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CALDERON'S REPRODUCING FORMULA FOR WATSON WAVELET TRANSFORM

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In this paper the inverse Watson wavelet transform is investigated, the Calderon reproducing formula of Watson convolution is obtained by generalizing the results of [6]. Some applications associated with Calderon's reproducing formula of Watson convolution are given.

Key words : Calderon's reproducing formula; Watson convolution; Watson transform; Watson wavelet transform.

1. INTRODUCTION

Calderon's reproducing formula [6] played an important role to find the inversion formula of wavelet transform using Fourier convolution transform. This formula can also be used to obtain several approximation results related to aforesaid transform.

Motivated from the results of [6], Calderon's reproducing formula associated with Hankel convolution was defined by Pathak and Pandey [8] and studied many properties using the theory of Hankel transform. By exploiting above formula we can find the inversion formula of Bessel wavelet transform.

Watson convolution and Watson transform are generalizations of many integral transforms. Recently continuous Watson wavelet transform is defined in [13] and found several properties. Using the results of [13], our main aim of this paper is to investigate the Calderon's reproducing formula associated with Watson convolution, Watson transform and Watson wavelet transform.

Now, we restate the theorem of [6] which is useful for present investigation.

***Theorem* 1.1** — *Let* $\psi \in L^1(\mathbb{R}), \psi_a(t) = \frac{1}{a}\,\psi\left(\frac{t}{a}\right)$ *for* $a > 0$ *and* $f \in L^2(\mathbb{R})$ *then*

$$f(x) = \int_0^\infty (\psi_a * \psi_a * f)\,(x)\,\frac{da}{a}, \tag{1}$$

and the above expression can be converted into the following form:

$$f(x) = \int_0^\infty \int_{-\infty}^\infty (W_\psi f)(y, a)\,\psi\left(\frac{x-y}{a}\right)\,\frac{dyda}{a^2} \tag{2}$$

where $(W_\psi f)(y,\, a) = \int_{-\infty}^\infty \psi_a(x-y)\,f(t)\,dt$ *and* $\psi_a(x-y) = \psi\left(\frac{x-y}{a}\right)$.

Now, we restate the definitions of Mellin transform and inverse Mellin transform and following theorem, which is given in [12, pp. 232-233] and [10, pp. 57-58].

Definition 1.1 — Let $f(t)$ be a function defined on the positive real axis $0 < t < \infty$. The Mellin transformation M is the operation mapping of the function f into the function F defined on the complex plane by the relation:

$$M[f : s] = F(s) = \int_0^\infty t^{s-1} f(t)\,dt. \tag{3}$$

The function $F(s)$ is called the Mellin transform of $f(t)$. In general, the integral exists only for complex values of $s = a + ib$ such that $a < a_1 < a_2$, where a_1 and a_2 depend on the function $f(t)$. This introduces the strip of definition of Mellin transform that will be denoted by $S(a_1, a_2)$.

Definition 1.2 — The inversion formula for Mellin transform is given by

$$f(t) = \frac{1}{2\pi j}\int_{a-i\infty}^{a+i\infty} t^{-s} F(s)\,ds, \tag{4}$$

where the integration is along a vertical line through $Re(s) = a$.

***Theorem* 1.2** – *Let* $(Mk)(s)$ *which is defined in (3) be regular in a strip,* $\sigma_1 < \sigma < \sigma_2$, *where* $\sigma_1 < 0, \sigma_2 > 1$ *except perhaps for a finite number of simple poles on the imaginary axis; and let* $(Mk)(s)$ *be of the forms:*

$$(Mk_0)(s)\left\{\alpha\frac{\beta}{s} + O\left(\frac{1}{|s|^2}\right)\right\}, (Mk_0)(s)\left\{\gamma\frac{\delta}{s} + O\left(\frac{1}{|s|^2}\right)\right\}$$

for large positive and negative t, $s = \sigma + it$, *respectively, where* $(Mk_0)(s) = \Gamma(s)cos\frac{1}{2}s\pi$ *is the Mellin transform of* $\cos x$.

Let $(Mk)(s)$ satisfy the condition

$$(Mk)\,(s)\,(Mk)\,(1-s) = 1$$

and from (4) let $k(x)$ be the inverse Mellin transform of $(Mk)(s)$.

