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LAPLACIAN SPECTRAL CHARACTERIZATION OF SOME GRAPH JOIN

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For two disjoint graphs G and H, the join of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. A graph is said to be DLS if there is no other non-isomorphic graph with the same Laplacian spectrum. For a connected DLS graph G with a cut vertex, we prove that $G \vee K_r$ is DLS, where K_r is a complete graph. For a disconnected DLS graph G with $n \geqslant 10$ vertices and $m \leqslant n-4$ edges, we show that $G \vee (K_r - e)$ is DLS, where $K_r - e$ is the graph obtained by deleting one edge of K_r. Applying these results we can obtain new DLS graphs.

Key words : Cospectral graphs; Laplacian spectrum; spectral characterization; join.

1. Introduction

Let G be a simple, undirected graph with n vertices. Let $A(G)$ be the adjacency matrix of G, $D(G)$ be the diagonal matrix of vertex degrees of G. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$. Since $L(G)$ is real, symmetric and positive semidefinite, its eigenvalues are nonnegative. We use $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0$ to denote the eigenvalues of $L(G)$. It is well known that $\mu_{n-1}(G) > 0$ if and only if G is connected. The multiset of the eigenvalues of $L(G)$ is called the *Laplacian spectrum* of G. Two graphs are said to be *L-cospectral* if they have the same Laplacian spectrum. A graph G is said to be *determined by the Laplacian spectrum* if there is no other non-isomorphic graph L-cospectral with G. We use "DLS" as an abbreviation for "determined by the Laplacian spectrum" in this paper.

For two disjoint graphs G and H, let $G \cup H$ denote the *disjoint union* of G and H, while rG denotes the disjoint union of r copies of G. Let $\overline{G}$ denote the complement of G. The *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. Clearly we have $\overline{G \vee H} = \overline{G} \cup \overline{H}$. As usual, P_r, C_r and K_r stand for the path, the cycle and the complete graph with r vertices, respectively. In particular, K_1 stands for an isolated vertex.

Which graphs are determined by their spectra is a difficult problem in the theory of graph spectra. Only some graphs with special structures are known to be determined by their spectra [1, 3-5, 11, 17, 22]. It is known that $G \vee K_r$ is DLS when G is a disconnected DLS graph (see [5, 21]). For a connected DLS graph G, some results on the Laplacian spectral characterization of $G \vee K_r$ are given in [10, 12-14, 20, 21].

For a connected DLS graph G with a cut vertex, we prove that $G \vee K_r$ is DLS. For a disconnected DLS graph G with $n \geqslant 10$ vertices and $m \leqslant n-4$ edges, we show that $G \vee (K_r - e)$ is DLS, where $K_r - e$ is the graph obtained by deleting one edge of K_r. We can obtain new DLS graphs from these results.

2. Preliminaries

In order to get our main results, some auxiliary lemmas are given in this section.

Lemma 2.1 [2] — Let G be a graph with n vertices. Then $\mu_i(G) + \mu_{n-i}(\overline{G}) = n$ for $i = 1, 2, \ldots, n-1$.

Lemma 2.2 [5, 21] — A graph G is DLS if and only if its complement $\overline{G}$ is DLS.

Lemma 2.3 [10] — The join $C_n \vee K_r$ is DLS when $n \neq 6$.

Lemma 2.4 [4] — Let G be a graph. The following invariants of G can be obtained from the Laplacian spectrum of G:

(1) The number of vertices.

(2) The number of edges.

(3) The number of components.

For a graph G, the *vertex connectivity* of G is the smallest number of vertices whose removal results in a disconnected or trivial graph. Let $\kappa(G)$ denote the vertex connectivity of G.

Lemma 2.5 [6] — Let G be a non-complete graph with n vertices. Then $\mu_{n-1}(G) \leqslant \kappa(G)$.

