_1|d^\infty\\d_1\leqslant y}}\sum_{\substack{d_1k_1'\equiv k_2 \bmod q,\, d_1k_1'\neq k_2\\(k_1',dk_2)=(k_1'k_2,d_1q)=1}}\gamma_{dd_1k_1'}\gamma_{dk_2}\sum_{\substack{m\sim M\\m\equiv a\overline{dd_1k_1'} \bmod q}}1\\&+O\big(x\mathcal{L}^{B^*}+x^2M^{-1}Q^{-1}x^{-\frac{\epsilon}{6}}\big).\end{aligned}\tag{53}$$

(Compare with [10, Form.(14)]). The main term in (53) is given by replacing the innermost sum by its approximation M/q. When this substitution is made we can forget the conditions $d_1k_1\neq k_2$ and $d_1\leqslant y$. The resulting error term is bounded $\ll x\mathcal{L}^{B^*}+K^2MQ^{-1}x^{-\frac{\epsilon}{6}}$ (same computations as for (49) & (51)). Let

$$B(Q):=\sum_{\substack{q\sim Q\\(q,a)=1}}\frac1q\sum_{\substack{k_1\equiv k_2 \bmod q\\(k_1k_2,q)=1}}\gamma_{k_1}\gamma_{k_2}=\sum_{\substack{q\sim Q\\(q,a)=1}}\frac1q\sum_{\substack{\kappa \bmod q\\(\kappa,q)=1}}\Big(\sum_{k\equiv\kappa \bmod q}\gamma_k\Big)^2.\tag{54}$$

The above discussion transforms (53) into the following equality, which has to be compared with [10, Form.(15)]

$$W(Q)=MB(Q)+W_1(M,Q)-W_1(2M,Q)+O\big(x\mathcal{L}^{B^*}+x^2M^{-1}Q^{-1}x^{-\frac{\epsilon}{6}}\big),\tag{55}$$

with

$$W_1(Y,Q)=\sum_{\substack{q\sim Q\\(q,a)=1}}\sum_{\substack{d\leqslant\Delta\\(d,q)=1}}\sum_{\substack{d_1|d^\infty\\d_1\leqslant y}}\sum_{\substack{d_1k_1'\equiv k_2 \bmod q,\, d_1k_1'\neq k_2\\(k_1',dk_2)=(k_1'k_2,d_1q)=1}}\gamma_{dd_1k_1'}\gamma_{dk_2}\psi\Big(\frac{Y-a\overline{dd_1k_1'}}{q}\Big),$$

where $\psi(t)+1/2$ is the fractional part of t. Our present formula of $W_1(Y,Q)$ coincides with the corresponding formula of $W_1(Y,Q)$ given in [10, p.369], with the tiny difference that the sum is over $d\leqslant x^{\epsilon}$ instead of $d\leqslant\Delta$. In [10], the problem of bounding $W_1(Y,Q)$ (with $Y=M$ or $2M$) is accomplished by appealing to Weil's bound for Kloosterman

sums. It is easy to check, that, in this paper, the summation over d is always made on the norms of the corresponding sums. Hence, since by (46), we have $\Delta \leqslant x^{\epsilon}$, we can apply [10, Form.(26)], in our case, giving the bound

$$W_1(Y,Q) \ll L^2MN^2Q^{-1}x^{-\frac{\epsilon}{2}} + L^3N^{\frac{5}{2}}Q^{-1}x^{3\epsilon} + L^{\frac{11}{4}}N^3Q^{-1}x^{5\epsilon}, \tag{56}$$

for $Y = M$ or $2M$. By the orthogonality of characters, the large sieve inequality and the assumption $(A_1(B,\kappa))$ for $\boldsymbol{\beta}$, we get (compare with [10, Form.(37) & (40)]) the inequality

$$\begin{aligned} 0 \leqslant MB(Q) - MA(Q) &\leqslant \sum_{\substack{q\sim Q\\(q,a)=1}} \frac{1}{q\,\varphi(q)} \sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \Big|\sum_k \chi(k)\gamma_k\Big|^2 \\ &\ll x^2M^{-1}Q^{-1}\mathcal{L}^{B^*-2C} + x\mathcal{L}^{B^*}, \end{aligned} \tag{57}$$

which is true for any $C > 0$. Gathering (41), (42), (43), (44), (55), (56) & (57), we can write

$$\begin{aligned} E^2(Q) \ll MQ\mathcal{L}^{B^*}\Big\{&L^2N^2Q^{-1} + LNQ^{-1}x^{1-\epsilon} + L^{\frac{5}{2}}N^{\frac{5}{2}}Q^{-1}x^{7\epsilon} + L^2MN^2Q^{-1}x^{-\frac{\epsilon}{6}} \\ &+ L^3N^{\frac{5}{2}}Q^{-1}x^{3\epsilon} + L^{\frac{11}{4}}N^3Q^{-1}x^{5\epsilon} + L^2MN^2Q^{-1}\mathcal{L}^{-2C} + x\Big\}, \end{aligned}$$

which simplifies into

$$E^2(Q) \ll x^2\mathcal{L}^{B^*-2C} + MQx\mathcal{L}^{B^*} + L^3MN^{\frac{5}{2}}x^{8\epsilon} + L^{\frac{11}{4}}MN^3x^{8\epsilon}.$$

This gives the bound claimed in Proposition 9, under the assumptions $(S3)$ and (39) after changing the value of ϵ and C.

□

3.5. **Particular cases of equidistribution.** We now finish with some particular cases where $\boldsymbol{\lambda}$ is the characteristic function of quasi primes. The first result is [4, Theorem 5*].

Proposition 10. *Let $a \neq 0$ be an integer. Let ϵ, z, B and C be positive numbers. Let x, K, L, M, $\boldsymbol{\eta}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ as in (37), and satisfying the extra conditions*

- $1 \leqslant z \leqslant \exp(\log 2x/(\log\log 2x))$,
- K, L, $M \geqslant x^{\epsilon}$,
- $\boldsymbol{\alpha}$ *and* $\boldsymbol{\eta}$ *satisfies* $(A_2(B))$,
- $\boldsymbol{\lambda}$ *satisfies* $(A_4(z))$.

Then there exists A_0 depending only on B and C, such that the following inequality holds

$$\sum_{\substack{q\sim Q\\(q,a)=1}} \Big|\, \Delta(\boldsymbol{\eta}*\boldsymbol{\lambda}*\boldsymbol{\alpha};q,a)\,\Big| \ll x\mathcal{L}^{-A}, \tag{58}$$

as soon as Q satisfies

$$Q\mathcal{L}^{A_0} < KL,\ MK^4Q < x^{2-\epsilon} \text{ and } MK^2Q^2 < x^{2-\epsilon}. \tag{S4}$$

The constant implied in the $\ll$–symbol in (58) depends at most on ϵ, a, B, C.

Finally we recall a consequence of bounds of exponential sums (coming either from Weil's or Deligne's work) and of the fundamental lemma in sieve theory. We have (see [4, Lemma 2*])

Proposition 11. *Let $\epsilon>0$. Let K, L and $M\geqslant 1$ and $x\geqslant KLM$, such that either $K=1$ or $K\geqslant x^{12\epsilon}$ and similarly either $L=1$ or $L\geqslant x^{12\epsilon}$ and either $M=1$ or $M\geqslant x^{12\epsilon}$. Let $\mathfrak{K}$, $\mathfrak{L}$ and $\mathfrak{M}$ be three intervals respectively included in $[K,2K[$, $[L,2L[$ and $[M,2M[$. Then there exists an absolute positive constant δ such that*

$$\Delta\big(\mathfrak{z}(\mathbf{1}_{\mathfrak{K}}*\mathbf{1}_{\mathfrak{L}}*\mathbf{1}_{\mathfrak{M}});q,a\big) \ll \frac{x}{\varphi(q)}\exp\Big(-\epsilon\frac{\log x}{\log z}\Big)$$

uniformly for $(q,a)=1$ and $q\leqslant x^{\frac12+\delta}$. The constant implied in the $\ll$–symbol depends at most on ϵ.

4. Proof of Theorem 4

We arrive now at the central part of our work. Of course, our proof follows closely the proof given in [4]. The combinatorics is heavy and we were unable to find shortcuts to simplify the technique of [4].

4.1. **Notations and first reductions of the proof of Theorem 4.** As in [4, §13], the notation

$$\sum_n{}^*$$

means that we are summing over integers n, with $\mathfrak{z}(n)=1$, and z now has the value

$$z:=\exp\Big(\frac{\log x}{(\log\log x)^2}\Big). \tag{59}$$

To prove Theorem 4, we consider

$$S(x,Q,P_1,P_2) := \sum_{\substack{q\sim Q\\(q,a)=1}} \Big| \sum_{\substack{P_1<p\leqslant P_2,\ pm\leqslant x\\ pm\equiv a \bmod q}} \sum \log p - \frac{1}{\varphi(q)} \sum_{\substack{P_1<p\leqslant P_2,\ pm\leqslant x\\(pm,q)=1}} \sum \log p \Big|.$$