Let $x > 0$, and let $f(t)$ be in $L^1(0, \infty)$ and be of bounded variation near $t = x$.

Then

$$\int_0^\infty k(xu)du \int_0^\infty k(ut)f(t)\,dt = \frac{1}{2}\left[f(x+0) + f(x-0)\right]. \tag{5}$$

Equivalent relations are

$$F(x) = (Wf)(x) = \int_0^\infty k(xt)f(t)\;dt, \tag{6}$$

$$f(t) = \left(W^{-1}F\right)(t) = \int_0^\infty k(xt)F(x)\;dx, \tag{7}$$

where $k(x)$ is called symmetrical Fourier kernel and (6) is often called Watson transform of $f(t)$ (7) is the corresponding inversion formula.

From [9, pp. 1224], we define the basic function

$$w(x, y, z) = \int_0^\infty k(xt)\,k(yt)\,k(zt)\,dt. \tag{8}$$

The above integral is convergent under the assumption $k \in L^1(0, \infty) \bigcap L^\infty(0, \infty)$.

The inversion of (8) is formally given by

$$k(xt)\,k(yt) = \int_0^\infty w(x, y, z)\,k(zt)\,dz. \tag{9}$$

Now, we assume that $k(0) \neq 0$, $\tilde{w}(x, y, z) = \frac{w(x,y,z)}{k(0)}$ and $\tilde{w}(x, y, z) > 0\,\forall\, x, y, z \in (0, \infty)$.

Then, setting $t = 0$ in (9), we have

$$\int_0^\infty \tilde{w}(x, y, z)\,dz = 1. \tag{10}$$

Let $\psi \in L^1(0, \infty) \cap L^2(0, \infty)$, then the Watson translation is defined by

$$\psi(x, y) = \tau_y\psi(x) = \tau_x\psi(y) = \int_0^\infty \psi(z)\tilde{w}(x, y, z)\,dz,\; 0 < x, y < \infty. \tag{11}$$

From [10, pp. 70] the Watson convolution of ϕ and $\psi \in\; L^1(0, \infty) \cap L^2(0, \infty)$ is defined by

$$(\phi\#\psi)(x) = \int_0^\infty \int_0^\infty \phi(y)\,\psi(z)\,\tilde{w}(x, y, z)\,dy\,dz,\; 0 < x < \infty. \tag{12}$$

Then, in view of (11), (12) becomes

$$(\phi\#\psi)(x) = \int_0^\infty \psi(x, y)\,\phi(y)\,dy. \tag{13}$$

Let $\phi\,,\,\psi\,\in\,L^1(0\,,\,\infty)$ and let $(\phi\#\psi)(x)$ be defined by (12). Then

$$W(\phi\#\psi)=(W\phi)\,(W\psi). \tag{14}$$

Now, from [13], we define the Watson wavelet as follows:

Let $\psi\,\in L^p(0,\,\infty)$ be given, for $b\geq 0$ and $a>0$, we have

$$\psi_{b,a}(x)=\frac{1}{a}\,\psi\left(\frac{b}{a},\frac{x}{a}\right)=\psi_a(b,x)=\frac{1}{a}\int_0^\infty\psi(z)\,\tilde{w}\left(\frac{b}{a},\frac{x}{a},z\right)\,dz. \tag{15}$$

From [13], we define the Watson wavelet transform

$$\begin{aligned} W(b,a) &= (W_\psi\phi)(b,a)\\ &= \int_0^\infty \phi(x)\,\overline{\psi_{b,a}(x)}\,dx\\ &= \frac{1}{a}\int_0^\infty\int_0^\infty\phi(x)\,\overline{\psi(z)}\,\tilde{w}\left(\frac{b}{a},\frac{x}{a},z\right)\,dz\,dx, \end{aligned} \tag{16}$$

provided the integral is convergent.

Now, we restate Lemma 2.3 from [13].

Let $\phi,\psi\in L^1(0,\infty)$ and $(W_\psi\phi)(b,a)$ be the continuous Watson wavelet transform. Then

$$(W_\psi\phi)(b,a)=(\phi\,\#\,\overline{\psi_a})(b). \tag{17}$$

Example 1.1 : We give an example of the kernel $k(x)$, which possesses the aforesaid properties.