Lemma 2.6 [9] — Let G be a non-complete, connected graph with n vertices. Then $\mu_{n-1}(G) = \kappa(G) = 1$ if and only if $G = H \vee K_1$, where H is a disconnected graph with $n-1$ vertices.

Lemma 2.7 [7] — Let G be a connected graph with n vertices, and G has at least one edge. Then $\mu_1(G) \geqslant \Delta(G)+1$, with equality if and only if $\Delta(G) = n-1$, where $\Delta(G)$ is the maximum degree of G.

Lemma 2.8 [2] — Let G be a connected graph with n vertices. Then $\mu_1(G) \leqslant n$, with equality if and only if $\overline{G}$ is disconnected.

For a graph G, let $E(G)$ denote the edge set of G. We use d_i to denote the degree of vertex i of G. Let m_i denote the average degree of all neighbors of the vertex i. For a vertex u of G, $N(u)$ stands for the set of all neighbors of u.

Lemma 2.9 [23] — For any graph G, we have

$$\mu_1(G) \leqslant \max\left\{\frac{d_i(d_i+m_i)+d_j(d_j+m_j)-2\sum_{k\in N(i)\cap N(j)} d_k}{d_i+d_j} : \{i,j\} \in E(G)\right\}.$$

Let G and H be two L-cospectral graphs. We say that H is a *cospectral mate* of G if H is not isomorphic to G. Clearly a graph G is DLS if and only if G has no cospectral mates.

Lemma 2.10 — For two disjoint graphs G_1 and G_2, if $G_1 \vee G_2$ is DLS, then G_1 and G_2 are DLS.

PROOF : If G_1 is not DLS, then G_1 has a cospectral mate H. From [20, Lemma 2.1], we know that $G_1 \vee G_2$ has a cospectral mate $H \vee G_2$. Similarly, if G_2 is not DLS, then $G_1 \vee G_2$ also has a cospectral mate. Hence if $G_1 \vee G_2$ is DLS, then G_1 and G_2 are DLS. □

3. MAIN RESULTS

The following result generalizes [5, Proposition 4].

***Theorem* 3.1** — *Let G be a DLS graph with n vertices, and $\mu_{n-1}(G) < 1$. Then $G \vee K_r$ is DLS.*

PROOF : Since $\mu_{n-1}(G) < 1$, by Lemma 2.1, we get $\mu_1(\overline{G}) > n-1$. If $\overline{G}$ is disconnected, then each component of $\overline{G}$ has at most $n-1$ vertices. Lemma 2.8 implies that $\mu_1(\overline{G}) \leqslant n-1$, a contradiction. So $\overline{G}$ is connected. Since G is DLS, by Lemma 2.2, $\overline{G}$ is DLS, and $G \vee K_r$ is DLS if and only if $\overline{G} \cup rK_1$ is DLS. Let H be any graph L-cospectral with $\overline{G} \cup rK_1$. Since $\mu_1(H) = \mu_1(\overline{G} \cup rK_1) = \mu_1(\overline{G}) > n-1$, by Lemma 2.8, H has a component H_0 with at least n vertices. By Lemma 2.4, H has $n+r$ vertices and $r+1$ components. Hence we have $H = H_0 \cup rK_1$. Since H and $\overline{G} \cup rK_1$ are L-cospectral, H_0 is L-cospectral with $\overline{G}$. Since $\overline{G}$ is DLS, we have $H_0 = \overline{G}$, $H = \overline{G} \cup rK_1$. Hence $\overline{G} \cup rK_1$ is DLS, i.e., $G \vee K_r$ is DLS. □