Since a is supposed to be fixed, we forget the dependency on a. We want to prove the inequality

$$S(x,Q,P_1,P_2) \ll x\cdot \frac{(\log y)^2}{\log x}\cdot (\log\log x)^{B_2}, \tag{60}$$

under the conditions of Theorem 4. For the rest of the proof we may assume that the inequalities $B_2\geqslant 2$ and $Q^2\leqslant xy$ hold with

$$\mathcal{L}^A \leqslant y \leqslant \exp\Big(\frac{\log x}{\log\log x}\Big) := y_0, \tag{61}$$

indeed for $y\geqslant y_0$, (60) is trivial, by (13). In (61), A is a constant whose definition will be given in §4.6 when applying the results of §3.2 – 3.5. We shall also assume that

$$Q\geqslant x^{\frac12-\epsilon},$$

otherwise, (60) is a direct consequence of Proposition 4. Of course Theorem 4 is trivial also when P_1 is too large ($P_1>x$, since the sum is empty). Using the classical formulas

$$\sum_{\substack{m\leqslant \frac{x}{p}\\ m\equiv b \bmod q}} 1 = \frac{x}{pq} + O(1),$$

and

$$\sum_{\substack{m\leqslant \frac{x}{p}\\(m,q)=1}} 1 = \frac{\varphi(q)}{q}\cdot\frac{x}{p} + O(\tau(q)), \tag{62}$$

we deduce the inequality

$$S(x,Q,P_1,P_2) \ll \sum_{q\sim Q} \tau(q) \sum_{P_1<p\leqslant P_2} \log p \ll P_2\ Q\mathcal{L}.$$

In particular (60) holds trivially if $P_2\leqslant x^{\frac12}y_0^{-1}$ by our assumption (61). By the same remark we may as well assume that

$$P_2\geqslant P_1\geqslant x^{\frac12}y_0^{-1}. \tag{63}$$

Replacing the factor $\log p$ by the van Mangoldt function $\Lambda(n)$ (with an acceptable error) and applying a dyadic dissection we are led to introduce the modified sum

$$\mathcal{E}(x,Q,P_1,P_2) := \sum_{\substack{q\sim Q\\(q,a)=1}} \Big| \sideset{}{^*}\sum_{\substack{P_1<n\leqslant P_2,\\ nt\equiv a \bmod q}} \sum_{x/2<nt\leqslant x} \Lambda(n) - \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\substack{P_1<n\leqslant P_2,\\ (nt,q)=1}} \sum_{x/2<nt\leqslant x} \Lambda(n)\Big|. \tag{64}$$

Here the variable m has been renamed t to prepare for the applications of some lemmas of §3.1. We also observe that because of (63) and because a is fixed, we have $nt-a\neq 0$ if x is sufficiently large. The sum $\mathcal{E}(x,Q,P_1,P_2)$ is the analogue of $\mathcal{E}(x,Q)$ introduced in [4, p.388]. Gathering the above remarls, the proof of (60) is equivalent to the proof of the inequality

$$\mathcal{E}(x,Q,P_1,P_2) \ll x\cdot \frac{(\log y)^2}{\log x}\cdot (\log\log x)^{B_2}, \tag{65}$$

under the restrictions (61) and (63) and the condition $Q^2\leqslant xy$.

4.2. **Preparation of the variable t.** In (64), the variable t may have prime divisors less than the parameter z defined in (59). In particular we cannot use the characteristic function of the set $\{t\}$ to build (by convolution with other variables) a sequence (β_n) in order to apply one the propositions of the §3.2 – 3.5 (the assumption $(A_3(x))$ would not be satisfied). To circumvent this difficulty, we proceed as follows. We factorize each t as $t=t^\dagger\cdot t^\ddagger$, where

$$t^\dagger = \prod_{\substack{p^\nu \parallel t\\ p<z}} p^\nu.$$

This factorization is unique. Usually, $t^\dagger$ is small compared with t since we have (see [16, Theorem 07 p.4])

Lemma 6. *There exists an absolute positive c_0, such that, uniformly for $v\geqslant u\geqslant 2$ and $x\geqslant 2$ we have the inequality*

$$\Theta(x;u,v) := \big|\{n\leqslant x\ ;\ \prod_{\substack{p^\nu\parallel n\\ p\leqslant u}} p^\nu \geqslant v\}\big| \ll x\exp\Big(-c_0\,\frac{\log v}{\log u}\Big).$$

Let W_0 be a number which satisfies

$$W_0 \sim \exp\Big(\frac{\log x}{\sqrt{\log\log x}}\Big). \tag{66}$$

By Lemma 6, the contribution of the triples (q,n,t) such that $t^\dagger > W_0$ to the right part of (64) is

$$\ll \mathcal{L}\Big(\sum_{\substack{x/2<nt\leqslant x\\ t^\dagger>W_0}}{}^{*}\sum\tau(nt-a)+\sum_{\substack{x/2<nt\leqslant x\\ t^\dagger>W_0}}{}^{*}\sum 1\Big). \tag{67}$$

We write $m=nt$, to see that the expression (67) is

$$\ll \mathcal{L}\sum_{\substack{x/2<m\leqslant x\\ m^\dagger>W_0}}\tau(m)\tau(m-a)\ll \mathcal{L}\cdot\Theta^{\frac{1}{3}}(x;z,W_0)\cdot\Big(\sum_{m\leqslant x}\tau^3(m)\Big)^{\frac{1}{3}}\cdot\Big(\sum_{x/2<m\leqslant x}\tau^3(m-a)\Big)^{\frac{1}{3}},$$

by Hölder's inequality. Appealing to Lemmas 1 & 6, we see that the above term is

$$\ll x\cdot\mathcal{L}^6\cdot\exp\Big(-\frac{c_0}{3}(\log\log x)^{\frac{3}{2}}\Big)\ll x\mathcal{L}^{-C}\text{ for all }C,$$

which is acceptable in view of (65). These considerations allow to replace t by $t=uw$ where u (resp. w) has all its prime factors greater (resp. smaller) than z. In others words we are reduced to prove the inequality (65) for the sum $\tilde{\mathcal{E}}(x,Q,P_1,P_2)$ defined by

$$\tilde{\mathcal{E}}(x,Q,P_1,P_2):=\sum_{\substack{q\sim Q\\(q,a)=1}}\Big|\sum_{\substack{P_1<n\leqslant P_2,\ x/2<nuw\leqslant x\\ nuw\equiv a\bmod q}}{}^{*}\sum{}^{*}\sum{}^{\dagger}\Lambda(n)-\frac{1}{\varphi(q)}\sum_{\substack{P_1<n\leqslant P_2,\ x/2<nuw\leqslant x\\ (nuw,q)=1}}{}^{*}\sum{}^{*}\sum{}^{\dagger}\Lambda(n)\Big|, \tag{68}$$

where the †–symbol means that w has all its prime factors less than z and where w is small, that is

$$w\leqslant W_0. \tag{69}$$

4.3. **Application of a combinatorial identity.** To transform the function Λ into bilinear forms, we appeal to the identity of Heath–Brown (see [18, Prop. 13.3] for instance)

Lemma 7. *Let $J\geqslant 1$ and $n<2x$. We then have the equality*

$$\Lambda(n)=\sum_{j=1}^{J}(-1)^j\binom{J}{j}\sum_{m_1,\dots,m_j\leqslant x^{1/J}}\mu(m_1)\cdots\mu(m_j)\sum_{m_1\dots m_jn_1\dots n_j=n}\log n_1.$$

We apply this lemma to $\Lambda(n)$ inside (68) with $J=7$. It gives the inequality

$$\tilde{\mathcal{E}}(x,Q,P_1,P_2)\leqslant\binom{7}{4}\sum_{j=1}^{7}\tilde{\mathcal{E}}_j(x,Q,P_1,P_2), \tag{70}$$

with

$$\tilde{\mathcal{E}}_j(x,Q,P_1,P_2) := \sum_{\substack{q\sim Q\\(q,a)=1}} \Big| \sum\nolimits^{*}\cdots\sum\nolimits^{*}\sum\nolimits^{\dagger}_{m_1\cdots m_jn_1\cdots n_juw\equiv a \bmod q} \mu(m_1)\cdots\mu(m_j)\log n_1 - \frac{1}{\varphi(q)}\sum\nolimits^{*}\cdots\sum\nolimits^{*}\sum\nolimits^{\dagger}_{(m_1\cdots m_jn_1\cdots n_juw,q)=1} \mu(m_1)\cdots\mu(m_j)\log n_1\Big|, \tag{71}$$

where the variables of summation satisfy the inequalities

$$x/2 < m_1\cdots m_jn_1\cdots n_juw \leqslant x,\ P_1 < m_1\cdots m_jn_1\cdots n_j \leqslant P_2 \text{ and } m_1,\dots,m_j \leqslant D, \tag{72}$$

where D is any number $\geqslant x^{\frac{1}{7}}$. and where w satisfies (69).

4.4. **Dissection of the set of summation.** In (72), the variables m_i, n_i, u and w have to satisfy several multiplicative inequalities. To make these variables independent we proceed as usual in such problems, see [4, p.388] for instance. We define a parameter δ satisfying $x^{-\epsilon} < \delta < 1$, and introduce the notation

$$g \simeq G$$

to mean that the integer variable g satisfies $G \leqslant g < (1+\delta)G$. Let

$$\mathcal{D} := \big\{(1+\delta)^{\nu};\ \nu = 0,\ 1,\ 2,\dots\big\}.$$

To transform (72), we precise (66) and the choice of D by imposing

$$D,\ W_0 \in \mathcal{D} \text{ and } D \simeq x^{\frac{1}{7}}.$$

The conditions (72) are equivalent to

$$m_i \simeq M_i,\ n_i \simeq N_i\ (1 \leqslant i \leqslant j),\ u \simeq U \text{ and } w \simeq W, \tag{73}$$

for some numbers M_1,..., M_j, N_1,..., N_j, U and W from $\mathcal{D}$ satisfying

$$\begin{cases} x/2 < M_1\cdots M_jN_1\cdots N_jUW \leqslant x,\\ P_1 < M_1\cdots M_jN_1\cdots N_j \leqslant P_2,\\ M_1,\dots,M_j \leqslant D/(1+\delta),\\ W \leqslant W_0/(1+\delta), \end{cases} \tag{74}$$

unless the $2j+2$–uple $(m_1,\cdots,m_j,n_1,\cdots,n_j,u,w)$ is too near from some edge of the dissection, that means satisfies at least one of the following four conditions

$$\begin{cases} x < m_1\cdots m_j n_1\cdots n_j uw \leqslant x(1+\delta)^{2j+2}, \\ x/2 < m_1\cdots m_j n_1\cdots n_j uw \leqslant (x/2)(1+\delta)^{2j+2}, \\ P_1 < m_1\cdots m_j n_1\cdots n_j \leqslant P_1(1+\delta)^{2j} \ \text{ and } \ m_1\cdots m_j n_1\cdots n_j uw \leqslant x, \\ P_2 < m_1\cdots m_j n_1\cdots n_j \leqslant P_2(1+\delta)^{2j} \ \text{ and } \ m_1\cdots m_j n_1\cdots n_j uw \leqslant x. \end{cases} \tag{75}$$

