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SOME PROPERTIES OF A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY THE MULTIPLIER TRANSFORMATIONS

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The main object of this article is to present a systematic investigation of a new class of harmonic univalent functions $S_H(n,\lambda,\alpha)$ defined by the multiplier transformations. We obtain coefficient bounds, extreme points, distortion theorem and covering result for this class. Further, we give a sufficient condition for a function defined by Srivastava-Owa fractional calculus operator belonging to this class. Apart of these results, many interesting properties on convolution, partial sums and neighborhoods are also obtained. Relevant connections of the results presented herewith various well-known results are briefly indicated.

Key words : Harmonic; univalent functions; multiplier transformations; fractional calculus; convolution; neighborhoods.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|,\ z \in D$. See Clunie and Sheill-Small [14].

Let S_H denotes the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for

$f = h + \overline{g} \in S_H$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

In 1984 Clunie and Sheil-Small [14] investigated the class S_H and as well as its geometric subclasses and obtained some coefficient bounds. After the work of Clunie and Sheil-Small [14], several significant research papers have been published related to S_H and its subclasses. In fact by introducing new subclasses Silverman [58], Silverman and Silvia [59], Jahangiri [27], Frasin [25], Yalcin [63], Joshi and Darus [29], Dixit and Porwal [16], [18] and Pathak *et al.* [43] etc. presented a systematic and unified study of harmonic univalent functions. For more basic results on harmonic functions one may refer to the following standard text book by Duren [19], see also (Ahuja [2], Ponnusamy and Rasila [45], [46]).

We note that for $g \equiv 0$, the class S_H reduces to the class S of analytic univalent functions for which f can be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

Further, we denote the class A of functions of the form (1.2) which are analytic in the open unit disc U.

A function $f(z)$ of the form (1.1) in S_H is said to be in the class $HP(\alpha)$, if and only if

$$\mathrm{R}\left\{h'(z) + g'(z)\right\} > \alpha, \quad z \in U, \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$.

We further denote by $HP^*(\alpha)$ the subclass of $HP(\alpha)$ such that the functions h and g in $f = h + \overline{g}$ are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k|\, z^k \text{ and } g(z) = -\sum_{k=1}^{\infty} |b_k|\, z^k. \tag{1.4}$$

The classes $HP(\alpha)$ and $HP^*(\alpha)$ were introduced and extensively studied by Karpuzoğullari *et al.* [30]. Very recently, some other interesting properties such as generalized convolution and partial sums for these classes ($HP(\alpha)$ and $HP^*(\alpha)$) have been studied by Porwal and Dixit in [47] and [49], respectively.

In the present paper, we generalize the classes $HP(\alpha)$ and $HP^*(\alpha)$ by using multiplier transformations. First, we recall the definition of multiplier transformation $I_\lambda^n : A \to A$

introduced by Cho and Srivastava [12], (see also [13]) as follows

$$I_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^n a_k z^k.$$

For $\lambda = 1$, the operator $I_\lambda^n \equiv I^n$ was studied by Uralegaddi and Somanatha [62] and for $\lambda = 0$ the operator I_λ^n reduce to the well-known Salagean operator introduced by Salagean [56].

Now, for $0 \leq \alpha < 1$, $-1 < \lambda \leq 1$, $n \in N$ and $z \in U$, suppose that $S_H(n, \lambda, \alpha)$ denote the family of harmonic univalent functions f of the form (1.1) such that

$$\mathrm{R} \left\{ \frac{I_\lambda^n h(z) + I_\lambda^n g(z)}{z} \right\} > \alpha. \tag{1.5}$$

Further, let the subclass $\overline{S_H}(n, \lambda, \alpha)$ consisting of harmonic functions $f = h + \overline{g}$ in $S_H(n, \lambda, \alpha)$ such that h and g are of the form (1.4). The classes $S_H(n, \lambda, \alpha)$ and $\overline{S_H}(n, \lambda, \alpha)$ with $b_1 = 0$ will be denoted by $S_H^0(n, \lambda, \alpha)$ and $\overline{S_H^0}(n, \lambda, \alpha)$, respectievely. We note that $S_H(1, 0, \alpha) = HP(\alpha)$, $\overline{S_H}(1, 0, \alpha) = HP^*(\alpha)$ studied by Karpuzoğullari *et al.* in [30], (see also [17]) and for $n = 1, \lambda = 0$ with $g \equiv 0$ the class $S_H(n, \lambda, \alpha)$ reduce to the class $B(\alpha)$. The functions in $B(\alpha)$ are called functions of bounded turning (cf. [26]).

Clearly, if $0 \leq \alpha_1 \leq \alpha_2 < 1$, then

$$S_H(n, \lambda, \alpha_2) \subseteq S_H(n, \lambda, \alpha_1). \tag{1.6}$$

In the present paper, results involving the coefficient inequalities, extreme points, distortion bounds, fractional calculus, convolution, partial sums and neighborhood are determined for the classes $S_H(n, \lambda, \alpha)$ and $\overline{S_H}(n, \lambda, \alpha)$. It is worthy to note that some of our results generalize the results of Karpuzoğullari *et al.* [30] and we also obtain some new results on convolution, partial sums and neighborhoods.

2. Coefficient and Distortion Inequalities

We first mention a sufficient condition for the function f of the form (1.1) belong to the class $S_H(n, \lambda, \alpha)$ given by the following result which can be established easily.

***Theorem* 2.1** — *Let the function $f = h + \overline{g}$ be such that h and g are given by* (1.1). *Furthermore, let*

$$\sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^n |a_k| + \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^n |b_k| \leq 1 - \alpha, \tag{2.1}$$

where $0 \le \alpha < 1$, $-1 < \lambda \le 1$ *and* $n \in N$. *Then* f *is harmonic univalent, sense-preserving in* U *and* $f \in S_H(n, \lambda, \alpha)$.

Remark 2.1 : The result of Theorem 2.1 is sharp for the harmonic univalent functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} (1-\alpha) \left(\frac{1+\lambda}{k+\lambda}\right)^n x_k z^k + \sum_{k=1}^{\infty} (1-\alpha) \left(\frac{1+\lambda}{k+\lambda}\right)^n \overline{y_k z^k}, \tag{2.2}$$

where $0 \le \alpha < 1$, $-1 < \lambda \le 1$, $n \in N$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (2.1) is sharp. It is worthy to note that the function of the form (2.2) belongs to the class $S_H(n, \lambda, \alpha)$ for all $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| \le 1$ because coefficient inequality (2.1) holds.

If we put $n = 1, \lambda = 0$ in Theorem 2.1, then we obtain the following result of Karpuzoğullari *et al.* [30].

Corollary 2.1 — Let the function $f = h + \overline{g}$ be such that h and g are given by (1.1). Furthermore, let

$$\sum_{k=2}^{\infty} k |a_k| + \sum_{k=1}^{\infty} k |b_k| \le 1 - \alpha,$$

where $0 \le \alpha < 1$. Then f is harmonic univalent, sense-preserving in U and $f \in HP(\alpha)$.

In the following theorem, it is proved that the condition (2.1) is also necessary for the functions $f = h + \overline{g}$, where h and g are of the form (1.4).

Theorem 2.2 — *Let* $f = h + \overline{g}$ *be given by* (1.4). *Then* $f \in \overline{S_H}(n, \lambda, \alpha)$, *if and only if*

$$\sum_{k=2}^{\infty} \frac{1}{1-\alpha} \left(\frac{k+\lambda}{1+\lambda}\right)^n |a_k| + \sum_{k=1}^{\infty} \frac{1}{1-\alpha} \left(\frac{k+\lambda}{1+\lambda}\right)^n |b_k| \le 1, \tag{2.3}$$

where $0 \le \alpha < 1$, $-1 < \lambda \le 1$ *and* $n \in N$.

PROOF : The if part follows from Theorem 2.1, so we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (1.4), we notice that the condition

$$\mathrm{R}\left\{\frac{I_\lambda^n h(z) + I_\lambda^n g(z)}{z}\right\} > \alpha$$

is equivalent to

$$\mathrm{R}\left\{1 - \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n |a_k| z^{k-1} - \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n |b_k| z^{k-1}\right\} > \alpha.$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis and making $z \to 1^{-}$, we must have

$$1-\sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|a_k|-\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|b_k|\geq\alpha$$

which is the required condition.