***Theorem* 3.2** — *Let G be a connected DLS graph with a cut vertex. Then $G \vee K_r$ is DLS.*

PROOF : Suppose G has n vertices. By Lemma 2.5, we have $\mu_{n-1}(G) \leqslant \kappa(G) = 1$. If $\mu_{n-1}(G) < \kappa(G) = 1$, by Theorem 3.1, $G \vee K_r$ is DLS. If $\mu_{n-1}(G) = \kappa(G) = 1$, by Lemma 2.6, we have $G = G_0 \vee K_1$, where G_0 is a disconnected graph. Since G is DLS, by Lemma 2.10, G_0 is also DLS. Since G_0 is disconnected, we have $\mu_{n-2}(G_0) = 0 < 1$. Hence by Theorem 3.1, $G \vee K_r = G_0 \vee K_{r+1}$ is DLS. □

Let $H_{g,k}$ denote the lollipop graph obtained by identifying a vertex of the cycle C_g and an end vertex of the path P_{k+1}, and let $l(H_{g,k})$ be the line graph of $H_{g,k}$. It is known that $l(H_{g,k})$ is DLS (see [8]). Note that $l(H_{g,k})$ has a cut vertex when $k \geqslant 2$. From Theorem 3.2 we can obtain the following result.

Corollary 3.3 — The graph $l(H_{g,k}) \vee K_r$ is DLS for $k \geqslant 2$.

Let K_n^m denote the graph obtained by attaching m pendent edges to a vertex of complete graph K_{n-m}. It is known that K_n^m is DLS (see [19]). Since K_n^m has a cut vertex, by Theorem 3.2, we can obtain the following result.

Corollary 3.4 — The graph $K_n^m \vee K_r$ is DLS.

Let $S(n, c, k)$ denote the graph on n vertices obtained by attaching $n-2c-2k-1$ pendant edges together with k hanging paths of length two at vertex v_0, where v_0 is the unique common vertex of c triangles (see [11]).

Corollary 3.5 — The graph $S(n, c, k) \vee K_r$ is DLS.

PROOF : If $k = 0$, then $S(n, c, 0) = (cK_2 \cup (n-2c-1)K_1) \vee K_1$ is DLS (see [12]). If $k \geqslant 1$, then $S(n, c, k)$ is DLS (see [11]). Since $S(n, c, k)$ has a cut vertex, by Theorem 3.2, $S(n, c, k) \vee K_r$ is DLS. □

Applying Theorem 3.2, we give a more simpler proof for two results in [13].

Corollary 3.6 [13, Theorem 3.3] — Let G be a DLS tree. Then $G \vee K_r$ is DLS.

PROOF : If G has at most 2 vertices, then $G \vee K_r$ is a complete graph, which is DLS. If G has at least 3 vertices, then G has a cut vertex. By Theorem 3.2, $G \vee K_r$ is DLS. □

Let G be a connected graph with n vertices and m edges. G is called a *unicyclic graph* if $m = n$.

Corollary 3.7 [13, Theorem 4.4] — Let G be a DLS unicyclic graph. Then $G \vee K_r$ is DLS if and only if $G \neq C_6$.

PROOF : If G is a DLS unicyclic graph with a cut vertex, then by Theorem 3.2, $G \vee K_r$ is DLS. If G is a DLS unicyclic graph with no cut vertices, then G is a cycle. By Lemma 2.3, $G \vee K_r$ is DLS when $G \neq C_6$. Since $(2K_2 \cup K_1) \vee 2K_1 \vee K_{r-1}$ is a cospectral mate of $C_6 \vee K_r$ (see [10]), $C_6 \vee K_r$ is not DLS. Hence $G \vee K_r$ is DLS if and only if $G \neq C_6$. □

Let $\delta(G)$ denote the minimum degree of a graph G. The following lemma will be used in the proof of Theorem 3.9.

Lemma 3.8 — Let G be a connected graph with $n \geqslant 11$ vertices and m edges such that $\overline{G}$ is connected. If $\delta(G) \geqslant 2$ and $\mu_1(G) = n-2$, then $m \geqslant 2n-6$.