Write $r := m_1\cdots m_j n_1\cdots n_j$ and $t := uw$ (note that each t can be written in an unique way in that form, since u (resp. w) has all its prime factors greater (resp. less) than z). It is easy to see that the contribution (denoted by $\mathcal{C}_{1,j}$) to $\tilde{\mathcal{E}}_j(x,Q,P_1,P_2)$ of the $(2j+2)$–uples $(m_1,\cdots,m_j,n_1,\cdots,n_j,u,w)$ which satisfy at least one of the four conditions of (75) is

$$\begin{aligned} \mathcal{C}_{1,j} \leqslant \mathcal{L}\Big\{ \sum_{\substack{x<rt\leqslant x(1+\delta)^{2j+2} \text{ or} \\ x/2<rt\leqslant (x/2)(1+\delta)^{2j+2}}} + \sum_{\substack{P_1<r\leqslant P_1(1+\delta)^{2j} \\ x/2<rt\leqslant x}} + \sum_{\substack{P_2<r\leqslant P_2(1+\delta)^{2j} \\ x/2<rt\leqslant x}} \Big\} \tau(rt-a)\tau_{2j}(r) \\ + \mathcal{L}\sum_{q\sim Q}\frac{1}{\varphi(q)}\Big\{ \sum_{\substack{x<rt\leqslant x(1+\delta)^{2j+2} \text{ or} \\ x/2<rt\leqslant (x/2)(1+\delta)^{2j+2}}} + \sum_{\substack{P_1<r\leqslant P_1(1+\delta)^{2j} \\ x/2<rt\leqslant x}} + \sum_{\substack{P_2<r\leqslant P_2(1+\delta)^{2j} \\ x/2<rt\leqslant x}} \Big\} \tau_{2j}(r). \end{aligned} \tag{76}$$

Write $s := rt$. Then we see that in the six inner summations of (76), s belongs to some set $\mathcal{S}\subset[x/2,2x]$, which satisfies

$$|\mathcal{S}| \ll x\big[(1+\delta)^{2j+2}-1\big] + x\log(1+\delta)^{2j} \ll \delta x. \tag{77}$$

Hence (76) implies that

$$\begin{aligned} \mathcal{C}_{1,j} &\ll \mathcal{L}\sum_{s\in\mathcal{S}}\tau(s-a)\tau_{2j}(s) + \mathcal{L}\sum_{s\in\mathcal{S}}\tau_{2j}(s) \\ &\ll \mathcal{L}\cdot|\mathcal{S}|^{\frac{1}{3}}\cdot\Big\{\sum_{x/2\leqslant s\leqslant 2x}\tau^3(s-a)\Big\}^{\frac{1}{3}}\cdot\Big\{\sum_{x/2\leqslant s\leqslant 2x}\tau_{2j}^3(s)\Big\}^{\frac{1}{3}} \\ &\ll \delta^{\frac{1}{3}}\,x\,\mathcal{L}^{B_3}, \end{aligned} \tag{78}$$

for some absolute positive B_3 by appealing to (77) and to Lemma 1.

Finally, remark that if $n_1\simeq N_1$, the function $n_1\mapsto\log n_1$ is almost constant, more precisely, we have

$$\log n_1 = \log N_1 + O(\delta). \tag{79}$$

Hence, replacing $\log n_1$ by $\log N_1$ in the expression $\tilde{\mathcal{E}}_j(x, Q, P_1, P_2)$, we create a global error $\mathcal{C}_{2,j}$ that we bound by

$$\begin{aligned}\mathcal{C}_{2,j} &\ll \delta \sum_{\substack{q\sim Q\\(q,a)=1}} \Big(\sum_{\substack{x/2<n\leqslant x\\ n\equiv a \bmod q}} \tau_{2j+1}(n) + \frac{1}{\varphi(q)} \sum_{x/2<n\leqslant x} \tau_{2j+1}(n)\Big)\\ &\ll \delta \sum_{x/2<n\leqslant x} \tau(n-a)\tau_{2j+1}(n) + \delta\, x\, \mathcal{L}^{2j}\\ &\ll \delta\, x\, \mathcal{L}^{B_4}, \qquad (80)\end{aligned}$$

for some positive absolute B_4. Here also we used the Cauchy–Schwarz inequality and Lemma 1 to prove (80). The constant δ is at our disposal, so we fix

$$\delta := \mathcal{L}^{-3(2+B_3+B_4)}. \qquad (81)$$

Hence, by (78) & (80), the error terms $\mathcal{C}_{1,j}$ and $\mathcal{C}_{2,j}$ both satisfy

$$\mathcal{C}_{1,j},\ \mathcal{C}_{2,j} \ll x\mathcal{L}^{-2},$$

which, in view of (70), is acceptable (compare with (65)).

It remains to prove, for $1 \leqslant j \leqslant 7$, the inequality

$$\mathcal{F}_j(x, Q, P_1, P_2) \ll x \Big(\frac{\log y}{\log x}\Big)^2 (\log\log x)^{B_2}, \qquad (82)$$

• where

$$\mathcal{F}_j(x, Q, P_1, P_2) := \sum_{\mathcal{M},\mathcal{N},U,W} \mathcal{F}_j(\mathcal{M}, \mathcal{N}, U, W, Q), \qquad (83)$$

• with $\mathcal{M} = (M_1, \dots, M_j)$, and $\mathcal{N} = (N_1, \dots, N_j)$,

• where the numbers M_i, N_i $(1 \leqslant i \leqslant j)$, U and W are taken in the set $\mathcal{D}$ and satisfy (74),

• and where

$$\mathcal{F}_j(\mathcal{M}, \mathcal{N}, U, W, Q) := \sum_{\substack{q\sim Q\\(q,a)=1}} \big|\mathcal{F}_j(\mathcal{M}, \mathcal{N}, U, W; q, a)\big|,$$

• with

$$\begin{aligned}\mathcal{F}_j(\mathcal{M}, \mathcal{N}, U, W; q, a) := {}& \sum_{\substack{m_i\simeq M_i,\dots,n_i\simeq N_i\ (1\leqslant i\leqslant j),\\ m_1\cdots m_j n_1\cdots n_j uw\equiv a \bmod q}}{}^{*} \cdots \sum^{*} \sum^{*} \sum^{\dagger}_{u\simeq U,\, w\simeq W} \mu(m_1)\cdots\mu(m_j)\\ &- \frac{1}{\varphi(q)} \sum_{\substack{m_i\simeq M_i,\dots,n_i\simeq N_i\ (1\leqslant i\leqslant j),\\ (m_1\cdots m_j n_1\cdots n_j uw, q)=1}}{}^{*} \cdots \sum^{*} \sum^{*} \sum^{\dagger}_{u\simeq U,\, w\simeq W} \mu(m_1)\cdots\mu(m_j). \qquad (84)\end{aligned}$$

Remark that in the conditions of summation (84), the variable w has all its prime factors smaller than z, this fact creates extra difficulty compared with [4]. Observe that the factor $\log n_1$ has been removed by (79), at the cost of the extra $\log x$–factor in the denominator on the right–hand side of (82).

4.5. **The boundary configuration.** We follow the technique of [4, § 13]. We fix

$$\Delta := y^6, \tag{85}$$

hence Δ satisfies $1 \leqslant \Delta < x^{\epsilon}$ by (61). Given an integer $r \geqslant 1$ and a r–uple $(D_1, \dots, D_r)$ of elements of $\mathcal{D}$ such that

$$D_1 \geqslant \cdots \geqslant D_r \geqslant D \text{ and } D_1 \cdots D_r < x,$$

we say that this r–uple is a *boundary configuration* if one of the following holds

$$r = 4, \quad D_1 \leqslant \Delta D_2 \quad \text{and} \quad D_3 \leqslant \Delta D_4, \tag{B_4}$$

$$r = 5, \quad D_3 \leqslant \Delta D_5, \tag{B_5}$$

$$r = 6, \quad D_4 \leqslant \Delta D_6. \tag{B_6}$$

Let $1 \leqslant j \leqslant 7$. A sequence $(\mathcal{M}, \mathcal{N}, U, W) = (M_1, \dots, M_j, N_1, \dots, N_j, U, W)$ is said to be *exceptional of type* B_r (with $4 \leqslant r \leqslant 6$) if it can be partioned into subsets whose products form a boundary configuration $(D_1, \dots, D_r)$ satisfying (B_r). A sequence is *generic* if it is not exceptional of any type. By (83) we get the inequality

$$\begin{aligned} &\mathcal{F}_j(x, Q, P_1, P_2) \\ &\quad = \sum_{r=4}^{6} \sum_{\substack{(\mathcal{M},\mathcal{N},U,W) \\ \text{exceptional of type } B_r}} \mathcal{F}_j(\mathcal{M},\mathcal{N},U,W,Q) + \sum_{\substack{(\mathcal{M},\mathcal{N},U,W) \\ \text{generic}}} \mathcal{F}_j(\mathcal{M},\mathcal{N},U,W,Q), \end{aligned} \tag{86}$$

where $(\mathcal{M}, \mathcal{N}, U, W)$ satisfy (74). We now treat the contribution of the $(\mathcal{M}, \mathcal{N}, U, W)$ of type (B_r) to the right–hand side of (86). We shall concentrate on $r = 4$ (necessarily, we have $j \geqslant 2$). We group the summation variables $m_1, \dots, m_j, n_1, \dots, n_j, u, w$ into four products d_1, d_2, d_3, d_4 which necessarily satisfy

$$n := m_1 \dots m_j n_1 \dots n_j uw = d_1 d_2 d_3 d_4 \leqslant 2x,$$

and

$$d_1 \geqslant d_2 \geqslant d_3 \geqslant d_4 \geqslant D \simeq x^{\frac{1}{7}}, \ d_1 \leqslant 2\Delta d_2, \ d_3 \leqslant 2\Delta d_4.$$