The harmonic univalent functions of the form

$$f(z)=z-\sum_{k=2}^{\infty}(1-\alpha)\left(\frac{1+\lambda}{k+\lambda}\right)^{n}x_k z^k-\sum_{k=1}^{\infty}(1-\alpha)\left(\frac{1+\lambda}{k+\lambda}\right)^{n}y_k\overline{z^k}, \qquad (2.4)$$

where $0\leq\alpha<1,\ -1<\lambda\leq1,\ x_k\geq0,\ y_k\geq0$ and $\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k\leq1$ belongs to the class $\overline{S_H}(n,\lambda,\alpha)$. □

Next, we determine the extreme points of closed convex hulls of $\overline{S_H}(n,\lambda,\alpha)$ denoted by clco $\overline{S_H}(n,\lambda,\alpha)$.

***Theorem* 2.3** — *Let the functions $f=h+\overline{g}$ be given by (1.4). Then $f\in\overline{S_H}(n,\lambda,\alpha)$, if and only if*

$$f(z)=\sum_{k=1}^{\infty}\left(x_k h_k(z)+y_k g_k(z)\right),$$

where $h_1(z)=z,\ h_k(z)=z-\frac{(1-\alpha)(1+\lambda)^n}{(k+\lambda)^n}z^k,\ (k=2,3,4,\ldots),\ g_k(z)=z-\frac{(1-\alpha)(1+\lambda)^n}{(k+\lambda)^n}\bar{z}^k$, $(k=1,2,3,\ldots),\ x_k\geq0,\ y_k\geq0,\ \sum_{k=1}^{\infty}(x_k+y_k)=1$. In particular, the extreme points of $\overline{S_H}(n,\lambda,\alpha)$ are $\{h_k\}$ and $\{g_k\}$.

The following theorem gives the bounds for functions in $\overline{S_H}(n,\lambda,\alpha)$ which yields a covering result for this class.

***Theorem* 2.4** — *Let $f\in\overline{S_H}(n,\lambda,\alpha)$. Then for $|z|=r<1$, we have*

$$|f(z)|\leq(1+|b_1|)\,r+\left(\frac{1+\lambda}{2+\lambda}\right)^{n}(1-|b_1|-\alpha)\,r^2,$$

and

$$|f(z)|\geqslant(1-|b_1|)\,r-\left(\frac{1+\lambda}{2+\lambda}\right)^{n}(1-|b_1|-\alpha)\,r^2.$$

PROOF : The proofs of the above Theorems 2.3 and 2.4 are analogues to the corresponding similar theorems proved in [30] and therefore we omit the details involved. □

The following covering result follows from the left hand inequality in Theorem 2.4.

Corollary 2.2 — Let f of the form (1.4) be such that $f \in \overline{S_H}(n,\lambda,\alpha)$. Then

$$\left\{\omega : |\omega| < \frac{(2+\lambda)^n - (1-\alpha)(1+\lambda)^n}{(2+\lambda)^n} - |b_1|\left(1 - \left(\frac{1+\lambda}{2+\lambda}\right)^n\right)\right\}.$$

Remark 2.2 : If we put $n = 1, \lambda = 0$ in Theorem 2.2-2.4 and Corollary 2.2, then we obtain the corresponding results of Karpuzoğullari *et al.* [30].

In our next result, we give a beautiful application of Theorem 2.1 to obtain a sufficient condition for the Srivastava-Owa fractional calculus operator $\Omega^\mu f(z)$ belong to the class $S_H(n,\lambda,\alpha)$. For this purpose, we recall the following definitions of fractional derivative operator.

Let $L(a,b)$ consists of Lebesgue measurable real or complex valued function $f(x)$ on $[a,b]$:

$$L(a,b) = \left\{f : ||f||_1 = \int_a^b |f(t)|dt < +\infty\right\}.$$

Definition 2.1 (see [36], page 84) — The left-sided Riemann-Liouville fractional derivative of order $\alpha \in C$, $Re(\alpha) \geq 0$ of the function $f(x)$ is defined by

$$\begin{aligned}({}_aD_x^\alpha f)(x) &= \left(D_{a+}^\alpha f\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt,\\ & n = [Re(\alpha)] + 1;\ x > a,\end{aligned}$$

where $[Re(\alpha)]$ means the integral part of $Re(\alpha)$.

The following definitions of fractional derivative operators are due to Owa [38] and Srivastava and Owa [61].

Definition 2.2 — The fractional derivative of order μ is defined for a function $f(z)$ of the form (1.2) by

$$D_z^\mu f(z) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z \frac{f(\varsigma)}{(z-\varsigma)^\mu}d\varsigma, \tag{2.5}$$

where $0 \leq \mu < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\varsigma)^{-\mu}$ is removed by requiring $\log(z-\varsigma)$ to be real when $(z-\varsigma) > 0$.

Definition 2.3 — Under the hypothesis of Definition 2.2, the fractional derivative of order $m+\mu$ is defined for a function $f(z)$ of the form (1.2) by

$$D_z^{m+\mu} f(z) = \frac{d^m}{dz^m}D_z^\mu f(z), \tag{2.6}$$

where $0 \leq \mu < 1$ and $m \in N_0 = \{0, 1, 2, \ldots\}$.

It is easy to see that the Definition 2.2 is a particular case of Definition 2.1 for $a = 0$ and $0 \leq \alpha < 1$.

Using the Definitions 2.2, 2.3, Srivastava and Owa [61] introduced the operator $\Omega^{\mu} : A \to A$ by

$$\Omega^{\mu} f(z) = \Gamma(2-\mu) z^{\mu} D_z^{\mu} f(z), \qquad (\mu \neq 2, 3, 4, \ldots). \tag{2.7}$$

Now, we define the operator Ω^{μ} for functions $f(z)$ of the form (1.1) as

$$\Omega^{\mu} f(z) = \Omega^{\mu} h(z) + \overline{\Omega^{\mu} g(z)}. \tag{2.8}$$

***Theorem* 2.5** — *Let the functions $f = h + \overline{g}$ be given by (1.1) with $b_1 = 0$ satisfies the inequality*

$$\sum_{k=2}^{\infty} \frac{k}{1-\alpha} \left(\frac{k+\lambda}{1+\lambda}\right)^n (|a_k| + |b_k|) \leq 2 - \mu, \tag{2.9}$$

for $0 \leq \alpha < 1$, $0 \leq \mu < 1$, $-1 < \lambda \leq 1$ and $n \in N$, then $\Omega^{\mu} f(z)$ belongs to the class $S_H(n, \lambda, \alpha)$.

PROOF : Using the definition of Srivastava-Owa fractional calculus operator, we have

$$\Omega^{\mu} f(z) = \Omega^{\mu} h(z) + \overline{\Omega^{\mu} g(z)}, \tag{2.10}$$

$$= z + \sum_{k=2}^{\infty} k\phi(k, \mu) a_k z^k + \sum_{k=2}^{\infty} k\phi(k, \mu) \overline{b_k z^k},$$

where

$$\phi(k, \mu) = \frac{\Gamma k \Gamma(2-\mu)}{\Gamma(k+1-\mu)}, \quad k \geq 2.$$

Since $\phi(k, \mu)$ is non-increasing on k, we see that

$$0 < \phi(k, \mu) \leq \phi(2, \mu) = \frac{1}{2-\mu}. \tag{2.11}$$

Therefore

$$\sum_{k=2}^{\infty} \frac{1}{1-\alpha} \left(\frac{k+\lambda}{1+\lambda}\right)^n \phi(k, \mu) k (|a_k| + |b_k|)$$

$$\leq \phi(2, \mu) \sum_{k=2}^{\infty} \frac{k}{1-\alpha} \left(\frac{k+\lambda}{1+\lambda}\right)^n (|a_k| + |b_k|) \leq 1.$$

Hence applying Theorem 2.1, we have $\Omega^{\mu} f(z) \in S_H(n, \lambda, \alpha)$.

