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FOURIER COEFFICIENTS OF FORMS OF CM-TYPE

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Abstract. Let f be a cuspidal normalized eigenform of weight ≥ 2 for $\Gamma_0(N)$,with Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}.$$

While the Galois representations associated to f can be used effectively to study the divisibility properties of the Fourier coefficients, it is very difficult to analyze the condition $a_f(p) \equiv 0 \pmod{p}$. In this paper, we show that the problem is accessible in the case that f has complex multiplication. Under some mild conditions on f, we show that for p sufficiently large, $a_f(p) \equiv 0 \pmod{p}$ in fact implies that $a_f(p) = 0$.

Key words : Cusp form; complex multiplication; Fourier coefficients

1. The τ function

Consider the cusp form of Ramanujan:

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z} = q \prod_{k=1}^{\infty} \left(1 - q^k\right)^{24},$$

where $q = e^{2\pi i z}$. The coefficients $\tau(n)$ have received extensive arithmetic scrutiny following the ground-breaking investigations of Ramanujan himself.

Of the many problems that are open, there is Lehmer's conjecture that asserts that for any prime p,

$$\tau(p) \neq 0.$$

Equivalently, for any $n \geq 1$,

$$\tau(n) \neq 0.$$

Indeed, if there exists an n for which $\tau(n) = 0$, then the least such positive integer must be a power of a prime, say $n = p^a$. This means that if α_p and β_p are the roots of the quadratic equation $X^2 - \tau(p)X + p^{11} = 0$, then $\alpha_p = \zeta\beta_p$ for a root of unity ζ which satisfies $\zeta^{a+1} = 1$. But since the roots generate a quadratic extension of $\mathbb{Q}$, and such extensions contain either 2, 4 or 6 roots of unity, we must have $\zeta^b = 1$ for $b = 2, 4$ or 6. This means that one of $\tau(p), \tau(p^3)$ or $\tau(p^5)$ vanishes. According to a table of Simon Plouffe [10]

$$\tau(2) = -24,\ \tau(8) = 84480,\ \tau(32) = -196706304$$

and

$$\tau(3) = 252,\ \tau(27) = -73279080,\ \tau(243) = 13400796651732,$$

and so, we may suppose that $p \geq 5$. Suppose $\tau(p) \neq 0$. We have

$$\tau(p^3) = \tau(p)\left(\tau(p)^2 - 2p^{11}\right).$$

Since $\tau(p)$ is a nonzero rational integer, we cannot have $\tau(p)^2 = 2p^{11}$ as the right hand side is not a square. Thus, $\tau(p^3) = 0$ implies $\tau(p) = 0$ which is a contradiction. Similarly, we have

$$\tau(p^5) = \tau(p)(\tau(p)^2 - p^{11})(\tau(p)^2 - 3p^{11}).$$

Thus again, $\tau(p^5) = 0$ forces $\tau(p) = 0$ which is a contradiction.

Another curious formulation of Lehmer's conjecture is the assertion that the set

$$\{n : \tau(n) = 0\}$$

has density zero. In other words,

$$\#\{n \leq x : \tau(n) = 0\} \ = \ o(x).$$

Indeed, we have just seen that if there is one n for which $\tau(n) = 0$, then there is a prime p for which $\tau(p) = 0$. Then $\tau(pm) = 0$ for all positive integers m not divisible by p. This is a set of positive density.

2. Other cusp forms

More generally, let

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be the Fourier expansion of a normalized eigenform of weight $k \geq 2$ and level N, and suppose $a(n) \in \mathbb{Z}$. We can ask a question about the vanishing of the coefficients $a(p)$ or $a(n)$. If the weight is 2, it is known that there are infinitely many primes p for which $a(p) = 0$. If the weight is ≥ 4 and f is not of CM-type, it is expected (but not known) that the set of primes p for which $a(p) = 0$ is finite.