The number of representations of n in the form $n = m_1 \dots m_j n_1 \dots n_j uw$ is bounded by $\tau_{2j+1}(n)$. The study must be seperated in four cases according to the index i ($1 \leqslant i \leqslant 4$) such that the variable w appears in the constitution of the variable d_i in the above partition. We shall only write the case where $i = 1$ in full details (the other cases $i = 2$, $i = 3$ or $i = 4$ can be similarly handled by using variants (17) quoted at the end of Lemma 3). Note that, when $i = 1$, the variables $d_1' := d_1/w$, d_2, d_3 and d_4 all have their prime factors greater than z, but such a condition does not apply to w. We fix $y' = 2\Delta$ and we write

$$\begin{aligned}
&\sum_{\substack{(\mathcal{M},\mathcal{N},U,W) \text{ of type } B_4\\ w \text{ appears in } d_1}} \mathcal{F}_j(\mathcal{M},\mathcal{N},U,W;q,a)\\
&\leqslant \sum \sum^{*} \sum^{*} \sum^{*} \sum^{*} \tau_{2j+1}(d_4)\tau_{2j+1}(d_1' d_2 d_3)\\
&\qquad \sum_{\substack{wd_1' d_2 d_3 d_4 \leqslant 2x\\ D \leqslant d_4 \leqslant d_3 \leqslant d_2 \leqslant wd_1'\\ d_3 \leqslant y' d_4,\, wd_1' \leqslant y' d_2\\ wd_1' d_2 d_3 d_4 \equiv a \bmod q}}\\
&\quad + \frac{1}{\varphi(q)} \sum \sum^{*} \sum^{*} \sum^{*} \sum^{*} \tau_{2j+1}(d_4)\tau_{2j+1}(d_1' d_2 d_3)\\
&\qquad \sum_{\substack{wd_1' d_2 d_3 d_4 \leqslant 2x\\ D \leqslant d_4 \leqslant d_3 \leqslant d_2 \leqslant wd_1'\\ d_3 \leqslant y' d_4,\, wd_1' \leqslant y' d_2}}\\
&:= F_{4,1}(q,a) + \frac{1}{\varphi(q)} F_{4,1}, \qquad (87)
\end{aligned}$$

by definition. The index 1 is to remember that the variable w is glued to d_1'. Formula (16) of Lemma 3 gives the bound

$$F_{4,1} \ll (\log\log x) \cdot \frac{x}{\log 2D} \cdot \frac{(\log 2y')^2}{\log 2x} \cdot \Big(\frac{\log x}{\log z}\Big)^{8j+4} \ll x \cdot \Big(\frac{\log y}{\log x}\Big)^2 (\log\log x)^{B_5}, \qquad (88)$$

by (59), (85) and the inequality $\tau_{2j+1}(d_1' d_2 d_3) \leqslant \tau_{2j+1}(d_1')\tau_{2j+1}(d_2)\tau_{2j+1}(d_3)$. Here B_5 is an absolute constant. Summing over $q \sim Q$, $(q,a) = 1$, we recognize the second term in the inequality (82). By (87), we must give an upperbound for

$$0 \leqslant \sum_{\substack{q\sim Q\\ (q,a)=1}} F_{4,1}(q,a) \leqslant \sum_{\substack{q\sim Q\\ (q,a)=1}} \Big(F_{4,1}(q,a) - \frac{1}{\varphi(q)} \sum_{\substack{b \bmod q\\ (b,q)=1}} F_{4,1}(q,b)\Big) + \sum_{\substack{q\sim Q\\ (q,a)=1}} \frac{1}{\varphi(q)} F_{4,1}. \qquad (89)$$

The second term on the right part of (89) is treated as (88). For the first one, we appeal to Proposition 6 with the following choice of the parameters

$$\xi(\ell, m) = \sum \sum^{*} \sum^{*} \sum^{*} \tau_{2j+1}(d_1' d_2 d_3)$$
$$\sum_{\substack{wd_1' d_2 = \ell, d_3 = m\\ d_3 \leqslant d_2 \leqslant wd_1' \leqslant y' d_2}}$$

and

$$\beta_n = \sum_{D\leqslant d_4=n<(3x)^{\frac{1}{4}}}^{*} \tau_{2j+1}(d_4), \tag{90}$$

$y_1 = 1$ and $y_2 = y'$. By (89), we deduce

$$\sum_{\substack{q\sim Q\\(q,a)=1}} F_{4,1}(q,a) \ll x\mathcal{L}^{-2} + x\cdot\Big(\frac{\log y}{\log x}\Big)^2\cdot(\log\log x)^{B_5}, \tag{91}$$

The same procedure applies when w participates to d_2 or d_3. We then apply (17) of Lemma 3. However, when w participates to d_4, the choice of the function β_n given in (90) is not adequate, since the assumption $(A_3(x))$ is not satisfied to apply Proposition 6. We then choose $n = d_3$, $m = d_4$, $y_1 = 1/y'$ and $y_2 = 1$ to apply this Proposition. Note the inequality $x^{\frac{1}{7}} \leqslant n \leqslant 3x^{\frac{2}{7}}$ in all the cases.

It remains to deal with the contribution of the cases $r = 5$ and $r = 6$ to the right part of (86). The study is the same as for $r = 4$, but with Lemma 4 replacing Lemma 3.

In conclusion, we proved that the contribution of the boundary is acceptable, which means that it satisfies the inequality

$$\sum_{r=4}^{6} \sum_{\substack{(\mathcal{M},\mathcal{N},U,W)\\ \text{exceptional of type } B_r}} \mathcal{F}_j(\mathcal{M},\mathcal{N},U,W,Q) \ll x\cdot\Big(\frac{\log y}{\log x}\Big)^2\cdot(\log\log x)^{B_5}, \tag{92}$$

for $1 \leqslant j \leqslant 7$, with an absolute B_5.

4.6. **The interior.** This subsection has to be compared with [4, §15]. For any $1 \leqslant j \leqslant 7$, the number of generic subsequences $(\mathcal{M},\mathcal{N},U,W)$ is $O((\delta^{-1}\mathcal{L})^{16}) = O(\mathcal{L}^{B_6})$, by the choice (81), for some absolute constant B_6. Hence, by (86) and (92), the inequality (82) will be proved as soon as, for each j, with $1 \leqslant j \leqslant 7$, for each generic sequence $(\mathcal{M},\mathcal{N},U,W) = (M_1,\dots,M_j,N_1,\dots,N_j,U,W)$ satisfying

$$\begin{cases} x/2 < M_1\cdots M_jN_1\cdots N_jUW \leqslant x,\\ M_1,\dots,M_j \leqslant D/(1+\delta),\\ W \leqslant W_0/(1+\delta), \end{cases} \tag{93}$$

we have the inequality

$$\mathcal{F}_j(\mathcal{M},\mathcal{N},U,W,Q) \ll x\mathcal{L}^{-B_7}, \tag{94}$$

where B_7 is some absolute constant, for instance $B_7 = 1 + B_6$. The conditions (73), (84) and (93) show that the variable u behaves like any variable n_i. So it is natural to replace the name of the variable u by n_{j+1} and U by N_{j+1}. We put $\tilde{\mathcal{N}} = (\mathcal{N}, U)$ and $(\mathcal{M}, \mathcal{N}, U, W)$ becomes $(\mathcal{M}, \tilde{\mathcal{N}}, W)$. If $(\mathcal{M}, \tilde{\mathcal{N}}, W)$ is a generic sequence, by symmetry, we may assume that

$$N_1 \geqslant N_2 \geqslant \cdots \geqslant N_{j+1},$$

and we denote by $N_{s+1}, \cdots, N_{j+1}$ those N_i (possibly none) which are $\leqslant x^{\frac{1}{6}-\epsilon}$.

As in [4], the rest of the proof consists in playing with the orders of magnitude of $M_i (1 \leqslant i \leqslant j)$, N_i $(1 \leqslant i \leqslant j+1)$ and W to apply one of the propositions of §3.2, §3.3, §3.4 or §3.5. However, W is always small ($\leqslant W_0 \ll x^{\epsilon}$) and the corresponding characteristic function of the integers $w \simeq W$ never satisfies $(A_3(x))$, we shall also play with this small variable. This creates unexpected problems.

• *Case 1.* $M_1 \dots M_j N_{s+1} \dots N_{j+1} \geqslant x^{\epsilon}$. Then, there exists a partial product, say V, of some of these M_i and N_i, which satisfies $x^{\epsilon} \leqslant V \leqslant x^{\frac{1}{6}-\epsilon}$. This is a consequence of the inequalities $M_i \leqslant D$ and $x^{\frac{1}{6}-\epsilon} \geqslant N_{s+1} \geqslant \cdots \geqslant N_{j+1}$. We apply Proposition 5 with $N = V$, and M such that $MN = M_1 \dots M_j N_1 \dots N_{j+1} W$ (which is $\sim x/2$ by (93)). We also define $\boldsymbol{\beta}$ as the convolution of the functions $(\mathfrak{z}\mu)$ and $(\mathfrak{z})$, with respective support $\simeq M_i$ and $\simeq N_i$, where the M_i and N_i are those parameters which participate to the partial product V. Hence, in that case (94) is proved.