Thus the proof of Theorem 2.5 is established. □

3. Convolution Properties

In this section, we study various convolution properties for the classes $S_H(n,\lambda,\alpha)$ and $\overline{S_H}(n,\lambda,\alpha)$. For this we shall require the following definitions of convolution. Let $f_j(z) = z+\sum_{k=2}^{\infty} a_{k,j}z^k$, $(j=1,2)$, be analytic in U, then the convolution (or Hadamard product) of $f_1(z)$ and $f_2(z)$, denoted by $(f_1 * f_2)(z)$, is defined as

$$(f_1 * f_2)(z) = f_1(z) * f_2(z) = z + \sum_{k=2}^{\infty} a_{k,1}a_{k,2}z^k. \tag{3.1}$$

Similarly, the convolution of two harmonic functions is defined as:

Let $f_j(z)$ $(j=1,2)$ in S_H be given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j}z^k + \sum_{k=1}^{\infty} \overline{b_{k,j}z^k}. \tag{3.2}$$

Then the convolution $(f_1 * f_2)(z)$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1}a_{k,2}z^k + \sum_{k=1}^{\infty} \overline{b_{k,1}b_{k,2}z^k} \tag{3.3}$$

and the quasi-convolution for two harmonic functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k|z^k - \sum_{k=1}^{\infty} |b_k|\bar{z}^k \tag{3.4}$$

and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k|z^k - \sum_{k=1}^{\infty} |B_k|\bar{z}^k, \tag{3.5}$$

is defined as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty} |a_kA_k|z^k - \sum_{k=1}^{\infty} |b_kB_k|\bar{z}^k. \tag{3.6}$$

In 1975 Schild and Silverman [57] studied the convolution of univalent functions with negative coefficients. Using this techniques several researchers e.g. (see [1], [3], [37]) obtained analogues result for various classes of univalent functions. In 1985 Kumar [33] studied the Hadamard product of certain starlike functions and improves some results of Owa [39]. Recently, motivated with the work of Kumar [33], Porwal *et al.* [50] improved the result of Yalcin [63] concering the convolution of two harmonic univalent functions in the class $\overline{S_H}(m,n,\alpha)$.

For detailed study of this class one may refer to [63]. Applying the techniques of ([33], [50]), we obtain the following result for the class $\overline{S_H}\,(n,\lambda,\alpha)$.

***Theorem* 3.1** — *Let the functions $f(z)$, $F(z)$ defined by* (3.4), (3.5) *are in the classes $\overline{S_H}(m,\lambda,\beta)$, $\overline{S_H}(n,\lambda,\alpha)$, respectively, where $m,n\in N$, $-1<\lambda\le 1$, $0\le\beta<1$, $0\le\alpha<1$, then $f*F$ defined by* (3.6) *is in the class $\overline{S_H}(m+n,\lambda,\eta)$, where $\eta=\alpha+\beta-\alpha\beta$.*

PROOF : Since $f(z)\in\overline{S_H}(m,\lambda,\beta)$, then by Theorem 2.2, we have

$$\sum_{k=2}^{\infty}\frac{1}{1-\beta}\left(\frac{k+\lambda}{1+\lambda}\right)^{m}|a_k|+\sum_{k=1}^{\infty}\frac{1}{1-\beta}\left(\frac{k+\lambda}{1+\lambda}\right)^{m}|b_k|\le 1. \tag{3.7}$$

Similarly $F(z)\in\overline{S_H}(n,\lambda,\alpha)$, we have

$$\sum_{k=2}^{\infty}\frac{1}{1-\alpha}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|A_k|+\sum_{k=1}^{\infty}\frac{1}{1-\alpha}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|B_k|\le 1. \tag{3.8}$$

Therefore

$$\frac{1}{1-\alpha}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|A_k|\le 1,\quad k=2,3,\dots \tag{3.9}$$

and

$$\frac{1}{1-\alpha}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|B_k|\le 1,\quad k=1,2,3,\dots \tag{3.10}$$

Now, for the convolution function $f*F$ we have

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{1}{1-\eta}\left(\frac{k+\lambda}{1+\lambda}\right)^{m+n}|a_kA_k|+\sum_{k=1}^{\infty}\frac{1}{1-\eta}\left(\frac{k+\lambda}{1+\lambda}\right)^{m+n}|b_kB_k|\\&\le\sum_{k=2}^{\infty}\frac{1}{1-\beta}\left(\frac{k+\lambda}{1+\lambda}\right)^{m}|a_k|+\sum_{k=1}^{\infty}\frac{1}{1-\beta}\left(\frac{k+\lambda}{1+\lambda}\right)^{m}|b_k|,\quad\text{(using (3.9) and (3.10))}\\&\le 1,\quad\text{(using (3.7))}.\end{aligned}$$

Thus the proof of Theorem 3.1 is established. □

***Theorem* 3.2** — *Let the functions $f_i(z)$ defined as*

$$f_i(z)=z-\sum_{k=2}^{\infty}|a_{k,i}|z^k-\sum_{k=1}^{\infty}|b_{k,i}|\overline{z^k},$$

belong to the class $\overline{S_H}(n_i,\lambda,\alpha_i)$ for every $i=1,2,\dots,q$, then the convolution $f_1*f_2*\dots*f_q$ belongs to the class $\overline{S_H}(\sum_{i=1}^{q}n_i,\lambda,\epsilon)$, where $\epsilon=1-\prod_{i=1}^{q}(1-\alpha_i)$.

Proof : The proof of the above theorem is much akin that of Theorem 3.1. Hence we omit the details involved. □

In 1958 Polya-Schoenberg [44] conjectured that the class of convex functions K is preserved under convolution with convex functions, i.e. $f, g \in K \Rightarrow f * g \in K$. In 1973, Ruscheweyh and Sheil-Small [55] proved the Polya-Schoenberg conjecture. Infact, they proved that the classes of convex functions, starlike functions and close-to-convex functions are closed under convolution with convex functions. For an interesting development of these ideas, see Ruscheweyh [54], (and also Duren ([20], pp.246-258), as well as Goodman ([26], pp. 129-130)). Using the techniques developed in Ruscheweyh [54] several researchers (see e.g. [32], [35], [42], [48], [60]) have proved that their classes (subclasses of analytic univalent and multivalent functions only) are closed under convolution with convex (and other related) functions. It is therefore natural to ask what is the analogues result for the subclasses of harmonic univalent functions. In this section, we show that the class $S_H(n, \lambda, \alpha)$ is closed under convolution with certain conditions.

For this, we shall require the following definition and lemmas.

Definition 3.1 — A sequence $\{c_k\}_0^\infty$ of non-negative numbers is said to be a convex null sequence if $c_k \to 0$ as $k \to \infty$ and

$$c_0 - c_1 \geq c_1 - c_2 \geq \ldots \geq c_k - c_{k+1} \geq \ldots \geq 0.$$

Lemma 3.1 — Let $\{c_k\}_0^\infty$ be a convex null sequence. Then the function

$$q_1(z) = \frac{c_0}{2} + \sum_{k=2}^{\infty} c_k z^k$$

is analytic in U and $\text{Re } q_1(z) > 0$, $z \in U$.

Lemma 3.2 — Let $P(z)$ be analytic in U, $P(0) = 1$ and $Re\{P(z)\} > \frac{1}{2}$ in U. For functions F analytic in U, the convolution function $P * F$ takes values in the convex hull of the image on U under F.

Lemmas 3.1 is due to Fejěr [21]. The assertion of Lemma 3.2 readily follows by using the Herglotz representation for $P(z)$.

