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BERNSTEIN TYPE INEQUALITIES FOR RATIONAL FUNCTIONS

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In this paper, we consider a more general class of rational functions $r(s(z))$ of degree mn, where $s(z)$ is a polynomial of degree m and prove some sharp results concerning to Bernstein type inequalities for rational functions.

Key words : Rational function; polynomials; inequalities; poles; zeros.

1. INTRODUCTION

Let P_n denote the space of complex polynomials of degree at most n and $T := \{z : |z| = 1\}$. we denote by D_- the region inside T and by D_+ the region outside T. If $P \in P_n$, then concerning the estimate of $|P'(z)|$ on the unit circle T, we have the following well known result which relates the norm of a polynomial to that of its derivative due to Bernstein [9].

$$\max_{z\in T} |P'(z)| \leq n \max_{z\in T} |P(z)|. \tag{1.1}$$

The inequality (1.1) is sharp and equality holds for polynomials having all zeros at the origin.

The inequality (1.1) was improved by Malik [6]. In fact he proved:

If $P \in P_n$ and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then

$$\max_{z\in T} |P'(z)| + \max_{z\in T} |Q'(z)| \leq n \max_{z\in T} |P(z)|. \tag{1.2}$$

If we consider the class of polynomials $P \in P_n$ having no zero in D_-, then the bounds in inequality (1.1) can be considerably improved. In fact, Erdös conjectured and later Lax [4] verified that if $P(z)$ does not vanish in D_-, then (1.1) can be replaced by

$$\max_{z\in T} |P'(z)| \leq \frac{n}{2} \max_{z\in T} |P(z)|. \tag{1.3}$$

Turán [10] reversed the hypothesis of the result proved by Erdös-Lax and showed that if $P \in P_n$ and $P(z) \neq 0$ in D_+, then

$$\max_{z\in T}|P'(z)| \geq \frac{n}{2}\max_{z\in T}|P(z)|. \tag{1.4}$$

In 1988, Mohapatra, O'Hara and Rodrigues [7] proved that, if $z_1, z_2, ..., z_{2n}$ are any $2n$ equally spaced points on T listed in order, say $z_k = ue^{\frac{k\pi i}{n}}$, where $u \in T$ and $k = 1, 2, ..., 2n$, then for $P \in P_n$

$$\max_{z\in T}|P'(z)| \leq \frac{n}{2}[\max_{k\ odd}|P(z_k)| + \max_{k\ even}|P(z_k)|]. \tag{1.5}$$

2. Rational Functions

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be n given points in D_+. Consider the following space of rational functions with prescribed poles and with finite limit at infinity.

$$R_n = \left\{\frac{p(z)}{w(z)} : p \in P_n\right\},$$

where

$$w(z) = \prod_{j=1}^{n}(z - \alpha_j).$$

The inequalities of Bernstein and Erdös-lax have been extended to the rational functions ([2], [5]) by replacing the polynomial $p(z)$ by a rational function $r(z)$ and z^n by Blaschke product $\mathcal{B}(z)$ defined by

$$\mathcal{B}(z) = \frac{w^*(z)}{w(z)} = \frac{z^n\overline{w(\frac{1}{\bar{z}})}}{w(z)} = \prod_{j=1}^{n}\frac{1-\bar{\alpha}_j z}{z-\alpha_j}.$$

Besides other things they proved, for any $r \in R_n$

$$|r'(z)| \leq |\mathcal{B}'(z)|||r||. \tag{2.1}$$

Furthermore, the inequality (2.1) is sharp and the equality holds if $r(z) = \alpha\mathcal{B}(z)$ with $\alpha \in T$. If we assume $r \in R_n$ does not vanish in D_-, then for $z \in T$, the inequality (2.1) can be strengthened to

$$|r'(z)| \leq \frac{1}{2}|\mathcal{B}'(z)|||r||. \tag{2.2}$$

The inequality is sharp and equality holds if $r(z) = \alpha\mathcal{B}(z) + \beta$ with $\alpha,\ \beta \in T$. Also, if $r(z)$ does not vanish in D_+, then

$$|r'(z)| \geq \frac{1}{2}|\mathcal{B}'(z)|||r||. \tag{2.3}$$

In this paper we consider a more general class of rational functions $r(s(z))$, defined by

$$(ros)(z) = r(s(z)) = \frac{p(s(z))}{w(s(z))},$$

where $s(z)$ is a polynomial of degree m and $r(z)$ is a rational function of degree n, so that $r(s(z)) \in R_{mn}$, and

$$w(s(z)) = \prod_{j=1}^{mn}(z - a_j).$$

Hence, Balachke product is given by

$$B(z) = \frac{w^*(s(z))}{w(s(z))} = \frac{\overline{w(s(\frac{1}{\bar{z}}))}}{w(s(z))} = \prod_{j=1}^{mn}\left(\frac{1 - \bar{a}_j z}{z - a_j}\right).$$

Thereby prove the following results which in turn generalizes the above inequalities.