A problem closely related to the vanishing of $a(p)$ is to ask whether we can have

$$a(p) \equiv 0 \pmod{p}.$$

In the case of the Ramanujan τ-function, it is known that

$$\tau(p) \equiv 0 \pmod{p}$$

holds for primes $p = 2, 3, 5, 7, 2411, 7758337633$. Moreover, these are the only primes up to 10^{10} that satisfy this congruence [6], but it is not known if there are infinitely many such primes.[1] Nor do we know any good upper bounds of the number of such primes. In particular, is it true that

$$\#\left\{p \leq x : \tau(p) \equiv 0 \pmod{p}\right\} = o\left(\pi(x)\right)?$$

Or in general, is it true that

$$\#\left\{p \leq x : 0 \neq a(p) \equiv 0 \pmod{p}\right\} = o\left(\pi(x)\right)?$$

Since we have the Ramanujan-Petersson estimate (proved by Deligne)

$$|a(p)| \leq 2p^{(k-1)/2},$$

we see that in the weight $k = 2$ case, for $p > 3$ the condition $p | a(p)$ is equivalent to $a(p) = 0$. Heuristically, if the weight is > 2, then the number of primes p up to x for

[1] More generally, we may ask whether for a fixed a, the congruence

$$\tau(p) \equiv a \pmod{p}$$

holds infinitely often. For example, for $a = 1$, it is known to hold for $p = 11, 23$ and 691 and Plouffe has checked that it does not hold for any other primes less than 314747.

which $p|a(p)$ may grow like $\log\log x$, though as we stated above, we do not even know if these are of density zero.

While there is not much that can be said about this question in general, the CM case is more amenable to study. We will prove the following result.

Theorem 2.1. Suppose $f = f_\Psi$ is a form of CM type corresponding to a Hecke character Ψ of the imaginary quadratic field K. Write

$$f_\Psi(z) = \sum_{n\geq 1} a_\Psi(n) e^{2\pi i n z}$$

for the Fourier expansion at infinity. Suppose that Ψ takes values in K and for any prime $\mathfrak{p}$ of K we have $\Psi(\mathfrak{p}) + \Psi(\overline{\mathfrak{p}}) \in \mathbb{Z}$. Then for p a sufficiently large prime that splits in K, we have

$$a_\Psi(p) \not\equiv 0 \pmod{p}.$$

3. The CM case

Let K be an imaginary quadratic field. Let $\mathfrak{f}$ be an integral ideal of K and denote by $I(\mathfrak{f})$ the group of fractional ideals of K coprime to $\mathfrak{f}$. We consider Hecke characters Ψ of K. Thus, Ψ is a map

$$\Psi \ : \ I(\mathfrak{f}) \to \overline{\mathbb{Q}}^\times.$$

We suppose that for a positive integer r,

$$\Psi((\alpha)) = \alpha^r$$

for all $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\mathfrak{f}}$.
In order for such Ψ to exist, we need to have

$$\zeta^r = 1$$

for any root of unity $\zeta \in K$ with $\zeta \equiv 1 \pmod{\mathfrak{f}}$. To such a Ψ, we associate the function $f = f_\Psi$ defined by

$$f_\Psi(z) = \sum_{\substack{\mathfrak{a}\subseteq \mathcal{O}_K \\ (\mathfrak{a},\mathfrak{f})=1}} \Psi(\mathfrak{a}) e^{2\pi i (N\mathfrak{a}) z}$$

for $z \in \mathbb{C}$, $\Im z > 0$. Regrouping terms, we also write this as

$$f_\Psi(z) = \sum_{n\geq 1} a_\Psi(n) e^{2\pi i n z}$$

where

$$a_\Psi(n) \;=\; \sum_{(\mathfrak{a},\mathfrak{f})=1, N\mathfrak{a}=n} \Psi(\mathfrak{a}).$$

In particular, we see as usual that $a_\Psi(p) = 0$ if p does not split in K.