PROOF : By Lemma 2.9, there exist two adjacent vertices u and v of G such that

$$n-2 \leqslant \frac{d_u(d_u+m_u)+d_v(d_v+m_v)-2\sum_{w\in N(u)\cap N(v)} d_w}{d_u+d_v},$$
$$(n-2-d_u)d_u+(n-2-d_v)d_v \leqslant d_u m_u + d_v m_v - 2\sum_{w\in N(u)\cap N(v)} d_w.$$

Let $c = |N(u)\cap N(v)|$, $r = |N(u)\cup N(v)|$, then $r = d_u + d_v - c \leqslant n$. By $\delta(G) \geqslant 2$, we get

$$\begin{aligned} d_u m_u + d_v m_v - 2\sum_{w\in N(u)\cap N(v)} d_w &\leqslant 2m - 2(n-r) - \sum_{w\in N(u)\cap N(v)} d_w \\ &\leqslant 2m-2(n-r)-2c. \end{aligned}$$

So we have

$$(n-2-d_u)d_u+(n-2-d_v)d_v \leqslant 2m-2(n-r)-2c,$$
$$\Rightarrow\ m \geqslant \frac{(n-2-d_u)d_u+(n-2-d_v)d_v}{2}+n-r+c.$$

Without loss of generality, assume that $d_v \leqslant d_u$. Since $\mu_1(G) = n-2$ and G is connected, by Lemma 2.7, we have $2 \leqslant d_v \leqslant d_u \leqslant n-4$.

Let $f(x) = (n-2-x)x$, where $x \in [2, n-4]$, $n \geqslant 11$. Then $f(x_0) = f(n-2-x_0)$ for any $x_0 \in [2, n-4]$. Taking derivative with respect to x, we have $f'(x) = n-2-2x$. Hence $f(x_0) \geqslant 3(n-5)$ for any $x_0 \in [3, n-5]$ and $f(x_0) \geqslant 2(n-4)$ for any $x_0 \in [2, n-4]$.

If $2 < d_v < n-4$ or $2 < d_u < n-4$, then

$$\begin{aligned} m &\geqslant \frac{(n-2-d_u)d_u+(n-2-d_v)d_v}{2}+n-r+c \\ &\geqslant \frac{3(n-5)+2(n-4)}{2}+n-r+c. \end{aligned}$$

Since $n \geqslant 11, c \geqslant 0$ and $r \leqslant n$, we have $m \geqslant \frac{3(n-5)+2(n-4)}{2} \geqslant 2n-6$.

If $d_u = d_v = 2$, then $m \geqslant 2(n-4)+n-r+c$. By $r = d_u + d_v - c = 4-c$, we have $m \geqslant 2(n-4)+n+2c-4 \geqslant 3n-12 \geqslant 2n-6$.

If $d_u = d_v = n-4$, then $m \geqslant 2(n-4)+n-r+c$. By $r = d_u+d_v-c = 2(n-4)-c$, we have $m \geqslant n+2c$. By $r = 2(n-4)-c \leqslant n$, we get $c \geqslant n-8$. So we have $m \geqslant n+2c \geqslant 3n-16 \geqslant 2n-6$.

If $d_v = 2, d_u = n-4$, then $m \geqslant 2(n-4)+n-r+c$. By $r = d_u+d_v-c = n-2-c$, we have $m \geqslant 2n-6+2c \geqslant 2n-6$. □

Let $K_r - e$ denote the graph obtained by deleting one edge of K_r.