In [7, p. 19], we take $\gamma=\frac{1}{2}$, then

$$k(x)=G^{m,p}_{p+q,m+n}\left(x|^{a_p,b_q}_{c_m,d_n}\right), \tag{18}$$

where $n-p=m-q>0$ and $\sum\limits_{j=1}^{p}a_j+\sum\limits_{j=1}^{q}b_j=\sum\limits_{j=1}^{m}c_j+\sum\limits_{j=1}^{n}d_j$.

Putting $m=1,p=0,n=1,q=0$ in (18), then from [3, p. 216], we get

$$k(x)=G^{1,0}_{0,2}\,(x|a,b)=x^{\frac{a+b}{2}}\,J_{a-b}\left(2x^{\frac{1}{2}}\right).$$

Now,

$$\begin{aligned} f_{y,z}(t) &= k(yt)k(zt)\\ &= (yt)^{\frac{a+b}{2}}J_{a-b}\left(2(yt)^{\frac{1}{2}}\right)(zt)^{\frac{a+b}{2}}J_{a-b}\left(2(zt)^{\frac{1}{2}}\right)\in L^1(0,\infty)\\ &\qquad \text{for}\quad a+b<-\frac{1}{2},a>-\frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} f_x(t) &= k(xt) \\ &= (xt)^{\frac{a+b}{2}} J_{a-b}(2(xt)^{\frac{1}{2}}) \in L^{\infty}(0,\infty) \quad \text{for} \quad a+b < \frac{1}{2} \quad \text{and} \quad a > 0. \end{aligned}$$

Thus, the sufficient condition for the existence of $w(x,y,z)$ in the present case, viz., $k \in L^1(0,\infty)$ $\bigcap L^{\infty}(0,\infty)$ holds for $a > 0, a+b < -\frac{1}{2}$.

2. CALDERON'S FORMULA

In this section we obtain Calderon's reproducing identity using the properties of Watson transform and Watson convolution.

***Theorem* 2.1** — *If $f \in L^1(0,\infty) \cap L^2(0,\infty)$ then f can be reconstructed by the formula*

$$f(x) = \frac{1}{A_\psi} \int_0^\infty \int_0^\infty (W_\psi f)(y,\, a)\, \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{dy\, da}{a^2}, \tag{19}$$

where $A_\psi = \int_0^\infty \frac{|(W\psi)(\omega)|^2}{\omega} d\omega < \infty$ and $(W_\psi f)(y,\, a)$ is Watson wavelet transform of the function f with respect to Watson wavelet ψ.

PROOF : Let $g \in L^1(0,\infty) \cap L^2(0,\infty)$, then by the Parseval formula [13] for the Watson wavelet transfrom, we have

$$\begin{aligned} & A_\psi \langle f, g \rangle \\ &= \int_0^\infty \int_0^\infty (W_\psi f)(y,a)\overline{(W_\psi g)(y,a)}\, \frac{dyda}{a} \\ &= \int_0^\infty \int_0^\infty (W_\psi f)(y,a) \left(\frac{1}{a} \int_0^\infty \overline{g(x)} \psi\left(\frac{y}{a}, \frac{x}{a}\right) dx \right) \frac{dyda}{a} \\ &= \int_0^\infty \left(\int_0^\infty \int_0^\infty (W_\psi f)(y,a) \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{dyda}{a^2} \right) \overline{g(x)}\, dx. \end{aligned}$$

Thus,

$$A_\psi \langle f, g \rangle = \left\langle \int_0^\infty \int_0^\infty (W_\psi f)(y,a) \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{dyda}{a^2}, g(x) \right\rangle. \tag{20}$$

Therefore

$$A_\psi f(x) = \int_0^\infty \int_0^\infty (W_\psi f)(y,\, a)\, \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{dy\, da}{a^2}. \tag{21}$$

If we put $f = g$ in (20), then

$$A_\psi \|f\|^2 = \int_0^\infty \int_0^\infty |(W_\psi f)(y,\, a)|^2 \frac{dy\, da}{a^2}.$$

Lemma 2.1 — Let $\psi \in L^2(0,\infty)$ be a basic Watson wavelet which satisfies the following admissibility condition

$$A_\psi = \int_0^\infty \frac{|(W\psi)(\omega)|^2}{\omega}\, d\omega = 1. \tag{22}$$

Then

$$\int_0^\infty \int_0^\infty (W_\psi f)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{dyda}{a^2} = \int_0^\infty (f\#\overline{\psi_a}\#\psi_a)(x)\frac{da}{a} \tag{23}$$

for $f \in L^1(0,\infty)\cap L^2(0,\infty)$.