Let us set $k = r + 1$ and $M = |d_K| N\mathfrak{f}$. Let us also define the Dirichlet character ϵ modulo M given by

$$\epsilon(a) = \left(\frac{d_K}{a}\right) \frac{\Psi((a))}{a^r}$$

for $a \in \mathbb{Z}$, $(a, M) = 1$. Then by a theorem of Hecke, f_Ψ is a cusp form of weight k, level M and character ϵ. In particular, if we set

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{M} \right\}$$

then

$$f_\Psi \left(\frac{az+b}{cz+d}\right) \;=\; \epsilon(d)(cz+d)^k f(z)$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \; \Gamma_0(M).$$

The L-function associated to Ψ is given by

$$L(s, \Psi) = \prod_{\mathfrak{p}} \left(1 - \frac{\Psi(\mathfrak{p})}{(N\mathfrak{p})^s}\right)^{-1}$$

where the product ranges over the primes $\mathfrak{p}$ of K coprime to $\mathfrak{f}$. We can actually drop the latter condition by extending Ψ by 0 at primes that divide $\mathfrak{f}$. Writing the above as an Euler product over rational primes gives the L-function of f_Ψ:

$$L(s, f_\Psi) = \prod_p \left(1 - \frac{a_\Psi(p)}{p^s} + \frac{\epsilon(p)p^r}{p^{2s}}\right)^{-1}.$$

4. Proof of Theorem 2.1

We would like to show that for a CM form $f = f_\Psi$ with rational integer coefficients and trivial character, if $a_\Psi(p) \neq 0$ and $p | a_\Psi(p)$ then p is bounded.

Denote by $\mathfrak{f}$ the conductor of Ψ. Choose representatives $\mathfrak{C}_1, \ldots, \mathfrak{C}_h$ for the elements of the $\mathfrak{f}$-ideal class group. Suppose that p splits in K and write $p = \mathfrak{p}\overline{\mathfrak{p}}$. Say

$$\mathfrak{p} = (\gamma_p)\mathfrak{C}, \tag{1}$$

with

$$\gamma_p \equiv 1 \pmod{\mathfrak{f}},$$

where $\mathfrak{C} = \mathfrak{C}_i$ (for some i) is a fractional ideal. Then

$$\Psi(\mathfrak{p}) = \Psi(\mathfrak{C})\gamma_p^{k-1}.$$

Since

$$\Psi(\mathfrak{p})\Psi(\overline{\mathfrak{p}}) = \Psi((p)) = \epsilon_p p^r$$

for some root of unity ϵ_p, and $|\Psi(\mathfrak{p})|^2 = p^r$, we have

$$\Psi(\overline{\mathfrak{p}}) \ = \epsilon_p \ \overline{\Psi(\mathfrak{p})}.$$

It follows that

$$a_\Psi(p) = \Psi(\mathfrak{C})\gamma_p^{k-1} + \ \epsilon_p \ \overline{\Psi(\mathfrak{C})}\overline{\gamma_p}^{k-1}.$$

Here $\Psi(\mathfrak{C})$ and γ_p need not be integers, but after multiplication of this equation by the common denominator Δ (say) of $\Psi(\mathfrak{C})$ and γ_p, we deduce that

$$a_\Psi(p) \cdot \Delta \ = A\delta_p^{k-1} + B\epsilon_p \ \overline{\delta_p}^{k-1} \tag{2}$$

where A, B and δ_p are integers. Equation (1) implies that $\mathfrak{p}|\text{numerator}(\mathfrak{C})$ or $\mathfrak{p}|(\delta_p)$. This means that $\mathfrak{p}|(\delta_p)$ except for finitely many primes (all the primes that divide the numerators of the $\mathfrak{C}_i$). This in turn, together with (2), shows that

$$p | a_\Psi(p) \ \Rightarrow \ \mathfrak{p}|B(\overline{\delta_p}).$$

Since $\mathfrak{p}|B$ is possible for finitely many primes $\mathfrak{p}$ only, this means that for p sufficiently large, $\mathfrak{p}|(\overline{\delta_p})$. Since $\mathfrak{p}|(\delta_p)$, we have

$$p|(\delta_p).$$

Moreover, apart from a finite number of primes p, we have δ_p is equal to the numerator of γ_p. Returning to the equation

$$\mathfrak{p} = \mathfrak{C}(\gamma_p)$$

gives a contradiction as the right hand side is divisible by p while the left hand side is not. Thus we have shown that if p splits in K and is sufficiently large, namely

$$p > \max_{1\le i\le h}\{\text{all numerators and denominators of } \mathfrak{C}_i\overline{\mathfrak{C}_i}\},\ p \nmid d_K N\mathfrak{f}$$

we have $a_\Psi(p) \not\equiv 0 \pmod p$.