3. Main Results

From now on, we shall always assume that all poles $a_1, a_2, ..., a_{mn}$ of $r(s(z))$ lie in D_+. For the case when all poles are in D_-, we can obtain analogous results with suitable modification.

***Theorem* 1** — *If $r(s(z)) \in R_{mn}$ and $z \in T$, then*

$$B'(z)r(s(z)) - s'(z)r'(s(z))[B(z) - \lambda] = \frac{B(z)}{z}\sum_{k=1}^{mn} c_k r(s(t_k))\left|\frac{B(z) - \lambda}{z - t_k}\right|^2, \tag{3.1}$$

where $c_k = c_k(\lambda)$ is defined for $k = 1, 2, 3, ..., mn$ by

$$c_k^{-1} = \sum_{j=1}^{mn}\frac{|a_j|^2 - 1}{|t_k - a_j|^2}. \tag{3.2}$$

Furthermore, for $z \in T$

$$\frac{zB'(z)}{B(z)} = \sum_{k=1}^{mn} c_k\left|\frac{B(z) - \lambda}{z - t_k}\right|^2, \tag{3.3}$$

where $t_k,\ k = 1, 2, 3, ..., mn$ are defined as in Lemma 1 (to be mentioned later).

Corollary 1 — Let c_k and t_k (for $k = 1, 2, 3, ..., mn$) be defined as in Theorem 1. If $u_1, u_2, ..., u_{mn}$ are the roots of $B(z) = -\lambda$ and d_k is defined as c_k with t_k replaced by u_k, for $k = 1, 2, ..., mn$. If

$$\min_{z \in T}|s(z)| = m' \tag{3.4}$$

and all zeros of $s(z)$ lie in $T \cup D_-$, then for $z \in T$

$$|r'(s(z))| \leq \frac{1}{2mm'}|B'(z)|\left\{\max_{1\leq k\leq mn}|r(s(t_k))| + \max_{1\leq k\leq mn}|r(s(u_k))|\right\}. \tag{3.5}$$

The inequality is sharp and equality holds for $r(s(z)) = uB(z)$ with $u \in T$, where $s(z) = z^m$.

Corollary 1 immediately yields the following generalization of inequality (2.1).

Corollary 2 — If $r(s(z)) \in R_{mn}$ and all zeros of $s(z)$ lie in $T \cup D_-$, then

$$|r'(s(z))| \leq \frac{1}{mm'}|B'(z)|||r(s)||, \tag{3.6}$$

where m' is defined by equation (3.4) and $||r(s)|| = \max_{z\in T} |r(s(z))|$.

The inequality is sharp in the sense that equality is obtained when $r(s(z)) = uB(z)$ with $u \in T$, where $s(z) = z^m$.

As a generalization of inequality (1.2), we prove:

***Theorem* 2** — *If $r(s(z)) \in R_{mn}$ and all zeros of $s(z)$ lie in $T \cup D_-$ then for $z \in T$,*

$$|r^{*\prime}(s(z))| + |r'(s(z))| \leq \frac{|B'(z)|}{mm'}||r(s)||, \tag{3.7}$$

where $r^*(s(z)) = B(z)\overline{r(s(\frac{1}{\bar{z}}))}$.

Also equality holds for $r(s(z)) = uB(z)$ with $u \in T$, where $s(z) = z^m$.

We next present the following generalization of inequality (2.2).

***Theorem* 3** — *Let $r(s(z)) \in R_{mn}$ be such that $r(s(z)) \neq 0$ in D_- and all zeros of $s(z)$ lie in $T \cup D_-$. If*

$$\min_{z\in T} |s(z)| = m',$$

then for $z \in T$, we have

$$|r'(s(z))| \leq \frac{1}{2mm'}|B'(z)|||r(s)||. \tag{3.8}$$

The inequality is sharp and equality holds for the rational functions of the form $r(s(z)) = \alpha B(z) + \beta$ with $\alpha,\ \beta \in T$ where $s(z) = z^m$.