Lemma 3.3 — Let $f(z) \in S_H^0(n, \lambda, \alpha)$, where $n \in N, n > 1, -1 < \lambda \leq 1$, $0 \leq \alpha < 1$. Then

$$\text{R}\left\{\frac{h(z) + g(z)}{z}\right\} > \frac{1}{2}, \quad z \in U.$$

PROOF : Let $f(z)$ be given by (1.1) with $b_1 = 0$. Since $f(z) \in S_H^0(n, \lambda, \alpha)$, hence by definition

$$\mathrm{R}\left\{\frac{I_\lambda^n h(z) + I_\lambda^n g(z)}{z}\right\} > \alpha, \quad z \in U,$$

which is equivalent to

$$\mathrm{R}\left\{1 - \alpha + \sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^{k-1} + \sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n b_k z^{k-1}\right\} > 0,$$

and hence

$$\mathrm{R}\left\{1 + \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{1-\alpha}\left(\frac{k+\lambda}{1+\lambda}\right)^n (a_k + b_k)\, z^{k-1}\right\} > \frac{1}{2}. \tag{3.11}$$

We observe that the sequence $\{c_k\}_0^\infty$ defined by $c_0 = 1$, $c_k = \frac{2(1-\alpha)(1+\lambda)^n}{(k+1+\lambda)^n}$, $k \geq 1$, is a convex null sequence, we have in view of Lemma 3.1

$$\mathrm{R}\left\{1 + 2\sum_{k=2}^{\infty}\frac{1-\alpha}{\left(\frac{k+\lambda}{1+\lambda}\right)^n} z^{k-1}\right\} > \frac{1}{2}. \tag{3.12}$$

Now

$$\frac{h(z)+g(z)}{z} = \left[1 + \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{1-\alpha}\left(\frac{k+\lambda}{1+\lambda}\right)^n (a_k + b_k)\, z^{k-1}\right] * \left[1 + 2\sum_{k=2}^{\infty}\frac{1-\alpha}{\left(\frac{k+\lambda}{1+\lambda}\right)^n} z^{k-1}\right]$$

and making use of (3.11), (3.12) and Lemma 3.2, we conclude that

$$\mathrm{R}\left\{\frac{h(z)+g(z)}{z}\right\} > \frac{1}{2}.$$

□

***Theorem* 3.3** — *If the functions $f_i(z)$ $(i = 1, 2)$ defined by (3.2) with $b_{1,i} = 0$ $(i = 1, 2)$ are in the classes $S_H(n_i, \lambda, \alpha_i)$, where $-1 < \lambda \leq 1$, $0 \leq \alpha_2 \leq \alpha_1 < 1$, $n_1 \in N$, $n_2 \in N$, $n_2 > 1$ and satisfy the condition $a_{k,1}b_{k,2} + a_{k,2}b_{k,1} = 0$, $(k = 2, 3, \ldots)$, then*

$$P(z) = H(z) + \overline{G(z)} = (f_1 * f_2)(z) \in S_H(n_1, \lambda, \alpha_1).$$

PROOF : To prove that $P(z) \in S_H(n_1, \lambda, \alpha_1)$, we have to show that

$$\mathrm{R}\left\{\frac{I_\lambda^n H(z) + I_\lambda^n G(z)}{z}\right\} > \alpha_1,$$

which is equivalent to

$$\mathrm{R}\left\{1+\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{1-\alpha_1}\left(\frac{k+\lambda}{1+\lambda}\right)^{n_1}a_{k,1}a_{k,2}z^{k-1}+\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{1-\alpha_1}\left(\frac{k+\lambda}{1+\lambda}\right)^{n_1}b_{k,1}b_{k,2}z^{k-1}\right\}>\frac{1}{2}. \tag{3.13}$$

Since $f_1(z) \in S_H(n_1, \lambda, \alpha_1)$ from (3.11) we have

$$\mathrm{R}\left\{1+\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{1-\alpha_1}\left(\frac{k+\lambda}{1+\lambda}\right)^{n_1}a_{k,1}z^{k-1}+\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{1-\alpha_1}\left(\frac{k+\lambda}{1+\lambda}\right)^{n_1}b_{k,1}z^{k-1}\right\}>\frac{1}{2}. \tag{3.14}$$

and since $f_2(z) \in S_H(n_2, \lambda, \alpha_2)$, from Lemma 3.3 we have

$$\mathrm{R}\left\{1+\sum_{k=2}^{\infty}a_{k,2}z^{k-1}+\sum_{k=2}^{\infty}b_{k,2}z^{k-1}\right\}>\frac{1}{2}. \tag{3.15}$$

From (3.14), (3.15) and Lemma 3.2 we immediately have (3.13).

This establishes the proof of the Theorem 3.3. □

In our next result, we give some beautiful applications of the convolution to the classes $S_H(n, \lambda, \alpha)$ and $\overline{S_H}\,(n, \lambda, \alpha)$. For this we shall require the following lemma due to Ruscheweyh and Sheil-Small [55], (see also [20]).

Lemma 3.4 — Let $\varphi(z)$ and $q(z)$ be analytic in U and satisfy the condition $\varphi(0) = q(0) = 0$, $\varphi'(0) = 1$, $q'(0) = 1$. Suppose that for each $\sigma\,(|\sigma| = 1)$ and $\rho\,(|\rho| = 1)$, we have

$$\varphi * \frac{1+\rho\sigma z}{1-\sigma z}q(z) \neq 0, \quad (0 < |z| < 1).$$

Then for each function $F(z)$ analytic in U, satisfying
$\mathrm{R}\{F(z)\} > 0$, $(z \in U)$, we have

$$\mathrm{R}\left\{\frac{\varphi * Fq(z)}{\varphi * q(z)}\right\} > 0, \quad (z \in U).$$

We now obtain the following result:

***Theorem* 3.4** — *Let* $f \in \overline{S_H^0}\,(n, \lambda, \alpha)$, $\delta < 1$ *and*

$$\psi * \frac{1+\rho\sigma z}{1-\sigma z}f(z) \neq 0, \quad (0 < |z| < 1),$$

for each $\sigma\,(|\sigma| = 1)$ *and* $\rho\,(|\rho| = 1)$, *where*

$$\psi(z) = z + \sum_{k=2}^{\infty}\frac{\Gamma(k+1)\,\Gamma(2-\delta)}{\Gamma(k+1-\delta)}z^k.$$

Then $\Omega^\delta f(z)$ *defined by* (2.8) *is also in the class* $\overline{S_H^0}\,(n,\lambda,\alpha)$.

Proof : From definition of $\Omega^\delta f(z)$, we have

$$\Omega^\delta f\,(z) = z - \sum_{k=2}^{\infty} \frac{\Gamma\,(k+1)\,\Gamma\,(2-\delta)}{\Gamma\,(k+1-\delta)}|a_k|z^k - \sum_{k=2}^{\infty} \frac{\Gamma\,(k+1)\,\Gamma\,(2-\delta)}{\Gamma\,(k+1-\delta)}|b_k|\bar{z}^k.$$

$$= \psi\,(z) * h\,(z) + \overline{\psi\,(z) * g\,(z)}.$$

Letting $\varphi\,(z) = \psi\,(z)$, $q\,(z) = z$, $F\,(z) = \frac{I_\lambda^n h(z)+I_\lambda^n g(z)}{z} - \alpha$ in Lemma 3.4, we see that

$$\begin{aligned} \mathrm{R}\left\{\frac{\varphi * Fq\,(z)}{\varphi * q\,(z)}\right\} &= \mathrm{R}\left\{\frac{\psi * \left(I_\lambda^n h\,(z) + I_\lambda^n g\,(z)/_z - \alpha\right) z}{\psi * z}\right\} \\ &= \mathrm{R}\left\{\frac{I_\lambda^n \psi * h\,(z) + I_\lambda^n \psi * g\,(z)}{z}\right\} - \alpha \\ &= \mathrm{R}\left\{\frac{I_\lambda^n \Omega^\delta h\,(z) + I_\lambda^n \Omega^\delta g\,(z)}{z}\right\} - \alpha \\ &> 0, \end{aligned}$$

which implies $\Omega^\delta f \in \overline{S_H^0}\,(n,\lambda,\alpha)$. □

In our next result, we study the mapping properties of the Bernardi integral operator $J_c(f(z))$ on the class $\overline{S_H}\,(n,\lambda,\alpha)$. For this purpose, we define the function $\phi\,(a,c;z)$ by

$$\phi\,(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k, \qquad (c \neq 0,-1,-2,\ldots), \tag{3.16}$$

where $(a)_k$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$(a)_k = \frac{\Gamma\,(a+k)}{\Gamma\,(a)}$$

$$= \begin{cases} 1, & (k=0), \\ a(a+1)(a+2)\ldots(a+k-1), & (k \in N = \{1,2,3,\ldots\}). \end{cases}$$

The function $\phi\,(a,c;z)$ is an incomplete function related to the Gauss hypergeometric function by

$$\phi\,(a,c;z) = zF\,(1,a;c;z)\,. \tag{3.17}$$

Carlson and Shaffer [11] defined a linear operator $L\,(a,c)$, corresponding to the function $\phi(a,c;z)$ on A via the convolution as:

$$L(a,c)h(z) = \phi(a,c;z) * h(z), \qquad (h(z) \in A). \tag{3.18}$$

If $c > a > 0$, $L(a,c)$ has the integral representation

$$L(a,c)h(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}h(uz)du. \tag{3.19}$$

Clearly, $L\left(a,a\right)$ is the identity operator and

$$L(a,c) = L(a,b).L(b,c) = L(b,c).L(a,b), \qquad (b, c \neq 0, -1, -2, \ldots).$$

Moreover if $a \neq 0, -1, -2, \ldots$ then $L(a,c)$ has an inverse $L(c,a)$ and is a one-one mapping of A onto itself, (see Owa and Srivastava [40]).