PROOF : From (17) we have

$$\int_0^\infty \int_0^\infty (W_\psi f)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{dyda}{a^2} = \int_0^\infty \int_0^\infty (f\#\overline{\psi_a})(y)\,\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{dyda}{a^2}. \tag{24}$$

Using the symmetry of $\tilde{w}(x,y,z)$ in (24), we get

$$\int_0^\infty \int_0^\infty (W_\psi f)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{dyda}{a^2} = \int_0^\infty \int_0^\infty (f\#\overline{\psi_a})(y)\,\psi_a\,(x,y)\,\frac{dyda}{a},$$

for $\psi\left(\frac{y}{a},\frac{x}{a}\right) = \psi_a\,(x,y)$.

Hence from (13), we obtain

$$\int_0^\infty \int_0^\infty (W_\psi f)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{dyda}{a^2} = \int_0^\infty (f\#\overline{\psi_a}\#\psi_a)(x)\frac{da}{a}.$$

***Theorem* 2.2** — *Let $\phi,\psi \in L^1(0,\infty)$ and $(W\phi),(W\psi) \in L^1(0,\infty)$ be such that the following admissibility condition holds:*

$$\int_0^\infty (W\phi)(\omega)(W\psi)(\omega)\frac{d\omega}{\omega} = 1. \tag{25}$$

Then the following Calderon's reproducing identity holds:

$$f(x) = \int_0^\infty (f\#\overline{\phi_a}\#\psi_a)(x)\frac{da}{a} \qquad \forall\, f \in L^1(0,\infty). \tag{26}$$

PROOF : If we put $\phi = \psi$ in Lemma 2.1, then we can find Theorem 2.2.

3. APPLICATIONS

In this section we give some applications related to Watson wavelet transform by using the theory of Watson convolution and Mellin transform.

***Theorem* 3.1** — *Let $\psi \in L^2(0,\infty)$ be a basic Watson wavelet and $W_\psi f(y,a)$ be the continuous Watson wavelet transform, then*

$$\int_0^\infty \int_0^\infty (W_\psi f)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\,\frac{dy\,da}{a^2} = A_\psi \int_0^\infty k(xu)(Wf)(u)\,du. \tag{27}$$

PROOF : From (15), we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty (W_\psi f)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\,\frac{dy\,da}{a^2}\\
&= \int_0^\infty \int_0^\infty (W_\psi f)(y,a)\left(\int_0^\infty \psi(z)\,\tilde{w}\left(\frac{y}{a},\frac{x}{a},z\right)\,dz\right)\frac{dyda}{a^2}\\
&= \int_0^\infty \int_0^\infty (W_\psi f)(y,a)\left\{\int_0^\infty \psi(z)\left(\int_0^\infty k\left(\frac{y\omega}{a}\right)k\left(\frac{x\omega}{a}\right)k(z\omega)\,d\omega\right)dz\right\}\frac{dyda}{a^2}\\
&= \int_0^\infty \int_0^\infty (W_\psi f)(y,a)\left\{\int_0^\infty k\left(\frac{y\omega}{a}\right)k\left(\frac{x\omega}{a}\right)\left(\int_0^\infty k(z\omega)\psi(z)\,dz\right)d\omega\right\}\frac{dyda}{a^2}\\
&= \int_0^\infty \int_0^\infty (W_\psi f)(y,a)\left\{\int_0^\infty k\left(\frac{y\omega}{a}\right)k\left(\frac{x\omega}{a}\right)(W\psi)(\omega)\,d\omega\right\}\frac{dyda}{a^2}\\
&= \int_0^\infty \int_0^\infty k\left(\frac{x\omega}{a}\right)(W\psi)(\omega)\left(\int_0^\infty k\left(\frac{y\omega}{a}\right)(W_\psi f)(y,a)\,dy\right)\frac{d\omega da}{a^2}\\
&= \int_0^\infty \int_0^\infty k\left(\frac{x\omega}{a}\right)(W\psi)(\omega)\left(W\left\{(W_\psi f)(y,a)\right\}\right)\left(\frac{\omega}{a}\right)\frac{d\omega da}{a^2}.
\end{aligned}$$