So we are left with the case

$$V := M_1 \dots M_j N_{s+1} \dots N_{j+1} \leqslant x^{\epsilon} \text{ and } N_1 \geqslant \cdots N_s \geqslant x^{\frac{1}{6}-\epsilon}. \tag{95}$$

By (93), we have $1 \leqslant s \leqslant 6$. The value of s is an important parameter of our discussion.

• *Case 2.* $s = 6$. We partition $(\mathcal{M}, \tilde{\mathcal{N}}, W)$ into

$$(N_1 V W) \geqslant N_2 \geqslant N_3 \geqslant N_4 \geqslant N_5 \geqslant N_6 \ (\geqslant x^{\frac{1}{6}-\epsilon}),$$

and since $(\mathcal{M}, \tilde{\mathcal{N}}, W)$ is generic, we necessarily have $N_4 > \Delta N_6$. We want to apply Proposition 9, with $L := (N_1 V W) N_2$, $M := N_4 N_5 N_6$ and $N := N_3$. We easily check the conditions $(S3)$, since, by (95), we have $L = x^{\frac{1}{3}+O(\epsilon)}$, $M = x^{\frac{1}{2}+O(\epsilon)}$ and $N = x^{\frac{1}{6}+O(\epsilon)}$.

It remains to check the inequality $Q \leqslant (LN)\mathcal{L}^{-A_0}$ to deduce that, in that case, (94) is satisfied. We write

$$x/2 \leqslant V W N_1 N_2 N_3 N_4 N_5 N_6 \leqslant V W N_1 N_2 N_3 N_4^2 N_5 \Delta^{-1} \leqslant \frac{VW}{\Delta} (N_1 N_2 N_3)^2,$$

which implies $x/2 \leqslant (LN)^2/\Delta$, from which we deduce

$$LN \geqslant (\Delta x/2)^{\frac{1}{2}} = (x/2)^{\frac{1}{2}} y^3 \geqslant Qy \geqslant Q\mathcal{L}^A,$$

by (61) and (85). Finally, if we choose A larger than $A_0(= A_0(B_7))$ where A_0 is defined in Proposition 9, we have $Q \leqslant (LN)\mathcal{L}^{-A_0}$. Proposition 9 is applicable and (94) is proved in that case.

• *Case 3.* $s = 5$. Recall the relation (see (93))

$$N_1 N_2 N_3 N_4 N_5 VW \asymp x.$$

We partition $(\mathcal{M}, \tilde{\mathcal{N}}, W)$ into

$$(N_1 VW) \geqslant N_2 \geqslant N_3 \geqslant N_4 \geqslant N_5 (\geqslant x^{\frac{1}{6}-\epsilon}). \tag{96}$$

Since $(\mathcal{M}, \tilde{\mathcal{N}}, W)$ is generic, we have, by (B_5), the inequality $N_3 > \Delta N_5$. We split the arguments in four cases.

• *Case 3.1.* $N_5 N_4 N_2 > Q\mathcal{L}^A$. As in [4], we apply Proposition 7 with the conditions $(S2)$. The choices are $K = N_4$, $L = N_2 N_5$ and $M = N_1 VW N_3$. We check the three inequalities of $(S2)$ as follows

$$KL = N_5 N_4 N_2 > Q\mathcal{L}^A,$$

by hypothesis,

$$\begin{aligned} KL^2 Q^2 = N_4 (N_2 N_5)^2 Q^2 &< \Delta^{-1} N_5 N_4 N_3 N_2^2 Q^2 \ll \Delta^{-1} x Q^2 \\ &\ll \Delta^{-1} x^2 y \ll x^2 y^{-3} \ll x^2 \mathcal{L}^{-A}, \end{aligned}$$

by the assumptions (61) and (85). Finally, we check

$$K^2 = N_4^2 \ll (x/N_5)^{\frac{1}{2}} \ll x^{\frac{5}{12}+\frac{\epsilon}{2}} \ll Q x^{-\epsilon}.$$

It suffices to choose A larger than $A_0(= A_0(B_7))$ as defined in Proposition 7 to conclude the proof of (94) in that case.

• *Case 3.2.* $xQ^{-1}\mathcal{L}^{-A} < N_5 N_4 N_2 \leqslant Q\mathcal{L}^A$. Remark that the last inequality implies

$$N_2 \leqslant x^{\frac{1}{6}+3\epsilon}.$$

As in [4], we apply Proposition 7 with the conditions $(S1)$ with $K = N_2$, $L = N_4 N_3$ and $M = N_1 VW N_5$. We check the three conditions of $(S1)$ as follows

$$KL = N_4\, N_3\, N_2 > \Delta\, N_5\, N_4\, N_2 > \Delta\, x\, Q^{-1} \mathcal{L}^{-A} > Q\mathcal{L}^A,$$

the last inequality being a consequence of (61) and (85). We also have

$$K^2L^3 = N_2^2(N_4N_3)^3 \leqslant N_2^8 \ll Qx\mathcal{L}^{-A},$$

and

$$K^4L^2(K+L) \ll K^4L^3 = N_2^4(N_4N_3)^3 \ll N_2^{10} \ll x^{\frac{5}{3}+30\epsilon} < x^{2-\epsilon}.$$

In that case also, we proved (94) by choosing A larger than $A_0 = A_0(B_7)$ as defined in Proposition 7.

• *Case 3.3.* $x^{\frac{10}{21}} < N_5N_4N_2 \leqslant xQ^{-1}\mathcal{L}^{-A}$. This case does not appear in [4]. We appeal to Proposition 9, with the choices $L = N_1VW$, $M = N_5N_4N_2$ and $N = N_3$. We directly check

$$LN = N_1VWN_3 \gg x/(N_5N_4N_2) \gg Q\mathcal{L}^A,$$

by assumption. To check the conditions of $(S3)$, we first notice that the inequalities (96) and $N_1N_2N_3N_4N_5VW \sim x$ imply

$$N_1 \leqslant x^{\frac{1}{3}+4\epsilon} \text{ and } N_3 \leqslant x^{\frac{2}{9}+\epsilon}. \tag{97}$$

It is easy to check that the first inequality $L^2N < M^{2-\epsilon}$ is satisfied when one has $L^4N^3 < x^{2-\epsilon}$. But this last inequality is true, since we write

$$L^4N^3 = L(LN)^3 \ll x^{\frac{1}{3}+6\epsilon}\ (x^{\frac{11}{21}})^3 \ll x^{\frac{40}{21}+6\epsilon},$$

by (97) and the hypothesis of this case.

We now see that the second condition of $(S3)$ $L^3N^4 < M^{4-\epsilon}$ is satisfied when one has $L^7N^8 < x^{4-\epsilon}$. To check this inequality, in our case, we write

$$L^7N^8 = (LN)^7N \ll (x^{\frac{11}{21}})^7x^{\frac{2}{9}+\epsilon} \ll x^{\frac{35}{9}+\epsilon} \ll x^{4-\epsilon},$$

by hypothesis of Case 3.3 and by (97). The last condition of $(S3)$ is trivial. In conclusion (94) is also proved in that case, by choosing A sufficiently large.

• *Case 3.4.* $N_5N_4N_2 \leqslant x^{\frac{10}{21}}$. We appeal to Proposition 7 (Conditions $(S1)$) with the choices $K = N_3$, $L = N_1$. We verify each of these conditions by writing

$$KL = N_3N_1 \gg x/(VWN_5N_4N_2) \gg x^{\frac{11}{21}}x^{-2\epsilon} \gg Q\mathcal{L}^{A_0},$$

by hypothesis of Case 3.4,

$$K^2L^3 = N_3^2N_1^3 \ll x^{\frac{13}{9}+14\epsilon} \ll Qx\mathcal{L}^{-A_0},$$

by (97) and, similarly

$$K^4L^2(K+L) \ll K^4L^3 = N_3^4N_1^3 \ll x^{\frac{17}{9}+16\epsilon} \ll x^{2-\epsilon}.$$

The proof of (94) is now complete in this case. It follows that it is complete in all the cases corresponding to $s = 5$.

• *Case 4.* $s = 4$. We start from the following configuration

$$N_1 \geqslant N_2 \geqslant N_3 \geqslant N_4 \ (\geqslant x^{\frac{1}{6}-\epsilon}). \tag{98}$$

• *Case 4.1.* We suppose that $N_4N_1 > x^{\frac{1}{2}+3\epsilon}$. As in [4], we apply Proposition 10 with $K = N_4$, $L = N_1$ and $M = N_2N_3VW$. We check that

$$KL = N_4N_1 > Q\mathcal{L}_0^A.$$

Using the trivial inequality $N_4 \ll x^{\frac{1}{4}}$, we check the last two conditions of $(S4)$ by writing

$$MK^4Q \ll xN_4^3N_1^{-1}Q \ll x^{\frac{1}{2}-3\epsilon}N_4^4Q \ll x^{2-\epsilon},$$

and

$$MK^2Q^2 \ll xN_4N_1^{-1}Q^2 \ll x^{\frac{1}{2}-3\epsilon}N_4^2Q^2 \ll x^{2-\epsilon}.$$

Here also (94) is proved in that case.

• *Case 4.2.* We suppose that we have $N_4N_1 < x^{\frac{1}{2}+3\epsilon}$ But the sequel of our discussion will depend on the effect of the factor VW on the inequality (98).