Bernardi [9] defined the integral operator $J_c f(z)$ for the function $f(z)$ of the form (1.2) as

$$J_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \qquad (c > -1)$$

$$= z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k$$

$$= L\left(c+1, c+2\right) f\left(z\right) \tag{3.20}$$

or

$$J_c\left(f\right) = \phi\left(c+1, c+2; z\right) * f\left(z\right). \tag{3.21}$$

Now, we define the Bernardi integral operator $J_c f(z)$ on the class S_H of harmonic univalent functions of the form (1.1) as follows:

$$J_c f(z) = J_c h(z) + \overline{J_c g(z)}$$

$$= z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k + \overline{\sum_{k=1}^{\infty} \frac{c+1}{c+k} b_k z^k} \tag{3.22}$$

$$= L(c+1, c+2)h(z) + \overline{L(c+1, c+2)g(z)}. \tag{3.23}$$

$$= \phi(c+1, c+2; z) * h(z) + \overline{\phi(c+1, c+2; z) * g(z)}. \tag{3.24}$$

Theorem 3.5 — *If the function $f\left(z\right)$ defined by (1.4) with $b_1 = 0$ is in the class $\overline{S_H}\left(n, \lambda, \alpha\right)$, then $J_c f\left(z\right)$ defined by (3.22) is in the class $\overline{S_H}\left(n, \lambda, \beta\right)$, where*

$$\beta = (2\alpha - 1) + 2(1-\alpha)(c+1) \sum_{k=1}^{\infty} \frac{(-1)^k}{c+k+1}. \tag{3.25}$$

Proof : By using (3.24), we have

$$I_\lambda^n J_c\left(h(z)\right) + I_\lambda^n J_c\left(g(z)\right) = \phi(c+1, c+2; z) * \left\{I_\lambda^n h(z) + I_\lambda^n g(z)\right\}.$$

A simple calculation shows that

$$\frac{I_\lambda^n J_c\left(h(z)\right) + I_\lambda^n J_c\left(g(z)\right)}{z} = \frac{1}{z}[L(c+1, c+2)(I_\lambda^n h(z) + I_\lambda^n g(z))].$$

Using (3.19), we obtain that

$$\mathrm{R}\left\{\frac{I_\lambda^n J_c h\left(z\right) + I_\lambda^n J_c g\left(z\right)}{z}\right\} = (c+1)\int_0^1 u^c \mathrm{R}\left\{\frac{I_\lambda^n h\left(zu\right) + I_\lambda^n g\left(zu\right)}{zu}\right\} du\,. \tag{3.26}$$

Since $f(z) \in \overline{S_H}(n, \lambda, \alpha)$, we put

$$\frac{I_\lambda^n h(z) + I_\lambda^n g(z)}{z} = G(z),$$

then $G(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in U and $\mathrm{R}\{G(z)\} > \alpha$. It is known that [52] if $q(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is analytic in U and $Re\{q(z)\} > \gamma,\ (0 \le \gamma < 1),$ then

$$\mathrm{R}\{q(z)\} \ge \frac{1 + (2\gamma - 1)r}{1+r}, \quad (|z| = r < 1). \tag{3.27}$$

Hence by using (3.26) and (3.27), we have

$$\begin{aligned}\mathrm{R}\left\{\frac{I_\lambda^n J_c\left(h\right) + I_\lambda^n J_c\left(g\right)}{z}\right\} &\ge (c+1)\int_0^1 u^c \frac{1 + (2\alpha - 1)u}{1+u}\, du \\ &= (2\alpha - 1) + 2(1-\alpha)\left(c+1\right)\int_0^1 \frac{u^c}{1+u} du \\ &= (2\alpha - 1) + 2(1-\alpha)\left(c+1\right)\sum_{k=0}^{\infty} \frac{(-1)^k}{c+k+1},\end{aligned}$$

that is $J_c(f(z)) \in \overline{S_H}(n, \lambda, \beta)$, where β is defined by (3.25). □

4. Partial Sums of the Libera Integral Operator

The study of partial sums of certain integral operator is one of the main interesting problems in Geometric Function Theory. In 1997, Li and Owa [34] proved that for a normalized univalent function $f(z)$ of the form (1.2) that the partial sums of the Libera integral operator of functions is starlike in $|z| < \frac{3}{8}$, the number $\frac{3}{8}$ is best possible. Jahangiri and Farahmand

[28] also shown that the partial sums of the Libera integral operator of functions of bounded turning are also of bounded turning. Adopting the same technique used in [28], Babalola [8] and Darus and Ibrahim [15] extend the results of [28] to the more general classes of functions. Very recently, Porwal and Dixit [49], Porwal *et al.* [51] studied the analogues results on the partial sums of certain integral operators of harmonic univalent functions for the classes $HP(\alpha)$ and $HP^*(\alpha)$. For $g \equiv 0$ the results of ([49], [51]) reduce to the corresponding results of ([28], [15]), respectively. It is worthy to note that they ([49], [51]) also improve the results of [28] and [15], when $f(z)$ has of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k. \tag{4.1}$$

In this section, motivated with the above mentioned work, an attempt has been made to systematically study on the partial sums of Libera integral operator for the classes $S_H(n, \lambda, \alpha)$ and $\overline{S_H}(n, \lambda, \alpha)$.

Now, we recall some definitions relevant to further study. For $f(z)$ of the form (1.2), the Libera integral operator $F(z)$ is given by

$$F(z) = \frac{2}{z}\int_0^z f(\varsigma)\,d\varsigma = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k. \tag{4.2}$$

Motivated with the definition (4.2), Porwal and Dixit [49] define the Libera integral operator for $f = h + \overline{g}$ in S_H, where h and g are given by (1.1) as

$$F(z) = \frac{2}{z}\int_0^z h(\varsigma)\,d\varsigma + \overline{\frac{2}{z}\int_0^z g(\varsigma)\,d\varsigma} = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k + \sum_{k=1}^{\infty} \overline{\frac{2}{k+1} b_k z^k}. \tag{4.3}$$

The j-th partial sums $F_j(z)$ of the integral operator $F(z)$ for functions f of the form (1.1) are given by

$$\begin{aligned} F_j(z) &= z + \sum_{k=2}^{j} \frac{2}{k+1} a_k z^k + \sum_{k=1}^{j} \overline{\frac{2}{k+1} b_k z^k}. \\ &= H_j(z) + \overline{G_j(z)}. \end{aligned} \tag{4.4}$$

The j-th partial sums $F_j(z)$ of the Libera integral operator $F(z)$ for analytic univalent functions of the form (1.2) have been studied by various authors in ([28], [34]) and for harmonic univalent functions of the form (1.1) have been studied in [49].

To derive our first main result of this section, we shall require the following lemma which is due to Jahangiri and Farahmand [28].

Lemma 4.1 — For $z \in U$,

$$\mathrm{R}\left(\sum_{k=1}^{m} \frac{z^k}{k+2}\right) > -\frac{1}{3}. \tag{4.5}$$

***Theorem* 4.1** — *If* $f(z)$ *of the form* (1.1) *with* $b_1 = 0$ *and* $f(z) \in S_H(n, \lambda, \alpha)$, *then* $F_j(z) \in S_H\left(n, \lambda, \frac{4\alpha-1}{3}\right)$ *for* $\frac{1}{4} \leq \alpha < 1$.

PROOF : Let f be of the form (1.1) and belong to $S_H(n, \lambda, \alpha)$ for $\frac{1}{4} \leq \alpha < 1$.