***Theorem* 4** — *Let $r(s(z)) \in R_{mn}$ and $r(s(z)) \neq 0$ in D^+. If*

$$\max_{z\in T} |s(z)| = M', \tag{3.9}$$

then for $z \in T$, we have

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| - m(n - n')\right\}|r(s(z))|, \tag{3.10}$$

where mn' and mn are respectively number of zeros and poles of $r(s(z))$.

The inequality is sharp and equality holds for rational functions of the form $r(s(z)) = \alpha B(z) + \beta$ *with* $\alpha,\ \beta \in T\ where\ s(z) = z^m$.

If $r(s(z))$ has exactly mn zeros then $n = n'$ and we get the following generalization of inequality (2.3).

Corollary 3 — Let $r(s(z)) \in R_{mn}$ and $r(s(z)) \neq 0$ in D^+. If

$$\max_{z \in T} |s(z)| = M'$$

and $r(s(z))$ has exactly mn zeros, then for $z \in T$, we have

$$|r'(s(z))| \geq \frac{1}{2mM'} |B'(z)| r(s(z))|. \tag{3.11}$$

The inequality is sharp and equality holds for rational functions of the form $r(s(z)) = \alpha B(z) + \beta$ with $\alpha,\ \beta \in T\ where\ s(z) = z^m$.

4. Lemmas

For the proofs of these Theorems we need the following lemmas.

The first two lemmas are due to Li, Mohapatra and Rodrigues [5].

Lemma 1 — Suppose $\lambda \in T$. Then the equation $B(z) = \lambda$ has exactly mn simple roots, say $t_1, t_2, ..., t_{mn}$ and all lie on the unit circle T. Moreover

$$\frac{t_k B'(t_k)}{\lambda} = \sum_{j=1}^{mn} \frac{|a_j|^2 - 1}{|t_k - a_j|^2} \ \ for\ k = 1, 2, 3, ..., mn. \tag{4.1}$$

Lemma 2 — If $|u| = |v| = 1$, then

$$(u - v)^2 = -uv|u - v|^2. \tag{4.2}$$

Next lemma is due to Aziz and Dawood [1].

Lemma 3 — If $p(z)$ is a polynomial of degree n, having all zeros in $T \cup D_{-}$, then

$$\min_{z \in T} |p'(z)| \geq n \min_{z \in T} |p(z)|. \tag{4.3}$$

The inequality is sharp and equality holds for polynomials having all zeros at the origin.

Lemma 4 — If $z \in T$, then

$$\frac{zB'(z)}{B(z)} = \sum_{j=1}^{mn} \frac{|a_j|^2 - 1}{|z - a_j|^2} = |B'(z)|. \tag{4.4}$$

PROOF : We have

$$B(z)=\frac{w^*(s(z))}{w(s(z))}=\prod_{j=1}^{mn}\frac{1-\bar{a}_jz}{z-a_j}.$$

This gives

$$\frac{zB'(z)}{B(z)}=\sum_{j=1}^{mn}\left\{\frac{-z\bar{a}_j}{1-\bar{a}_jz}-\frac{z}{z-a_j}\right\}.$$

Hence for $z\in T$, we have

$$\frac{zB'(z)}{B(z)}=\sum_{j=1}^{mn}\frac{|a_j|^2-1}{|z-a_j|^2}.$$

Since $|a_j|>1\ \forall\ 1\le j\le mn$, it follows from above that $\frac{zB'(z)}{B(z)}$ is real and positive. Therefore for $z\in T$, we have

$$\frac{zB'(z)}{B(z)}=\left|\frac{zB'(z)}{B(z)}\right|=|B'(z)|.$$

This completes the proof of Lemma 4.

Lemma 5 — Let $r(s(z))\in R_{mn}$. If all zeros of $r(s(z))$ lie in $T\cup D^+$, then for $z\in T$ and $r(s(z))\neq 0$

$$Re\left(\frac{z(r(s(z)))'}{r(s(z))}\right)\le\frac{1}{2}|B'(z)|. \tag{4.5}$$

PROOF : If $p(z)$ has n' zeros and $s(z)$ has m zeros, then $p(s(z))$ has mn' zeros. Let $b_1,b_2,...,b_{mn'}$ be the zeros of $p(s(z))$, $mn'\le mn$. Now

$$r(s(z))=\frac{p(s(z))}{w(s(z))}.$$

This gives

$$z\frac{(r(s(z)))'}{r(s(z))}=\sum_{j=1}^{mn'}\frac{z}{z-b_j}-\sum_{j=1}^{mn}\frac{z}{z-a_j}. \tag{4.6}$$