***Theorem* 3.9** — *Let G be a disconnected DLS graph with $n \geqslant 10$ vertices and $m \leqslant n-4$ edges. Then $G \vee (K_r - e)$ is DLS.*

PROOF : Since G is disconnected, $\overline{G}$ is connected. By Lemma 2.8, we get $\mu_1(\overline{G}) = n$. By Lemma 2.2, $\overline{G}$ is DLS, and $G \vee (K_r - e)$ is DLS if and only if $\overline{G} \cup K_2 \cup (r-2)K_1$ is DLS. Let H be any graph L-cospectral with $\overline{G} \cup K_2 \cup (r-2)K_1$. By Lemma 2.4, H has $n+r$ vertices, $\frac{n(n-1)}{2} - m + 1$ edges and r components. Suppose $H = H_0 \cup H_1 \cup \cdots \cup H_{r-1}$, where H_i is a connected graph with n_i vertices, and $\sum_{i=0}^{r-1} n_i = n+r$. Without loss of generality, assume that $n_0 \geqslant n_1 \geqslant \cdots \geqslant n_{r-1} \geqslant 1$. By $\mu_1(\overline{G}) = n \geqslant 10$, we have $\mu_1(H) = \mu_1(\overline{G} \cup K_2 \cup (r-2)K_1) = n$. Lemma 2.8 implies that $n_0 \geqslant n$. By $\sum_{i=0}^{r-1} n_i = n+r$, we get $n_0 \leqslant n+1$. So $n \leqslant n_0 \leqslant n+1$.

If $n_0 = n$, by $\sum_{i=0}^{r-1} n_i = n+r$, we have $H_1 \cup \cdots \cup H_{r-1} = K_2 \cup (r-2)K_1$. Since H and $\overline{G} \cup K_2 \cup (r-2)K_1$ are L-cospectral, H_0 and $\overline{G}$ are L-cospectral. Since $\overline{G}$ is DLS, we have $H_0 = \overline{G}$, $H = \overline{G} \cup K_2 \cup (r-2)K_1$.

If $n_0 = n+1$, by $\sum_{i=0}^{r-1} n_i = n+r$, we have $H_1 = H_2 = \cdots = H_{r-1} = K_1$. Since H has $\frac{n(n-1)}{2} - m + 1$ edges, H_0 has $\frac{n(n-1)}{2} - m + 1$ edges. Since H and $\overline{G} \cup K_2 \cup (r-2)K_1$ are L-cospectral, $\overline{G} \cup K_2$ and $H_0 \cup K_1$ are L-cospectral. Since $m \leqslant n-4$, G has at least 4 components. Lemma 2.8 implies that $\mu_1(G) \leqslant n-3$. By Lemma 2.1, we get $\mu_{n-1}(\overline{G}) \geqslant 3$. Since $\overline{G} \cup K_2$ and $H_0 \cup K_1$ are L-cospectral, by $\mu_1(\overline{G}) = n \geqslant 10$ and $\mu_{n-1}(\overline{G}) \geqslant 3$, we get $\mu_1(H_0) = n$, $\mu_n(H_0) = \mu_n(\overline{G} \cup K_2) = 2$. By Lemma 2.1, we have $\mu_1(\overline{H_0}) = n-1, \mu_n(\overline{H_0}) = 1$. Since $\mu_n(\overline{H_0}) = 1$ and H_0 has $\frac{n(n-1)}{2} - m + 1$ edges, $\overline{H_0}$ is a connected graph with $n+1$ vertices and $n+m-1$ edges. If $\delta(\overline{H_0}) = 1$, then $\kappa(\overline{H_0}) = 1 = \mu_n(\overline{H_0})$. Lemma 2.6 implies that H_0 is disconnected, a contradiction. Hence we have $\delta(\overline{H_0}) \geqslant 2$. Since $\mu_1(\overline{H_0}) = n-1$ and $n_0 = n+1 \geqslant 11$, by Lemma 3.8, we have $n+m-1 \geqslant 2(n+1)-6$, $m \geqslant n-3$, a contradiction to $m \leqslant n-4$.

Hence $\overline{G} \cup K_2 \cup (r-2)K_1$ is DLS, i.e., $G \vee (K_r - e)$ is DLS. □

Remark 3.1 : Some disconnected DLS graphs are given in [3, 4, 15-18]. We can obtain new DLS graphs from Theorem 3.9.

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