Since

$$\mathrm{R}\left\{\frac{I_\lambda^n h(z) + I_\lambda^n g(z)}{z}\right\} > \alpha,$$

we have

$$\mathrm{R}\left\{1 + \frac{1}{2(1-\alpha)}\left(\sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^{k-1} + \sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n b_k z^{k-1}\right)\right\} > \frac{1}{2}. \tag{4.6}$$

Applying the convolution properties of power series to $\frac{I_\lambda^n H_j(z)+I_\lambda^n G_j(z)}{z}$, we may write

$$\begin{aligned}\frac{I_\lambda^n H_j(z) + I_\lambda^n G_j(z)}{z} &= 1 + \sum_{k=2}^{j} \frac{2}{k+1}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^{k-1} + \sum_{k=2}^{j} \frac{2}{k+1}\left(\frac{k+\lambda}{1+\lambda}\right)^n b_k z^{k-1} \\ &= \left(1 + \frac{1}{2(1-\alpha)}\left(\sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n (a_k + b_k) z^{k-1}\right)\right) * \left(1 + (1-\alpha)\sum_{k=2}^{j} \frac{4}{k+1} z^{k-1}\right) \qquad (4.7) \\ &= P(z) * Q(z).\end{aligned}$$

From Lemma 4.1 for $m = j-1$, we obtain

$$\mathrm{R}\left(\sum_{k=2}^{j} \frac{z^{k-1}}{k+1}\right) > -\frac{1}{3}. \tag{4.8}$$

By applying a simple algebra to inequality (4.8) and $Q(z)$ in (4.7), one may obtain

$$\mathrm{R}(Q(z)) = \mathrm{R}\left\{1 + (1-\alpha)\sum_{k=2}^{j} \frac{4}{k+1} z^{k-1}\right\} > \frac{4\alpha - 1}{3}.$$

On the other hand, the power series $P(z)$ in (4.7) in conjunction with the condition (4.6) yields

$$\mathrm{R}\left(P\left(z\right)\right)>\frac{1}{2}.$$

Therefore, by Lemma 3.2, $\mathrm{R}\left\{\frac{I_\lambda^n H_j(z)+I_\lambda^n G_j(z)}{z}\right\}>\frac{4\alpha-1}{3}$.

This completes the proof of Theorem 4.1. □

Remark 4.1 : If we put $n=1,\lambda=0$ in Theorem 4.1, then we obtain the corresponding result of Porwal and Dixit [49].

Remark 4.2 : If f of the form (1.2) with $n=1,\lambda=0$ in Theorem 4.1 then we obtain the corresponding result of Jahangiri and Farahmand in [28].

Theorem 4.2 — *Let f be of the form (1.4) with $b_1=0$ and $f\in\overline{S_H}\left(n,\lambda,\alpha\right)$, then the functions $F\left(z\right)$ defined by (4.3) belongs to $\overline{S_H}\left(n,\lambda,\rho\right)$, where $\rho=\frac{1+2\alpha}{3}$. The result is sharp. Further, the converse need not to be true.*

Proof : Since $f\in\overline{S_H}\left(n,\lambda,\alpha\right)$, Theorem 2.2 ensures that

$$\sum_{k=2}^{\infty}\frac{1}{1-\alpha}\left(\frac{k+\lambda}{1+\lambda}\right)^n\left(|a_k|+|b_k|\right)\leq 1. \tag{4.9}$$

Also, from (4.3) we have

$$F\left(z\right)=z-\sum_{k=2}^{\infty}\frac{2}{k+1}\left|a_k\right|z^k-\sum_{k=2}^{\infty}\frac{2}{k+1}\left|b_k\right|\bar{z}^k.$$

Let $F\left(z\right)\in\overline{S_H}\left(n,\lambda,\sigma\right)$, then by Theorem 2.2, we have

$$\sum_{k=2}^{\infty}\left(\frac{1}{1-\sigma}\left(\frac{k+\lambda}{1+\lambda}\right)^n\right)\left(\frac{2}{k+1}\left|a_k\right|+\frac{2}{k+1}\left|b_k\right|\right)\leq 1.$$

Thus we have to find largest value of σ so that the above inequality holds. Now this inequality holds if

$$\sum_{k=2}^{\infty}\left(\frac{1}{1-\sigma}\left(\frac{k+\lambda}{1+\lambda}\right)^n\right)\left(\frac{2}{k+1}\left|a_k\right|+\frac{2}{k+1}\left|b_k\right|\right)\leq\sum_{k=2}^{\infty}\frac{1}{1-\alpha}\left(\frac{k+\lambda}{1+\lambda}\right)^n\left(|a_k|+|b_k|\right).$$

or, if

$$\left(\frac{1}{1-\sigma}\left(\frac{k+\lambda}{1+\lambda}\right)^n\right)\frac{2}{k+1}\leq\frac{1}{1-\alpha}\left(\frac{k+\lambda}{1+\lambda}\right)^n,\qquad\text{for each }\ k=2,3,4,\ldots$$

which is equivalent to

$$\sigma \leq \frac{k-1+2\alpha}{k+1} = \rho_k, \qquad k = 2, 3, 4, \ldots.$$

It is easy to verify that ρ_k is an increasing function of k. Therefore $\rho = \inf_{k\geq 2} \rho_k = \rho_2$ and, hence

$$\rho = \frac{1+2\alpha}{3}.$$

To show the sharpness, we take the function $f(z)$ given by

$$f(z) = z - \frac{(1-\alpha)(1+\lambda)^n}{(2+\lambda)^n} |x| z^2 - \frac{(1-\alpha)(1+\lambda)^n}{(2+\lambda)^n} |y| \bar{z}^2, \text{ where } |x| + |y| = 1.$$

Then

$$\begin{aligned} F(z) =& z - \frac{2(1-\alpha)(1+\lambda)^n}{3(2+\lambda)^n} |x| z^2 - \frac{2(1-\alpha)(1+\lambda)^n}{3(1+\lambda)^n} |y| \bar{z}^2 \\ =& H(z) + \overline{G(z)} \end{aligned}$$

and therefore

$$\begin{aligned} \frac{I_\lambda^n H(z) + I_\lambda^n G(z)}{z} &= 1 - \frac{2(1-\alpha)}{3} |x| z - \frac{2(1-\alpha)}{3} |y| z \\ &= \frac{3 - 2(1-\alpha)(|x|+|y|) z}{3} \\ &= \frac{1+2\alpha}{3}, \quad \text{for } z \to 1. \end{aligned}$$

Hence, the result is sharp.

We now show that the converse of above theorem need not to be true. To this end, we consider the function

$$F(z) = z - \frac{(1+\lambda)^n (1-\sigma)}{(3+\lambda)^n} |x| z^3 - \frac{(1+\lambda)^n (1-\sigma)}{(3+\lambda)^n} |y| \bar{z}^3,$$

where

$$|x| + |y| = 1, \qquad \sigma = \frac{2\alpha+1}{3}.$$

Theorem 2.2 guarantees that $F(z) \in \overline{S_H}(n, \lambda, \sigma)$.

But the corresponding function

$$f(z) = z - \frac{2(1-\sigma)(1+\lambda)^n}{(3+\lambda)^n} |x| z^3 - \frac{2(1-\sigma)(1+\lambda)^n}{(3+\lambda)^n} |y| \bar{z}^3,$$

does not belong to $\overline{S_H}(n,\lambda,\alpha)$, since for this the function $f(z)$ does not satisfy the coefficient inequality of Theorem 2.2. □

In our next theorem, we improve the result of Theorem 4.1 for functions $f(z)$ of the form (1.4).

***Theorem* 4.3** — *Let f of the form (1.4) with $b_1 = 0$ and $f \in \overline{S_H^0}(n,\lambda,\alpha)$, then the function $F_j(z)$ defined by (4.4) belong to $\overline{S_H^0}\left(n,\lambda,\frac{2\alpha+1}{3}\right)$.*

Proof : Since

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=2}^{\infty} |b_k| \bar{z}^k.$$

Then

$$F(z) = z - \sum_{k=2}^{\infty} \frac{2}{k+1} |a_k| z^k - \sum_{k=2}^{\infty} \frac{2}{k+1} |b_k| \bar{z}^k.$$

By using Theorem 4.2, we have

$$F(z) \in \overline{S_H^0}(n,\lambda,\sigma), \quad \text{where } \sigma = \frac{2\alpha+1}{3}.$$

Now

$$F_j(z) = z - \sum_{k=2}^{j} \frac{2}{k+1} |a_k| z^k - \sum_{k=2}^{j} \frac{2}{k+1} |b_k| \bar{z}^k.$$

To show that $F_j(z) \in \overline{S_H}(n,\lambda,\sigma)$, we have

$$\begin{aligned} &\sum_{k=2}^{j} \left(\frac{1}{1-\sigma}\left(\frac{k+\lambda}{1+\lambda}\right)^n\right)\left(\frac{2}{k+1}|a_k| + \frac{2}{k+1}|b_k|\right) \\ \leq &\sum_{k=2}^{\infty} \left(\frac{1}{1-\sigma}\left(\frac{k+\lambda}{1+\lambda}\right)^n\right)\left(\frac{2}{k+1}|a_k| + \frac{2}{k+1}|b_k|\right) \\ \leq &1. \end{aligned}$$

Thus $F_j(z) \in \overline{S_H}(n,\lambda,\sigma)$. □

Remark 4.3 : Since $\frac{2\alpha+1}{3} > \frac{4\alpha-1}{3}$, for $\frac{1}{4} \leq \alpha < 1$. From (1.6), we see that

$$\overline{S_H}\left(n,\lambda,\frac{2\alpha+1}{3}\right) \subset \overline{S_H}\left(n,\lambda,\frac{4\alpha-1}{3}\right).$$

Hence the result of Theorem 4.3 provides a smaller class in comparison to the class given by Theorem 4.1.