• *Case 4.2.1.* We now suppose

$$N_1 \geqslant VWN_2 \geqslant N_3 \geqslant N_4 \ (> x^{\frac{1}{6}-\epsilon}).$$

Since we are not in a boundary configuration, we have $N_1 > \Delta(VWN_2)$ or $N_3 > \Delta N_4$. We follow the technique of [4] by applying Proposition 7 (Conditions (S1)), with the choice $K = N_3$ and $L = N_1$. We check that

$$KL = N_1N_3 > (\Delta N_1(VWN_2)N_3N_4)^{\frac{1}{2}} \gg (\Delta x)^{\frac{1}{2}} > x^{\frac{1}{2}}y^2 > Q\mathcal{L}^A.$$

It remains to check the inequalities

$$K^2L^3 < Qx\mathcal{L}^{-A}, \tag{99}$$

and

$$K^4L^2(K+L) \ < x^{2-\epsilon}. \tag{100}$$

It is easy to see that both (99) & (100) are implied by the two inequalities

$$K^2L^3 < x^{\frac{3}{2}-\epsilon} \text{ and } K^4L^3 < x^{2-\epsilon}. \tag{101}$$

In other words, (94) is proved if we assume that (101) holds (under the hypothesis of Case 4.2.1).

We now suppose that at least one of the inequalities of (101) is not satisfied. Then we turn our attention to Proposition 10. To follow the notations of that Proposition, we fix $K = N_3$, $L = N_1$ and $M = N_2N_4VW$. In order to check the conditions $(S4)$, we easily see that they are satisfied if one has

$$N_1N_3 > x^{\frac{1}{2}+\epsilon},\ N_3^3 < N_1x^{\frac{1}{2}-2\epsilon} \text{ and } N_3 < N_1x^{-2\epsilon}. \tag{102}$$

Also recall that the inequalities $N_1N_2N_3N_4 < x$ and (98) imply

$$N_1N_3^2 \leqslant x^{\frac{5}{6}+\epsilon}. \tag{103}$$

• Suppose that $K^2L^3 = N_1^3N_3^2 \geqslant x^{\frac{3}{2}-\epsilon}$. Then, we deduce that $N_1^3N_3^3 \geqslant (x^{\frac{1}{6}-\epsilon}) \cdot (x^{\frac{3}{2}-\epsilon})$, which implies that $N_1N_3 > x^{\frac{1}{2}+\epsilon}$. This is the first condition of (102).

To check the second condition, we combine with the square of (103) to write $(N_1^3N_3^2) \cdot (x^{\frac{5}{6}+\epsilon})^2 \geqslant x^{\frac{3}{2}-\epsilon}(N_1N_3^2)^2$. This is equivalent to $N_1x^{\frac{1}{6}+3\epsilon} \geqslant N_3^2$. This certainly implies the second condition of (102) since (103) gives $N_3 \leqslant x^{\frac{5}{18}+\epsilon}$.

For the last one, we start from $N_3 \leqslant x^{\frac{5}{18}+\epsilon}$. However, by hypothesis, we have $N_1 \geqslant (x^{\frac{3}{2}-\epsilon}N_3^{-2})^{\frac{1}{3}} \geqslant x^{\frac{17}{54}-\epsilon}$. This gives the third inequality of (102) since $17/54 > 5/18$.

• Suppose that $K^4L^3 = N_1^3N_3^4 \geqslant x^{2-\epsilon}$. Then we deduce that $N_1^4N_3^4 > (x^{2-\epsilon}) \cdot (x^{\frac{1}{6}-\epsilon})$, which implies that $N_1N_3 > x^{\frac{1}{2}+\epsilon}$. This is the first condition of (102).

For the second condition, we raise the inequality (103) to the power $5/2$, so we have $(N_1^3N_3^4) \cdot (x^{\frac{5}{6}+\epsilon})^{\frac{5}{2}} \geqslant x^{2-\epsilon}(N_1N_3^2)^{\frac{5}{2}}$. This is equivalent to $N_1x^{\frac{1}{6}+7\epsilon} \geqslant N_3^2$, from which we deduce the second inequality of (102) as a consequence of $N_3 \leqslant x^{\frac{5}{18}+\epsilon}$.

To check the last inequality, we start from $N_3 \leqslant x^{\frac{5}{18}+\epsilon}$. However, by hypothesis, we have $N_1 \geqslant (x^{2-\epsilon}N_3^{-4})^{\frac{1}{3}} \geqslant x^{\frac{8}{27}-2\epsilon}$. This gives the third inequality of (102) since $8/27 > 5/18$.

• *Case 4.2.2.* We now investigate the following situation

$$N_1 \geqslant N_2 \geqslant N_3 \geqslant VWN_4\ (> x^{\frac{1}{6}-\epsilon}).$$

Since we are not in a boundary configuration, we have $N_1 > \Delta N_2$ or $N_3 > \Delta VWN_4$. We apply the same technique as in Case 4.2.1, with $K = N_3$, $L = N_1$ and $M = N_2N_4VW$. The calculations are the same.

• *Case 4.2.3.* We now suppose that we have both conditions

$$N_1 \leqslant VWN_2 \text{ and } N_3 \leqslant VWN_4. \tag{104}$$

In other words, N_1 has almost the same order of magnitude as N_2, the same is true also for N_3 and N_4. Combining (104) with the relation $N_1N_2N_3N_4VW \sim x$, we get

$$\Big(\frac{x}{2VW}\Big)^{\frac{1}{2}} \leqslant N_1N_3 \leqslant (xVW)^{\frac{1}{2}}. \tag{105}$$

Consider the sequence $VWN_1 \geqslant N_2 \geqslant N_3 \geqslant N_4$. From the inequalities $(VWN_1) \leqslant (VW)^2N_2$ and $N_3 \leqslant VWN_4$, we deduce that, necessarily, we have $VW > \Delta^{\frac{1}{2}}$ otherwise, the above configuration would be a boundary configuration of type (B_4). We apply Proposition 9 with $L = N_3$ and $N = N_1VW$. It easy to check that

$$x \ll N_1N_2N_3N_4VW \leqslant N_1^2N_3^2VW = (LN)^2(VW)^{-1}$$

which implies that $LN > x^{\frac{1}{2}}(VW)^{\frac{1}{2}} \geqslant x^{\frac{1}{2}}\Delta^{\frac{1}{4}} > Q\mathcal{L}^A$, by the definition of Δ.

The two first inequalities of the set of conditions $(S3)$, are consequences of the inequalities $L^4N^3 < x^{2-\epsilon}$ and $L^7N^8 < x^{4-\epsilon}$ (see discussion in Case 3.3). Hence, by (105), we write

$$L^4N^3 = N_3^4(N_1VW)^3 = N_3(N_1N_3)^3(VW)^3 \ll x^{\frac{3}{2}+10\epsilon}N_3 < x^{\frac{7}{4}+11\epsilon},$$

and

$$L^7N^8 = N_3^7(N_1VW)^8 = N_1(N_1N_3)^7(VW)^8 \ll x^{\frac{7}{2}+20\epsilon}N_1 \leqslant x^{\frac{23}{6}+25\epsilon},$$

since, by (104), we deduce $N_1 < x^{\frac{1}{3}+2\epsilon}$ and $N_3 < x^{\frac{1}{4}+\epsilon}$, from the inequality $N_1N_2N_3N_4 \leqslant x$. The last inequality of $(S3)$ is trivially verified.

• *Case 5.* $s = 1$, 2 or 3. Recall the inequalities (95). For $s = 3$, the corresponding sum $\mathcal{F}_j(\mathcal{M},\mathcal{N},U,W,Q) = \mathcal{F}_j(\mathcal{M},\tilde{\mathcal{N}},W,Q)$ satisfies the inequality

$$\mathcal{F}_j(\mathcal{M},\mathcal{N},U,W,Q) \leqslant \sum_{\substack{q\sim Q\\(q,a)=1}}\sum_{\substack{v\leqslant 2V\\(v,q)=1}}\sum_{\substack{w\simeq W\\(w,q)=1}} \Big| \sum_{\substack{n_1n_2n_3\equiv a\overline{vw} \bmod q\\ n_1\simeq N_1,\, n_2\simeq N_2,\, n_3\simeq N_3}}\sum\sum\sum \mathfrak{z}(n_1n_2n_3) - \frac{1}{\varphi(q)} \sum_{\substack{(n_1n_2n_3,q)=1\\ n_1\simeq N_1,\, n_2\simeq N_2,\, n_3\simeq N_3}}\sum\sum\sum \mathfrak{z}(n_1n_2n_3) \Big|.$$

Since we have $N_1N_2N_3 > x^{1-2\epsilon} > Q^{\frac{1}{1/2+\delta}}$, we apply Proposition 11 to the expression inside $|\cdots|$ giving

$$\begin{aligned}\mathcal{F}_j(\mathcal{M},\mathcal{N},U,W,Q) &\ll \sum_{\substack{q\sim Q\\(q,a)=1}}\sum_{\substack{v\leqslant 2V\\(v,a)=1}}\sum_{\substack{w\simeq W\\(w,a)=1}}\frac{N_1N_2N_3}{\varphi(q)}\cdot\exp\big(-\frac{\epsilon}{2}(\log\log x)^2\big)\\ &\ll x\mathcal{L}^{-B_7},\end{aligned}$$

since $N_1N_2N_3VW \ll x$. This gives (94). The case $s = 2$ is treated in a similar way. The case $s = 1$ is trivial.

This completes the proof of Theorem 4.

5. Proof of Theorem 1

Theorem 1 is an easy consequence of Proposition 2.