5. Examples of CM Hecke eigenforms

In our work, we assume that $\epsilon = 1$ and all the $a(p)$ are rational integers. Such CM forms do arise 'in nature' as the following examples illustrate.

Example 1: Let us consider an elliptic curve E over $\mathbb{Q}$ with CM by K. Let Ψ be the corresponding Hecke character. Then it satisfies the above hypotheses with f_Ψ of weight 2, i.e. f_Ψ is a normalized Hecke eigenform with CM and integer coefficients. We may also consider Ψ^m for some $1 \le m \in \mathbb{Z}$. The corresponding f_{Ψ^m} is of weight $m+1$. Note that if m is even, the above construction gives a form f_{Ψ^m} with a nontrivial Nebentypus character. We will thus restrict ourselves to the case m is odd, but a large part of our discussion will apply in the general case.

Let us write

$$f_{\Psi^m}(z) = \sum_{n\ge 1} a_m(n)e^{2\pi i n z}.$$

To see that the Fourier coefficients $a_m(n)$ of f_{Ψ^m} are rational integers, we observe first that $a_m(p) = 0$ if p does not split in K. If $p = \mathfrak{p}\overline{\mathfrak{p}}$, then

$$(3) \qquad a_m(p) \;=\; \Psi^m(\mathfrak{p}) + \Psi^m(\overline{\mathfrak{p}}) = (\Psi(\mathfrak{p}) + \Psi(\overline{\mathfrak{p}}))^m - \sum_{j=1}^{m-1} \binom{m}{j} \Psi(\mathfrak{p})^j \Psi(\overline{\mathfrak{p}})^{m-j}.$$

The sum on the right is (up to a root of unity)

$$\sum_{1\le j<m/2} \binom{m}{j} p^j \left(\Psi(\mathfrak{p})^{m-2j} + \Psi(\overline{\mathfrak{p}})^{m-2j}\right).$$

Now we use induction on m to deduce that the left hand side is a rational integer.

We can also ask when the left hand side of (3) is divisible by p. The sum over j on the right hand side of (3) is clearly divisible by p and so the left hand side is divisible by p if and only if

$$p | a_1(p) \text{ where } a_1(p) = \Psi(\mathfrak{p}) + \Psi(\overline{\mathfrak{p}}).$$

Since Ψ corresponds to an elliptic curve, we know from a result of Deuring (see [3], Chapter 13, Theorem 12) that if $p \geq 5$, then $a_1(p) = 0$ if and only if either p is inert or ramified in K or divides the conductor of Ψ (equivalently, the corresponding elliptic curve has bad reduction). Note that at primes of bad reduction, we have $a_1(p) = 0$ as an elliptic curve with CM has additive reduction at such primes. The primes of bad reduction are exactly those that divide the conductor which is the same as the conductor of the Hecke character Ψ, namely $|d_K|N\mathfrak{f}$. We have thus proved the following result.

Proposition 5.1. The following are equivalent:

- $a_m(p) \equiv 0 \pmod{p}$
- $a_1(p) \equiv 0 \pmod{p}$.

Moreover, if $p \geq 5$, the above two statements are equivalent to the following:

- $a_m(p) = 0$
- $a_1(p) = 0$
- p is inert or ramified in K or divides the conductor of Ψ.