In the following corollary, we improve a result of Jahangiri and Farahmand in [28] when $f(z)$ has of the form (4.1). For this we shall require the following lemma.

Lemma 4.2 — If $0 \leq \alpha_1 \leq \alpha_2 < 1$, then

$$B(\alpha_2) \subseteq B(\alpha_1).$$

PROOF : The proof of the above lemma is straight forward, so we omit the details. □

If we put $n = 1, \lambda = 0, g \equiv 0$ in Theorem 4.3, then we obtain the following corollary.

Corollary 4.1 — Let $f(z)$ be defined by (4.1) is in the class $B(\alpha)$, then $F_j(z) = z - \sum_{k=2}^{j} \frac{2}{k+1} |a_k| z^k$ belongs to the class

$$B\left(\frac{2\alpha+1}{3}\right).$$

Remark 4.4 — For $\frac{1}{4} \leq \alpha < 1$, $f(z) \in B(\alpha)$ Jahangiri and Farahmand [28] shows that $F_j(z) \in B\left(\frac{4\alpha-1}{3}\right)$ and our result states that $F_j(z) \in B\left(\frac{2\alpha+1}{3}\right)$.

Since $\frac{2\alpha+1}{3} > \frac{4\alpha-1}{3}$, for $\frac{1}{4} \leq \alpha < 1$, and using Lemma 4.2, we have

$$B\left(\frac{2\alpha+1}{3}\right) \subset B\left(\frac{4\alpha-1}{3}\right).$$

Hence our result provides a smaller class in comparison to the class given by Jahangiri and Farahmand [28].

5. Inclusion Relationship Involving Neighborhoods

The concept of $\delta-$ neighborhoods $N_\delta(f(z))$ of analytic functions $f(z)$ of the form (1.2) with normalization $f(0) = f'(0) - 1 = 0$ was first introduced by Ruscheweyh [53] and was studied by Fournier [22]-[24] and by Brown [10]. Motivated with the above mentioned work several researchers e.g. see ([4]-[7], [16], [30], [31], and [41]) studied the neighborhood properties for various subclasses of analytic and harmonic functions.

In this section, we study some neighborhood properties for the class $\overline{S_H}(n, \lambda, \alpha)$.

Now, we define the δ- neighborhood of f is the set

$$N_\delta(f) = \left\{ F_1 : F_1(z) = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k z^k} \text{ and } \sum_{k=1}^{\infty} k(|a_k - A_k| + |b_k - B_k|) \leq \delta \right\}. \tag{5.1}$$

***Theorem* 5.1** — *Let* $f \in \overline{S_H}(n, \lambda, \alpha)$. *If* $\delta \leqslant (1-\alpha-|b_1|)\left\{1-\left(\frac{1+\lambda}{2+\lambda}\right)^{n-1}\right\}$, *then* $N_\delta(f) \subset HP(\alpha)$.

PROOF : Let

$$F_1(z) = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k z^k} \tag{5.2}$$

belongs to $N_\delta(f)$.

Now

$$\begin{aligned}
&|B_1| + \sum_{k=2}^{\infty} k(|A_k| + |B_k|) \\
&\leq |B_1 - b_1| + |b_1| + \sum_{k=2}^{\infty} k(|A_k - a_k| + |B_k - b_k|) + \sum_{k=2}^{\infty} k(|a_k| + |b_k|) \\
&\leq \delta + |b_1| + \left(\frac{1+\lambda}{2+\lambda}\right)^{n-1} \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n} (|a_k| + |b_k|) \\
&\leq \delta + |b_1| + \left(\frac{1+\lambda}{2+\lambda}\right)^{n-1} (1-\alpha-|b_1|) \\
&\leq 1-\alpha,
\end{aligned}$$

if $\delta \leqslant (1-\alpha-|b_1|)\left\{1-\left(\frac{1+\lambda}{2+\lambda}\right)^{n-1}\right\}$.

Thus $F_1(z) \in HP(\alpha)$. □

***Theorem* 5.2** — *Let* $f \in \overline{S_H}(n, \lambda, \alpha)$. *If* $\delta \leqslant (1-\alpha-|b_1|)\left(1-\frac{1}{(2-\mu)}\left(\frac{1+\lambda}{2+\lambda}\right)^{n-2}\right)$, *then* $N_\delta(\Omega^\mu f) \subset HP(\alpha)$.

PROOF : Let

$$F_1(z) = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k z^k} \tag{5.3}$$

belongs to $N_\delta(\Omega^\mu f)$.

Now

$$\begin{aligned}
&|B_1| + \sum_{k=2}^{\infty} k(|A_k| + |B_k|) \\
&\leq |B_1 - b_1| + |b_1| + \sum_{k=2}^{\infty} k(|A_k - k\phi(k,\mu)a_k| + |B_k - k\phi(k,\mu)b_k|) \\
&+ \sum_{k=2}^{\infty} k(\phi(k,\mu)k\,|a_k| + \phi(k,\mu)k\,|b_k|)
\end{aligned}$$

$$\leq \delta + |b_1| + \frac{1}{(2-\mu)} \left(\frac{1+\lambda}{2+\lambda}\right)^{n-2} \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n} (|a_k| + |b_k|)$$
$$\leq \delta + |b_1| + \frac{1}{(2-\mu)} \left(\frac{1+\lambda}{2+\lambda}\right)^{n-2} (1-\alpha-|b_1|)$$
$$\leq 1-\alpha,$$

if $\delta \leqslant (1-\alpha-|b_1|)\left(1-\frac{1}{(2-\mu)}\left(\frac{1+\lambda}{2+\lambda}\right)^{n-2}\right)$.

Thus $F_1(z) \in HP(\alpha)$. □

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References

1. M. Acu, On some analytic functions with negative coefficients, *Gen. Math.*, **15**(2-3) (2007), 190-200.

2. O. P. Ahuja, Planar harmonic univalent and related mappings, *J. Inequal. Pure Appl. Math.*, **6**(4) (2005), Art. 122, 1-18.

3. R. M. Ali, M. H. Khan, V. Ravichandran and K. G. Subramanian, A class of multivalent functions with negative coefficients defined by convolution, *Bull. Korean Math. Soc.*, **43**(1) (2006), 179-188.

4. O. Altintas and S. Owa, Neighborhoods of certain analytic functions with negative coefficients, *Int. J. Math. Math. Sci.*, **19**(4) (1996), 797-800.

5. O. Altintas, Ö. Özkan and H. M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, *Appl. Math. Lett.*, **13**(3) (2000), 63-67.

6. M. K. Aouf, Inclusion and neighborhood properties for certain subclasses of analytic functions associated with convolution structure, *The Aust. J. Math. Anal. Appl.*, **7**(1) (2010), Art. 4, 1-10.

7. M. K. Aouf and J. Dziok, Inclusion and neighborhood properties of certain subclasses of analytic and multivalent functions, *European J. Pure Appl. Math.*, **2**(4) (2009), 544-553.

8. K. O. Babalola, Quasi-partial sums of the generalized Bernardi integral of certain analytic functions, *J. Nigerian Assoc. Math. Phy.*, **11** (2007), 67-70.