Since all zeros of $p(s(z))$ lie in $T\cup D_+$, therefore for $z\in T$ with $z\neq b_k$, we have

$$\left|\frac{z}{z-b_j}\right|\le\left|\frac{z}{z-b_j}-1\right| \ for\ j=1,2,3,...,mn'. \tag{4.7}$$

Using the fact that $Re(z)\le\frac{1}{2}$ if and only if $|z|\le|z-1|$, we get from inequality (4.7)

$$Re\left(\frac{z}{z-b_j}\right)\le\frac{1}{2}\ \ for\ \ j=1,2,...,mn'.$$

Hence from equation (4.6), we have

$$Re\left(z\frac{(r(s(z)))'}{r(s(z))}\right) \leq \sum_{j=1}^{mn'}\frac{1}{2} - \sum_{j=1}^{mn} Re\left(\frac{z}{z-a_j}\right)$$
$$\leq \sum_{j=1}^{mn} Re\left(\frac{1}{2} - \frac{z}{z-a_j}\right)$$
$$= \sum_{j=1}^{mn} \frac{|a_j|^2-1}{2|z-a_j|^2}.$$

This with the help of equation (4.4) gives

$$Re\left(z\frac{(r(s(z)))'}{r(s(z))}\right) \leq \frac{1}{2}|B'(z)|.$$

This completes the proof of Lemma 5.

Lemma 6 — Let $r(s(z)) \in R_{mn}$. If all zeros of $r(s(z))$ lie in $T \cup D^-$, then for $z \in T$ and $r(s(z)) \neq 0$, we have

$$Re\left(\frac{z(r(s(z)))'}{r(s(z))}\right) \geq \frac{1}{2}\left\{|B'(z)| - m(n-n')|\right\}, \tag{4.8}$$

where mn' and mn are respectively the number of zeros and poles of $r(s(z))$.

PROOF : Suppose all the zeros of $r(s(z))$ lie in $T \cup D_-$ and $z \in T$ with $z \neq b_j \;\forall\; 1 \leq j \leq mn'$.

Then as in lemma 5, we obtain

$$Re\left(\frac{z}{z-b_j}\right) \geq \frac{1}{2} \text{ for } j = 1, 2, ..., mn'.$$

Using equation (4.6), we get

$$Re\left(z\frac{(r(s(z)))'}{r(s(z))}\right) \geq \sum_{j=1}^{mn'}\frac{1}{2} - \sum_{j=1}^{mn} Re\left(\frac{z}{z-a_j}\right)$$
$$= \sum_{j=1}^{mn} Re\left(\frac{1}{2} - \frac{z}{z-a_j}\right) - \frac{(mn-mn')}{2}$$
$$= \sum_{j=1}^{mn} \frac{|a_j|^2-1}{2|z-a_j|^2} - \frac{m}{2}(n-n')$$
$$= \frac{1}{2}\left\{|B'(z)| - m(n-n')\right\}.$$

This completes the proof of Lemma 6.

5. Proofs of Theorems

Proof of Theorem 1 : Let $q(z) = w^*(s(z)) - \lambda(w(s(z)))$. Since the solution of $B(z) = \lambda$ is same as polynomial equation $w^*(s(z)) - \lambda w(s(z)) = 0$ which has degree exactly mn, it follows that it has exactly mn roots counting multiplicities. If these roots are denoted by $t_1, t_2, ..., t_{mn}$, then

$$q(z) = w(s(z))[B(z) - \lambda] = K \prod_{k=1}^{mn} (z - t_k).$$

For $r(s(z)) = \frac{p(s(z))}{w(s(z))} \in R_{mn}$, let $p(s(z)) = \mu z^{mn} + ...$, then

$$p(s(z)) - \frac{\mu}{K} q(z) \in P_{mn-1}.$$

The numbers $t_1, t_2, ..., t_{mn}$ are distinct, so by Lagrange interpolation formula we obtain

$$p(s(z)) - \frac{\mu}{K} q(z) = \sum_{k=1}^{mn} \frac{p(s(t_k))q(z)}{q'(t_k)(z - t_k)}.$$

Dividing both sides by $q(z)$ and differentiating, we get

$$\left(\frac{p(s(z))}{q(z)}\right)' = \sum_{k=1}^{mn} \frac{q'(t_k)(z - t_k)(p(s(t_k)))' - p(s(t_k))q'(t_k)}{(q'(t_k)(z - t_k))^2}$$

$$= - \sum_{k=1}^{mn} \frac{p(s(t_k))}{q'(t_k)(z - t_k)^2}. \tag{5.1}$$

Next recall that, $q(z) = w(s(z))[B(z) - \lambda]$ and $p(s(z)) = w(s(z))r(s(z))$.