Putting $\frac{\omega}{a} = u$ in the above expression we get

$$\begin{aligned}
&\int_0^\infty \int_0^\infty (W_\psi f)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\,\frac{dy\,da}{a^2}\\
&= \int_0^\infty \int_0^\infty k(xu)W[(W_\psi f)(y,a)]\,(u)\,(W\psi)(au)\,du\,\frac{da}{a}\\
&= \int_0^\infty \int_0^\infty k(xu)\overline{(W\psi)(au)}(Wf)(u)(W\psi)(au)\,du\,\frac{da}{a}\\
&= \int_0^\infty k(xu)(Wf)(u)\left(\int_0^\infty \frac{|(W\psi)(au)|^2}{a}da\right)du\\
&= A_\psi \int_0^\infty k(xu)(Wf)(u)\,du.
\end{aligned}$$

Example 3.1 — Assume that $k(t) = J_0(2\sqrt{t})$ and $f(t) = e^{-t}$, then from [4, p. 185] we have $(Wf)(u) = e^{-u}$.

Now, from (27) we get

$$\frac{1}{A_\psi}\int_0^\infty\int_0^\infty (W_\psi f)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{dy\,da}{a^2} \quad = \quad e^{-x}.$$

***Theorem* 3.2** — *Let* $f \in L^1(0,\infty)$ *and* $\psi \in L^1(0,\infty)$. *Then*

$$\begin{aligned}(f\#\overline{\psi_a})(\omega) &= M^{-1}[(f\#\overline{\psi_a})(1-s).(Mk)(s)](\omega)\\ &= \int_0^\infty k(\omega y)W(f\#\overline{\psi_a})(y)\,dy,\end{aligned}$$

where $(Mk)(s)(Mk)(1-s)=1$.

PROOF : The Watson wavelet transform (16) can be expressed in the following form:

$$W_\psi f(y,a) = (f\#\overline{\psi_a})(y) = \int_0^\infty k(\omega y)W(f\#\overline{\psi_a})(\omega)\,d\omega.$$

Therefore

$$\int_0^\infty y^{s-1}(f\#\overline{\psi_a})(y)\,dy = \int_0^\infty y^{s-1}\left(\int_0^\infty k(\omega y)W(f\#\overline{\psi_a})(\omega)\,d\omega\right)dy.$$

From (3),we have

$$M[(f\#\overline{\psi_a})(y)](s) \quad = \quad \int_0^\infty W(f\#\overline{\psi_a})(\omega)\left(\int_0^\infty k(\omega y)y^{s-1}\,dy\right)d\omega.$$

Putting $\omega y = u$, we get

$$\begin{aligned}M[(f\#\overline{\psi_a})(y)](s) &= \int_0^\infty W(f\#\overline{\psi_a})(\omega)\left(\int_0^\infty k(u)\left(\frac{u}{\omega}\right)^{s-1}\frac{du}{\omega}\right)d\omega\\ &= \int_0^\infty(\omega)^{1-s-1}W(f\#\overline{\psi_a})(\omega)\left(\int_0^\infty k(u)u^{s-1}du\right)d\omega\\ &= (Mk)(s)M[W(f\#\overline{\psi_a})(\omega)](1-s).\end{aligned}$$

Replacing s by $1-s$, we get

$$M[(f\#\overline{\psi_a})(y)](1-s) = (Mk)(1-s)M[W(f\#\overline{\psi_a})(\omega)](s).$$

Hence

$$M[W(f\#\overline{\psi_a})(\omega)](s) = M[(f\#\overline{\psi_a})(y)](1-s).(Mk)(s),$$

where $(Mk)(s) = \frac{1}{(Mk)(1-s)}$.

Taking inverse Mellin transform in both sides of above expression and from [2, pp. 217], we get

$$[W(f\#\overline{\psi_a})](\omega) = \int_0^\infty (f\#\overline{\psi_a})(y)k(\omega\,y)\,dy.$$

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