9. S. D. Bernardi, Convex and starlike univalent functions, *Trans Amer. Math. Soc.*, **135** (1969), 429-446.

10. J. E. Brown, Some sharp neighborhoods of univalent functions, *Trans Amer. Math. Soc.*, **287** (1985), 475-482.

11. B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15** (1984), 737-745.

12. N. E. Cho and H. M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, *Math. Comput. Modell.*, **37**(1-2) (2003), 39-49.

13. N. E. Cho and T. H. Kim, Multiplier transformations and strongly close-to-convex functions, *Bull. Korean Math. Soc.*, **40**(3) (2003), 399-410.

14. J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. AI Math.*, **9** (1984), 3-25.

15. M. Darus and R. W. Ibrahim, Partial sums of analytic functions of bounded turning with applications, *Comput. Appl. Math.*, **29**(1) (2010), 81-88.

16. K. K. Dixit and Saurabh Porwal, On a subclass of harmonic univalent functions, *J. Inequal. Pure Appl. Math.*, **10**(1) (2009), Art. 27, 1-18.

17. K. K. Dixit and Saurabh Porwal, A subclass of harmonic univalent functions with positive coefficients, *Tamkang J. Math.*, **41**(3) (2010), 261-269.

18. K. K. Dixit and Saurabh Porwal, Some properties of harmonic functions defined by convolution, *Kyungpook Math. J.*, **49** (2009), 751-761.

19. P. Duren, *Harmonic mappings in the plane*, Camb. Univ. Press, (2004).

20. P. L. Duren, *Univalent Functions*, Grundleherem der Mathematischen Wissenchaften 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, (1983).

21. L. Fejĕr, Uber die positivitat von summen die nach trigonometrischen order Legendreschen funktionen fortschreiten, *Acta Litt. Ac Sci. Szeged*, (1925), 75-86.

22. R. Fournier, A note on neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **87** (1983), 117-120.

23. R. Fournier, On neighbourhoods of univalent starlike functions, *Ann. Polon. Math.*, **47** (1986), 189-202.

24. R. Fournier, On neighbourhoods of univalent convex functions, *Rocky Mount. J. Math.*, **16** (1986), 579-589.

25. B. A. Frasin, Comprehensive family of harmonic univalent functions, *SUT J. Math.*, **42**(1) (2006), 145-155.

26. A. W. Goodman, *Univalent functions*, Vol. I, II, Marnier Publishing, Florida, (1983).

27. J. M. Jahangiri, Harmonic functions starlike in the unit disc, *J. Math. Anal. Appl.*, **235** (1999), 470-477.

28. J. M Jahangiri and K. Farahmand, Partial sums of functions of bounded turning, *J. Inequal. Pure Appl. Math.*, **4**(4) (2003), Art. 79, 1-3.

29. S. B. Joshi and M. Darus, Unified treatment for harmonic univalent functions, *Tamsui Oxford J. Math. Sci.*, **24**(3) (2008), 225-232.

30. S. Y. Karpuzoğullari, M. Öztürk and M. Yamankaradeniz, A subclass of harmonic univalent functions with negative coefficients, *Appl. Math. Comput.*, **142** (2003), 469-476.

31. B. S. Keerthi, B.A. Stephen, A. Ganagadharan and S. Sivasubramanian, Neighborhoods of certain subclasses of analytic functions of complex order with negative coefficients, *The Aust. J. Math. Anal. Appl.*, **7**(1) (2010), Art. 6, 1-7.

32. Y. C. Kim, J. H. Choi and T. Sugawa, Coefficient bounds and convolution properties for certain classes of close-to-convex functions, *Proc. Japan Acad. Ser. A Math. Sci.*, **76**(6) (2000), 95-98.

33. V. Kumar, Hadamard product of certain starlike functions, *J. Math. Anal. Appl.*, **110** (1985), 425-428.

34. J. L. Li and S. Owa, On partial sums of the Libera integral operator, *J. Math. Anal. Appl.*, **213**(2) (1997), 444-454.

35. A. Y. Lashin, Some convolution properties of analytic functions, *Appl. Math. Lett.*, **18** (2005), 135-138.

36. A. M. Mathai, R. K. Saxena and H.J. Haubold, *The H-function theory and applications*, Springer, New York, Dordrecht, Heidelberg, London, (2010), ISBN 978-1-4419-0915-2.

37. J. Nishiwaki and S. Owa, Convolutions for certain analytic functions, *Gen. Math.*, **15**(2-3), (2007), 38-51.

38. S. Owa, On the distortion theorem I, *Kyungpook Math. J.*, **18** (1978), 53-59.

39. S. Owa, On the classes of univalent functions with negative coefficients, *Math. Japon.*, **27**(4) (1982), 409-416.

40. S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, **39** (1987), 1057-1077.

41. S. Owa, H. Saitoh and M. Nunokawa, Neighborhoods of certain analytic functions, *Appl. Math. Lett.*, **6**(4) (1993), 73-77.

42. R. Parvatham and S. Radha, On $\alpha-$ starlike and $\alpha-$ close-to-convex functions with respect to n-symmetric points, *Indian J. Pure Appl. Math.*, **17**(9) (1986), 1114-1122.

43. A. L. Pathak, S. Porwal, R. Agarwal and R. Misra, A subclass of harmonic univalent functions with positive coefficients associated with fractional calculus operator, *J. Nonlinear Anal. Appl.*, (2012), Article ID jnaa-00108, 1-11.

44. G. Polya and I. J. Schoenberg, Remarks on de la Vallee Poussin means and convex conformal maps of the circle, *Pacific J. Math.*, **8** (1958), 295-334.

45. S. Ponnusamy and A. Rasila, Planar harmonic mappings, *RMS Mathematics Newsletters*, **17**(2) (2007), 40-57.

46. S. Ponnusamy and A. Rasila, Planar harmonic and quasi-conformal mappings, *RMS Mathematics Newsletters*, **17**(3) (2007), 85-101.

47. Saurabh Porwal and K. K. Dixit, Some properties of generalized convolution of harmonic univalent functions, *Demonstratio Math.*, **46**(1) (2013), 63-74.

48. Saurabh Porwal and K. K. Dixit, A note on convolution of analytic functions, *Bull. Allahabad Math. Soc.*, **27**(2) (2012), 219-225.

49. Saurabh Porwal and K. K. Dixit, Partial sums of harmonic univalent functions, *Studia Univ. Babes Bolayi*, **58**(1) (2013), 15-21.

50. Saurabh Porwal, K. K. Dixit and S. B. Joshi, Convolution of Salagean-type harmonic univalent functions, *Punjab University J. Math.*, **43** (2011), 69-73.

51. Saurabh Porwal, B. A. Frasin and Ajay Singh, *Partial sums of certain integral operator on harmonic univalent functions*, Analele Universitatii Oradea Fasc. Matematica, **XX** (2) (2013), 145-152.

52. M. S. Robertson, On the theory of univalent functions, *Ann. of Math.*, **37** (1936), 374-408.

53. S. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81** (1981), 521-528.

54. S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Presses Univ. Montreal, Montreal, Que., (1982).

55. S. Ruscheweyh and T. Sheil-Small, Hadamard products of schlicht functions and the Polya-Schoenberg conjecture, *Comment. Math. Helv.*, **48** (1973), 119-135.

56. G. S. Salagean, *Subclasses of univalent functions*, Complex Analysis-Fifth Romanian Finish Seminar, Bucharest, **1** (1983), 362-372.

57. A. Schild and H. Silverman, Convolution of univalent functions with negative coefficients, *Ann. Univ. Mariae Curie-Sklodowska Sect.*, A **29** (1975), 99-107.

58. H. Silverman, Harmonic univalent functions with negative coefficients, *J. Math. Anal. Appl.*, **220** (1998), 283-289.

59. H. Silverman and E. M. Silvia, Subclasses of harmonic univalent functions, *New Zealand J. Math.*, **28** (1999), 275-284.

60. R. Singh and S. Singh, Convolution properties of a class of starlike functions, *Proc. Amer. Math. Soc.*, **106**(1) (1989), 145-152.

61. H. M. Srivastava and S. Owa, An application of the fractional derivative, *Math. Japon.*, **29** (1984), 383-389.

62. B. A. Uralegaddi and C. Somanatha, *Certain classes of univalent functions*, in Current topics in analytic function theory, 371-374, World Sci. Publishing, River Edge, NJ.

63. S. Yalcin, A new class of Salagean-type harmonic univalent functions, *Appl. Math. Lett.*, **18** (2005), 191-198.