Hence $q'(t_k) = w(s(t_k))B'(t_k)$ and $p(s(t_k)) = w(s(t_k))r(s(t_k))$. Moreover, since t_k are the zeros of $B(z) = \lambda$. Therefore $q(t_k) = 0$. Using these in equation (5.1), we get

$$\left(\frac{r(s(z))}{B(z) - \lambda}\right)' = - \sum_{k=1}^{mn} \frac{r(s(t_k))}{B'(t_k)(z - t_k)^2}. \tag{5.2}$$

Which implies

$$\frac{[B(z) - \lambda](r(s(z)))' - r(s(z))B'(z)}{[B(z) - \lambda]^2} = - \sum_{k=1}^{mn} \frac{r(s(t_k))}{B'(t_k)(z - t_k)^2}.$$

Multiplying both sides by $-[B(z) - \lambda]^2$, we get

$$r(s(z))B'(z) - s'(z)r'(s(z))[B(z) - \lambda] = \sum_{k=1}^{mn} \frac{r(s(t_k))[B(z) - \lambda]^2}{B'(t_k)(z - t_k)^2}. \tag{5.3}$$

For $z \in T$, $|B(z)| = 1$ and $|\lambda| = 1$. Therefore by virtue of lemma 2, we obtain $[B(z) - \lambda]^2 = -B(z)\lambda|B(z) - \lambda|^2$. Similarly, $(z - t_k)^2 = -zt_k|z - t_k|^2$. Hence it follows from equation (5.3)

$$B'(z)r(s(z)) - s'(z)r'(s(z))[B(z) - \lambda] = \frac{B(z)}{z} \sum_{k=1}^{mn} \frac{\lambda r(s(t_k))}{t_k B'(t_k)} \left| \frac{B(z) - \lambda}{z - t_k} \right|^2. \tag{5.4}$$

Using Lemma (1) and definition of c_k, we get

$$B'(z)r(s(z)) - s'(z)r'(s(z))[B(z) - \lambda] = \frac{B(z)}{z} \sum_{k=1}^{mn} c_k r(s(t_k)) \left| \frac{B(z) - \lambda}{z - t_k} \right|^2. \tag{5.5}$$

Which completely proves Theorem 1.

PROOF OF COROLLARY 1 : Applying Theorem 1 after replacing λ by $-\lambda$, we get

$$B'(z)r(s(z)) - s'(z)r'(s(z))[B(z) + \lambda] = \frac{B(z)}{z} \sum_{k=1}^{mn} d_k r(s(u_k)) \left| \frac{B(z) + \lambda}{z - u_k} \right|^2. \tag{5.6}$$

Subtract (5.5) and (5.6), we get

$$\frac{zs'(z)r'(s(z))[B(z) + \lambda - B(z) + \lambda]}{B(z)} = \sum_{k=1}^{mn} c_k r(s(t_k)) \left| \frac{B(z) - \lambda}{z - t_k} \right|^2 - \sum_{k=1}^{mn} d_k r(s(u_k)) \left| \frac{B(z) + \lambda}{z - u_k} \right|^2.$$

Hence for $z \in T$, we have

$$\begin{aligned} |2s'(z)r'(s(z))| &\le \sum_{k=1}^{mn} |c_k||r(s(t_k))| \left| \frac{B(z) - \lambda}{z - t_k} \right|^2 + \sum_{k=1}^{mn} |d_k||r(s(u_k))| \left| \frac{B(z) + \lambda}{z - u_k} \right|^2 \\ &\le \max_k |r(s(t_k))| \sum_{k=1}^{mn} |c_k| \left| \frac{B(z) - \lambda}{z - t_k} \right|^2 + \max_k |r(s(u_k))| \sum_{k=1}^{mn} |d_k| \left| \frac{B(z) + \lambda}{z - u_k} \right|^2. \end{aligned}$$

Since both c_k and d_k are positive by definition, therefore using (3.3) we get

$$|2s'(z)r'(s(z))| \le \frac{zB'(z)}{B(z)} \left\{ \max_{1 \le k \le n} |r(s(t_k))| + \max_{1 \le k \le n} |r(s(u_k))| \right\}.$$

Finally, by virtue of lemma 3 and lemma 4, we obtain

$$|r'(s(z))| \le \frac{1}{2mm'} |B'(z)| \left\{ \max_{1 \le k \le n} |r(s(t_k))| + \max_{1 \le k \le n} |r(s(u_k))| \right\},$$

where m' is defined by equation (3.4).