Example 2: Consider the Dedekind η - function given by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = e^{\frac{1}{12}\pi i z} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right).$$

It is a modular form of weight $1/2$ and $\eta^r(z)$ is a modular form of weight $r/2$. For $r = 2, 4, 6, 8, 10, 14, 26$ it is of CM type. This was considered by Ramanujan [11] for $r = 2, 4, 6, 8$ and by Knopp, Lehner, Newman and Rankin for the remaining cases (see [14] for the details). Also, it is a fact that

$$\eta(z) = \sum_{n \equiv 1 \pmod{6}} (-1)^{\frac{n-1}{6}} q^{\frac{n^2}{24}},$$

which shows that the powers of η have integer coefficients. However, they may not be Hecke eigenforms. In [11] Ramanujan shows that the form $\eta^8(3z)$ is a form of weight 4 for $\Gamma_0(9)$ (in fact, a newform) with CM by $\mathbb{Q}(\sqrt{-3})$. Let us write (as in [14], p. 210)

$$\eta^8(3z) = q\prod_{m=1}^{\infty}(1-q^{3m})^8 = q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + \ldots = f_{K,\Psi}(z),$$

where $K = \mathbb{Q}(\sqrt{-3})$ and Ψ is a Hecke character of K with conductor $\mathfrak{f} = \sqrt{-3}\cdot\mathcal{O}_K$, defined as follows. If $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$ coprime with $\mathfrak{f}$, then $\Psi(\mathfrak{a}) = \alpha^3$, where α is the generator of $\mathfrak{a}$ for which $\alpha \equiv 1(\text{mod } \mathfrak{f})$. Notice that as $\bar{\mathfrak{f}} = \mathfrak{f}$, it follows that $\Psi(\overline{\mathfrak{a}}) = \overline{\alpha}^3$.

For this form we can show that there are no primes p such that $p|a_\Psi(p)$ and $a_\Psi(p) \neq 0$. Indeed, if $\mathfrak{p} = (\alpha)$, $\alpha \equiv 1 \pmod{\mathfrak{f}}$,

$$\alpha = \frac{a+b\sqrt{-3}}{2}, \quad a, b \in \mathbb{Z}$$

then

$$4p = a^2 + 3b^2,$$

and

$$a_\Psi(p) = \alpha^3 + \overline{\alpha}^3 = \frac{1}{8}(a^3 - 9ab^2) \equiv 0 \pmod{p}$$

implies $p|a$, and so $p|b$. This is a contradiction.

For this form, let us make a few more observations about the vanishing of Fourier coefficients. Consider $p = \mathfrak{p}\overline{\mathfrak{p}}$ and write α for a generator of $\mathfrak{p}$ which is congruent to 1 (mod $\mathfrak{f}$). If $a_\Psi(p^i) = 0$ for some $i \geq 1$, then $\overline{\alpha} = \zeta\alpha$ for some root of unity ζ, which must necessarily lie in K. Thus α and $\overline{\alpha}$ generate the same ideal so $\overline{\mathfrak{p}} = \mathfrak{p}$ which is only possible if $p = 3$. From the Fourier expansion, we know that $a_\Psi(3) = 0$. In fact, from the L-function description given at the end of section 3, it follows that $a_\Psi(3^i) = 0$ for all $i \geq 1$. Finally for primes that don't split in K, namely $p \equiv 2 \pmod 3$, we have $a_\Psi(p) = 0$ and $\Psi((p)) = -p^3$. We summarize this discussion in the result below.

Proposition 5.2. For the p-th coefficient $a_\Psi(p)$ of $\eta^8(3z)$, we have the following equivalent statements:

- $a_\Psi(p) \equiv 0 \pmod{p}$
- $a_\Psi(p) = 0$
- p is inert or ramified in $\mathbb{Q}(\sqrt{-3})$.

6. Variant of Lehmer's Conjecture

Analogous to the vanishing of $a(p)$ is the vanishing of $a(n)$. Analogous to the question whether $p|a(p)$ is whether $(n, a(n)) \neq 1$. In particular, we might ask whether

$$\# \{n \leq x : (n, a(n)) \neq 1\} = o(x). \tag{4}$$

In fact, as explained in [7], this is not true in general.