Proof. Under the assumptions of Theorem 1, let

$$Y = N\Big(1-\frac{1}{2(\log N)^A}\Big), Z_1 = N^2\Big(1-\frac{1}{2(\log N)^A}\Big) \text{ and } Z_2 = N^2\Big(1-\frac{1}{(\log N)^A}\Big).$$

Now consider the quantity

$$S := \sum_{Y\leqslant a\leqslant N}\Big(\pi(Z_1;a,1)-\pi(Z_2;a,1)\Big).$$

This sum is counting (with multiplicities of representations) the primes p satisfying $Z_2 < p \leqslant Z_1$ congruent to 1 mod a, with $Y \leqslant a \leqslant N$. Hence, such p are of the form $p = 1 + ab$ with $a \leqslant N$ and $b \leqslant (Z_1 - 1)/Y \leqslant N$.

Proposition 2 and the Prime Number Theorem give the relations

$$\begin{aligned}S &= \big(\pi(Z_1)-\pi(Z_2)\big)\sum_{Y\leqslant a\leqslant N}\frac{1}{\varphi(a)} + O\big(N^2(\log N)^{-2A-2}\big)\\ &\geqslant \big(\pi(Z_1)-\pi(Z_2)\big)\cdot\frac{N-Y}{N} - O\Big(\frac{N^2}{(\log N)^{2A+2}}\Big)\\ &\geqslant \frac{N^2}{8(\log N)^{2A+1}}(1-o(1)) - O\Big(\frac{N^2}{(\log N)^{2A+2}}\Big)\\ &\geqslant 1,\end{aligned}$$

for $N \geqslant N_0(A)$. Hence, the sum S is not empty, this implies the existence of a p with the required property. □

6. Proof of Theorem 2

This proof has many similarities with the proof of Theorem 1. The main difference, is that we shall appeal to Proposition 3 instead of Proposition 2. We are searching for primes of the form $p = ab+1$, with $b \in \mathcal{B}$, a, $b \leqslant N$, and $(1-2\delta)N^2 < p \leqslant (1-\delta)N^2$. This set of primes p certainly contains the set

$$\mathcal{E}(\mathcal{B}, N) := \big\{p \ ; \ (1-2\delta)N^2 < p \leqslant (1-\delta)N^2, \\ p \equiv 1 \text{ modulo some } b \in \mathcal{B} \text{ satisfying } (1-\delta)N < b \leqslant N\big\},$$

and $\mathcal{E}(\mathcal{B}, N)$ is non empty if and only if the sum S_1 defined by

$$S_1 := \sum_{\substack{b \in \mathcal{B} \\ (1-\delta)N < b \leqslant N}} \Big\{\pi\big((1-\delta)N^2; b, 1\big) - \pi\big((1-2\delta)N^2; b, 1\big)\Big\},$$

is non zero. By (12) deduced from Proposition 3, we see that S_1 satisfies the equality

$$S_1 = \Big\{\pi\big((1-\delta)N^2\big) - \pi\big((1-2\delta)N^2\big)\Big\} \sum_{\substack{b \in \mathcal{B} \\ (1-\delta)N < b \leqslant N}} \frac{1}{\varphi(b)} + O\Big(\frac{N^2}{\log^3 N} \cdot (\log\log N)^{B_1}\Big). \tag{106}$$

By the Prime Number Theorem, we know that the term inside $\{\cdots\}$ in (106) is $\sim \delta N^2/(2\log N)$, as $N \to \infty$. Inserting this formula into (106) and using the assumption (6), we obtain $S_1 \geqslant 1$, by choosing $c_6 = B_1 + 1$ and $N \geqslant c_7(\delta)$. This completes the proof of Theorem 2.

7. Proof of Theorem 3

Both sequences $\mathcal{A}$ and $\mathcal{B}$ are now quite general and to shorten notations, we write

$$A := |\mathcal{A}| \text{ and } B := |\mathcal{B}|.$$

Hence we have the inequalities

$$A \geqslant B \geqslant N/(\log N)^{\delta} \text{ with } 0 < \delta < 1, \tag{107}$$

as a consequence of (7). We are using the Tchebychev–Hooley method as it is done in [24] & [27]. Consider the product

$$E = E(\mathcal{A}, \mathcal{B}, N) := \prod_{\substack{a \in \mathcal{A}, \ b \in \mathcal{B} \\ a, \, b \leqslant N}} \prod (ab+1). \tag{108}$$

Taking logarithms, we deduce the lower bound

$$\log E \geqslant \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \log(ab) \geqslant \sum_{a\leqslant A} \sum_{b\leqslant B} \log(ab),$$

which leads to the lower bound

$$\log E \geqslant (2 - o_\delta(1))AB \log N, \tag{109}$$

uniformly under the condition (107).

Let

$$P = P^+(E),$$

be the number which we would like to bound from below in terms of A, B and N. As in [27, §6], we introduce

$$E_1 = \prod_{p\leqslant N} p^{v_p(E)},$$

where v_p is the p–adic valuation. We first prove (compare with [27, Lemma 3])

Proposition 12. *For every subsets $\mathcal{A}$ and $\mathcal{B}$ of $[1, \dots, N]$, with cardinalities A and B satisfying (107), we have the inequality*

$$\log E_1 \leqslant (1 + o_\delta(1))\, AB \log N. \tag{110}$$

as $N \to \infty$.

Proof. We first recall the following lemma (see [24, Lemma 4])

Lemma 8. *Let N be a positive integer and $\mathcal{U} \subset [1, \dots, N]$. Let h and m be integers with $m \geq 1$ and let*

$$r(\mathcal{U}, h, m) := \big|\{u \in \mathcal{U}\,;\ u \equiv h \bmod m\}\big|.$$

We then have the inequality

$$\sum_{p\leqslant N} \log p \sum_{k\leqslant \frac{\log N}{\log p}} \sum_{h=1}^{p^k} \big(r(\mathcal{U}, h, p^k)\big)^2 \leqslant |\mathcal{U}|\ \big(|\mathcal{U}| - 1 + \pi(N)\big) \log N.$$

The proof of Proposition 12 now follows the proof of [24, 4.14]. We have

$$\begin{aligned}\log E_1 &= \sum_{p\leqslant N} v_p(E)\log p\\ &= \sum_{p\leqslant N}\log p \sum_{k\leqslant \frac{\log(N^2+1)}{\log p}} \Big|\big\{(a,b)\in\mathcal{A}\times\mathcal{B};\ ab\equiv -1 \bmod p^k\big\}\Big|\\ &= \Sigma_1+\Sigma_2, \qquad (111)\end{aligned}$$

where Σ_1 and Σ_2 respectively correspond to the cases $1\leqslant k\leqslant \frac{\log N}{\log p}$ and $\frac{\log N}{\log p}< k\leqslant \frac{\log(N^2+1)}{\log p}$. Denoting by $\overline{h}$ the multiplicative inverse of $h \bmod p^k$ and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\Sigma_1 &= \sum_{p\leqslant N}\log p \sum_{k\leqslant \frac{\log N}{\log p}} \sum_{\substack{1\leqslant h\leqslant p^k\\ (h,p^k)=1}} r(\mathcal{A},h,p^k)r(\mathcal{B},-\overline{h},p^k)\\ &\leqslant \Big\{\sum_{p\leqslant N}\log p \sum_{k\leqslant \frac{\log N}{\log p}} \sum_{\substack{1\leqslant h\leqslant p^k\\ (h,p^k)=1}} r^2(\mathcal{A},h,p^k)\Big\}^{\frac12}\cdot\Big\{\sum_{p\leqslant N}\log p \sum_{k\leqslant \frac{\log N}{\log p}} \sum_{\substack{1\leqslant h\leqslant p^k\\ (h,p^k)=1}} r^2(\mathcal{B},h,p^k)\Big\}^{\frac12}\\ &\leqslant \Big\{A\big(A-1+\pi(N)\big)\Big\}^{\frac12}\cdot\Big\{B\big(B-1+\pi(N)\big)\Big\}^{\frac12}(\log N)\\ &\leqslant (1+o(1))\,AB\,\log N, \qquad (112)\end{aligned}$$

the last lines being a consequence of Lemma 8 and of the condition (107).

For Σ_2, we remark that we have the inequality $p^k>N$, hence when b is fixed, the equation $ab\equiv -1 \bmod p^k$ has at most one solution in a. From this, we deduce the inequality

$$\begin{aligned}\Sigma_2 &\leqslant \sum_{p\leqslant N}\log p \sum_{\frac{\log N}{\log p}<k\leqslant \frac{\log(N^2+1)}{\log p}} B\\ &\leqslant B\sum_{p\leqslant N}\log(N^2+1) = B\cdot\pi(N)\cdot\log(N^2+1). \qquad (113)\end{aligned}$$

Putting together (107), (111), (112) and (113), we complete the proof of Proposition 12. □

7.1. **Use of Theorem 4.** Let

$$E_2 = \prod_{\substack{N<p\leqslant P\\ p\mid E}} p^{v_p(E)},$$

where E is defined by (108). We then have

$$\log E_2 = \log E - \log E_1.$$

By (109) & (110) we have the lower bound

$$\log E_2 \geqslant (1 - o_\delta(1))\, AB \log N. \tag{114}$$

We are now searching for an upper bound of $\log E_2$. Since $v_p(ab+1) \leqslant 1$ for any p satisfying $N < p \leqslant P$ and $a,\, b \leqslant N$, we have the equality

$$v_p(E_2) = \big|\big\{(a,b) \in \mathcal{A} \times \mathcal{B};\ ab + 1 \equiv 0 \bmod p\big\}\big|.$$

We now use the property $\mathcal{A} \subset [1, \dots, N]$ to write the inequality

$$\log E_2 \leqslant \sum_{(a,b,m,p)\in\mathcal{Q}} \log p, \tag{115}$$

where the sum is over the set $\mathcal{Q}$ of the quadruples (a, b, m, p) defined by

$$\mathcal{Q} := \big\{(a,b,m,p);\ ab + 1 = pm,\ N < p \leqslant P,\ 1 \leqslant a,\, b \leqslant N,\ b \in \mathcal{B}\big\}.$$

We want to drop the inequality $a \leqslant N$ in order to apply Theorem 4. So we include $\mathcal{Q}$ in the disjoint union

$$\mathcal{Q} \subset \bigcup_{\ell=0}^{\ell_0} \mathcal{R}_\ell,$$

where

$$\mathcal{R}_\ell := \Big\{(a,b,m,p);\ ab + 1 = pm,\ pm \leqslant (1-\kappa)^\ell N^2 + 1,\ N < p \leqslant P,\\ b \in \mathcal{B},\ (1-\kappa)^{\ell+1} N < b \leqslant (1-\kappa)^\ell N\Big\},$$

where $\kappa = \kappa(N)$ is a function of N tending to zero as N tends to ∞ and ℓ_0 is the integer defined by $(1-\kappa)^{\ell_0+1} N < 1 \leqslant (1-\kappa)^{\ell_0} N$. This integer ℓ_0 satisfies $\ell_0 = O(\kappa^{-1} \log N)$. Using this decomposition, we transform (115) into the inequality

$$\log E_2 \leqslant \sum_{\ell=0}^{\ell_0} T_\ell, \tag{116}$$

with

$$T_\ell := \sum_{\substack{b\in\mathcal{B}\\ (1-\kappa)^{\ell+1} N < b \leqslant (1-\kappa)^\ell N}} \ \sum_{\substack{N < p \leqslant P,\ pm \leqslant (1-\kappa)^\ell N^2 + 1\\ pm \equiv 1 \bmod b}} \log p. \tag{117}$$

Note that we have lost the information that $a \in \mathcal{A}$ in the process of replacing the identity $pm = 1 + ab$ by the congruence condition $pm \equiv 1 \bmod b$. On the other hand,

since we still have the inequalities $\{(1-\kappa)^\ell N\}^2 \leqslant 3\big((1-\kappa)^\ell N^2+1\big)$ for $\ell \leqslant \ell_0$, we are now in good position to apply Theorem 4 (with $y=3$) to T_ℓ. Let $\rho(N^2) := N^2(\log\log N)^{B_2}(\log N)^{-1}$. With this theorem we have the equality

$$T_\ell = \sum_{\substack{b\in\mathcal{B}\\ (1-\kappa)^{\ell+1}N<b\leqslant(1-\kappa)^\ell N}} \frac{1}{\varphi(b)} \Big\{ \sum_{N<p\leqslant P,\ pm\leqslant(1-\kappa)^\ell N^2+1} \sum_{\substack{(pm,b)=1}} \log p \Big\} + O\big(\rho((1-\kappa)^\ell N^2+1)\big)$$

(118)
$$= T_\ell^{(1)} + O\big(T_\ell^{(2)}\big),$$

by definition. Using the equality (62) and the Prime Number Theorem, we transform $T_\ell^{(1)}$ as follows

$$\begin{aligned} T_\ell^{(1)} &= \sum_b \frac{1}{\varphi(b)} \Big\{ \sum_{N<p\leqslant P} \log p\Big(\frac{\varphi(b)}{b}\cdot\frac{(1-\kappa)^\ell N^2}{p} + O(\tau(b))\Big)\Big\} \\ &= N^2\cdot\Big(\log\frac{P}{N}+o(1)\Big)\cdot\Big((1-\kappa)^\ell\sum_b\ \frac{1}{b}\Big) + O\Big(P\sum_b\frac{\tau(b)}{\varphi(b)}\Big) \\ &\leqslant N^2\cdot\Big(\log\frac{P}{N}+o(1))\Big)\cdot\Big(\frac{1}{(1-\kappa)N}\cdot\sum_b\ 1\Big) + O\Big(P\sum_b\frac{\tau(b)}{\varphi(b)}\Big). \end{aligned} \tag{119}$$

In (119), the conditions of summation are always: $b\in\mathcal{B}$ and $(1-\kappa)^{\ell+1}N < b \leqslant (1-\kappa)^\ell N$. We now sum over all the $\ell\leqslant\ell_0$. We first write that

$$\sum_{\ell=0}^{\ell_0} T_\ell^{(2)} \ll \frac{N^2}{\kappa\log N}\cdot(\log\log N)^{B_2}.$$

Combining this relation with (116), (118) & (119), and summing $T_\ell^{(1)}$ over ℓ, we have the inequality

$$\log E_2 \leqslant \frac{BN}{1-\kappa}\cdot\Big(\log\frac{P}{N}+o(1)\Big) + O\Big(\frac{N^2}{\kappa\log N}\cdot(\log\log N)^{B_2}\Big) + O\big(P\log^2 N\big).$$

Choosing $\kappa = (\log N)^{\frac{\delta-1}{2}}$ and recalling (107) we obtain

$$\log E_2 \leqslant \big(1+o_\delta(1)\big)\cdot\Big(\log\frac{P}{N}+o(1)\Big)\cdot BN.$$

Comparing with (114), we obtain the inequality

$$\log\frac{P}{N} \geqslant (1-o_\delta(1))\frac{A}{N}\log N - o(1) \geqslant (1-o_\delta(1))\frac{A}{N}\log N.$$

This completes the proof of Theorem 3.

References

[1] E. Bombieri, On the large sieve, *Mathematika* 12: 201–225, 1965.

[2] E Bombieri, Le Grand Crible Dans La Théorie Analytique Des Nombres (Seconde édition). *Astérisque,* vol. 18, S.M.F., 1987.

[3] E. Bombieri, J.B. Friedlander and H. Iwaniec, Primes in arithmetic progressions to large moduli, *Acta Math.* 156: 203–251, 1986.

[4] E. Bombieri, J.B. Friedlander and H. Iwaniec, Primes in arithmetic progressions to large moduli. II, *Math. Annalen* 277: 361–393, 1987.

[5] E. Bombieri, J.B. Friedlander and H. Iwaniec, Primes in arithmetic progressions to large moduli. III, *J. of the Amer. Math. Soc.* 2: 215–224, 1989.

[6] J–M. Deshouillers and H. Iwaniec, Kloosterman sums and Fourier coefficients of cusp forms, *Inv. math.* 70: 219–288, 1982.

[7] P.D.T.A. Elliott and H. Halberstam, A conjecture in prime number theory, *Symp. Math.* 4: 59–72, 1968–69.

[8] K. Ford, The distribution of integers with a divisor in a given interval, *Ann. of Math. (2)* 168: 367–433, 2008.

[9] É. Fouvry, Répartition des suites dans les progressions arithmétiques. Résultats du type Bombieri–Vinogradov avec exposant supérieur à $\frac{1}{2}$, *Thèse de l'Université de Bordeaux I,* 1981.

[10] É. Fouvry, Répartition des suites dans les progressions arithmétiques, *Acta Arith.* 41: 359–382, 1982.

[11] É. Fouvry, Autour du théorème de Bombieri–Vinogradov, *Acta Math.* 152: 219–244, 1984.

[12] É. Fouvry, Sur le problème des diviseurs de Titchmarsh, *J. Reine Angew. Math.,* 357: 51–76, 1985.

[13] É. Fouvry, Autour du Théorème de Bombieri-Vinogradov.II, *Ann. Scient. École Norm. Sup. (4)* 20: 617–640, 1987.

[14] É. Fouvry and H. Iwaniec, On a theorem of Bombieri–Vinogradov type, *Mathematika* 27: 135–152, 1980.

[15] É. Fouvry and H. Iwaniec, Primes in arithmetic progressions, *Acta Arith.* 42: 197–218, 1983.

[16] R.R. Hall and G. Tenenbaum, Divisors. *Cambridge University Press,* vol. 90, Cambridge, 1988.

[17] K.–H. Indlekofer and N.M. Timofeev, Divisors of shifted primes, *Pub. Math. Debrecen,* 60: 307–345, 2002.

[18] H. Iwaniec and E. Kowalski, Analytic Number Theory, *Colloquium Publications*, 53, AMS, 2004.

[19] D. Koukoulopoulos, Divisors of shifted primes, *Int. Math. Res. Not. IMRN 2010*, no. 24, 4585–4627

[20] Ju.V. Linnik, The Dispersion Method In Binary Additive Problems, *Translation of Mathematical Monographs*, 4, AMS, 1963.

[21] K. Matomäki, On the greatest prime factor of $ab + 1$, *Acta Math. Hungar.* 124: 115–123, 2009.

[22] M. Nair, Multiplicative functions of polynomial values in short intervals, *Acta Arith.* 62: 257–269, 1992.

[23] M. Nair and G. Tenenbaum, Short sums of certain arithmetic functions, *Acta Math.* 180: 119–144, 1998.

[24] A.Sárközy and C.L. Stewart, On prime factors of integers of the form $ab+1$, *Pub. Math. Debrecen,* 56: 559–573, 2000.

[25] P. Shiu, A Brun-Titchmarsh theorem for multiplicative functions, *J. Reine Angew. Math.,* 313: 161–170, 1980.

[26] C.L. Stewart, On the greatest prime factor of integers of the form $ab + 1$ *Period. Math. Hungar,* 43: 81–91, 2001.

[27] C.L. Stewart, On prime factors of integers which are sums or shifted products, *Anatomy of integers,* (Ed. J.–M. de Koninck, A. Granville, F. Luca), CRM Proceedings & Lecture Notes A.M.S., vol. 46: 275–287, 2008.

[28] G. Tenenbaum, Introduction à la théorie analytique et probabiliste des nombres. *Cours Spécialisés, S.M.F.,* vol. 1, 1995.

[29] A.I. Vinogradov, On the density hypothesis for Dirichlet L–series, *Izv. Akad. Nauk. SSSR ser. Mat.* 29: 903–934, 1965; correction ibid. 30, 719–720, 1966.

[30] Y. Zhang, Bounded gaps between primes. *Ann. of Math. (2)* 179, no. 3, 1121–1174, 2014.