For example, consider the case of the τ function. From the results of Serre [13], we know that given a prime ℓ sufficiently large, the set $\mathcal{P}$ (say) of primes p for which $\ell|\tau(p)$ has positive density, roughly proportional to $1/\ell$. Given a set of primes of positive density, a simple sieve argument shows that the set of integers n which are divisible by a prime in this set has density 1. If, in addition, we consider only those integers m which are divisible by an element of $\mathcal{P}$ to the first power, in other words those that can be written as $m = pr$ with $p \in \mathcal{P}$ and r not divisible by any element of $\mathcal{P}$, we still get a set $\mathcal{S}$ (say) of positive density. Now consider the set of integers n of the form $n = \ell m$ where $m \in \mathcal{S}$. This is clearly a set of positive density. Moreover, for any such integer, we have $\ell|(n, \tau(n))$.

It turns out that the correct question to ask is the opposite of (4), namely whether it is true that

$$\# \{n \leq x : (n, a(n)) = 1\} = o(x). \tag{5}$$

This variant of Lehmer's conjecture was discussed in Murty [7], in Gun and Murty [2] and in Laptyeva [4] and Laptyeva and Murty [5].

Define $L_1(x) = \log x$ and $L_i(x) = \log L_{i-1}(x)$ for $i \geq 2$.
In [7], the following result was proved.

Theorem 6.1. For a normalized Hecke eigenform f with rational integer Fourier coefficients $a(n)$, one has

$$\#\{n \leq x|\ (n, a(n)) = 1\} \ll \frac{x}{L_3(x)}.$$

In [2], S. Gun and the second author showed that if f has complex multiplication (CM) and is of weight 2, one can in fact obtain an asymptotic formula. This was generalized to CM forms of all weight ≥ 2 by the authors of the current paper. In particular, they showed [4], [5] the following result.

Theorem 6.2. (N. Laptyeva, V. K. Murty) Let f be a normalized eigenform of weight ≥ 2 with rational integer Fourier coefficients $\{a(n)\}$. If f is of CM-type, then there is a constant $U_f > 0$ so that

$$\#\{n \leq x |\ (n, a(n)) = 1\} = (1 + o(1)) \frac{U_f x}{\sqrt{L_1(x) L_3(x)}}.$$

The constant U_f can be given explicitly.

As an illustration, we describe the constant U_f for the form $f(z) = \eta^8(3z)$ discussed above. Based on the discussion at the end of section 5, it follows that if $p \equiv 1 \pmod 3$, then $a_f(p^i) \neq 0$ for all $i \geq 1$. Moreover, $a_f(3) = 0$ and if $p \equiv 2 \pmod 3$, then $a_f(p^i) = 0$ if and only if i is odd. In the notation of [2] and [5], we have $i_f(p) \leq 1$. (The integer $i_f(p)$ is defined to be the least $i \geq 1$ for which $a_f(p^i) = 0$ if such an i exists, and otherwise, it is set to 0.) Let us define two constants u_f and μ_f as follows. We have

$$\prod_{\substack{p \leq x \\ p \equiv 1 \pmod 3}} \left(1 - \frac{1}{p}\right) \sim \frac{\mu_f}{(\log x)^{\frac{1}{2}}}.$$

Let us set

$$M_f(x) = \#\{n \leq x, a_f(n) \neq 0 \text{ and } p | n \Rightarrow a_f(p) \neq 0\}.$$

Then

$$M_f(x) \sim \frac{u_f x}{(\log x)^{\frac{1}{2}}}.$$

The fact that such constants exist is proved in [2] and [5]. It is further shown in [5] that

$$U_f = \frac{8}{9} u_f \mu_f \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^2}\right)$$

In [2], essential use is made of a result of Schaal [12] in order to obtain an estimate (of the form $\frac{1}{p} \log \log x$) for the sum

$$\sum_{\substack{q \leq x \\ 0 \neq a(q) \equiv 0 \pmod p}} \frac{1}{q}.$$

For weight > 2, the argument given in [2] breaks down and we need to find a replacement. In particular, we are not able to use Schaal's estimate. Rather, we rely on a clever use of the Chebotarev density theorem.

A new product that emerges in our estimates is:

$$\prod_{0 \neq p \equiv 0 (\bmod\ p)} \left(1 - \frac{1}{p}\right). \tag{6}$$

By the results of the previous section, this is in fact a finite product.

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