This proves Corollary 1 completely.

PROOF OF THEOREM 2 : From Theorem 1, we have for $z \in T$

$$|B'(z)r(s(z)) - s'(z)r'(s(z))[B(z) - \lambda]| = \left| \frac{B(z)}{z} \sum_{k=1}^{mn} c_k r(s(t_k)) \left| \frac{B(z) - \lambda}{z - t_k} \right|^2 \right|$$

$$\leq \left|\frac{B(z)}{z}\right|\left|\sum_{k=1}^{mn}|r(s(t_k))|\right|c_k\left|\frac{B(z)-\lambda}{z-t_k}\right|^2\Bigg|$$

$$\leq \max_{z\in T}|r(s(z))|\left|\frac{zB'(z)}{B(z)}\right|$$

$$= |B'(z)|||r(s)||.$$

Since right hand side is independent of λ, therefore we can suitably choose λ such that

$$|B'(z)r(s(z)) - s'(z)r'(s(z))B(z)| + |s'(z)r'(s(z))| \leq |B'(z)|||r(s)||. \tag{5.7}$$

Next recall that

$$r^*(s(z)) = B(z)\overline{r(s(\frac{1}{\bar{z}}))}.$$

So that

$$(r^*(s(z)))' = B'(z)\overline{r(s(\frac{1}{\bar{z}}))} - \frac{1}{z^2}B(z)\overline{r'(s(\frac{1}{\bar{z}})).s'(\frac{1}{\bar{z}})}.$$

Which implies

$$\left|(r^*(s(z)))'\right| = \left|B'(z)\overline{r(s(\frac{1}{\bar{z}}))} - \frac{1}{z^2}B(z)\overline{r'(s(\frac{1}{\bar{z}}))s'(\frac{1}{\bar{z}})}\right|.$$

Hence for $z \in T$, we have

$$\left|(r^*(s(z)))'\right| = \left|z\frac{B'(z)}{B(z)}\overline{r(s(z))} - \overline{zr'(s(z)).s'(z)}\right|.$$

Using the fact that $\frac{zB'(z)}{B(z)}$ is real, we get

$$\left|(r^*(s(z)))'\right| = \left|z\frac{B'(z)}{B(z)}r(s(z)) - zr'(s(z)).s'(z)\right|$$

$$= |B'(z)r(s(z)) - r'(s(z))s'(z)B(z)|. \tag{5.8}$$

Hence we have from inequality (5.7)

$$|r^{*\prime}(s(z))s'(z)| + |r'(s(z))s'(z)| \leq |B'(z)|||r(s)||.$$

Which gives the desired result by use of Lemma 3.

PROOF OF THEOREM 3 : From equation (4.4), we have

$$z\frac{B'(z)}{B(z)} = |B'(z)| > 0$$

Hence for $z \in T$, with $r(s(z)) \neq 0$, we have from (5.8)

$$\begin{aligned}\left|(r^*(s(z)))'\right| &= |B'(z)r(s(z)) - r'(s(z))s'(z)B(z)| \\ &= \left|z\frac{B'(z)}{B(z)}r(s(z)) - zr'(s(z))s'(z)\right| \\ &= ||B'(z)|r(s(z)) - zr'(s(z))s'(z)| \\ &= \left|\frac{zr'(s(z))s'(z)}{|B'(z)|r(s(z))} - 1\right||B'(z)r(s(z))|. \qquad (5.9)\end{aligned}$$

From lemma 5, we have

$$Re\left(\frac{zr'(s(z))s'(z)}{|B'(z)|r(s(z))}\right) \leq \frac{1}{2}.$$

Which further implies

$$\left|\frac{zr'(s(z))s'(z)}{|B'(z)|r(s(z))}\right| \leq \left|\frac{zr'(s(z))s'(z)}{|B'(z)|r(s(z))} - 1\right|.$$

Using in (5.9), we get

$$|(r^*(s(z)))'| \geq \left|\frac{zr'(s(z))s'(z)}{|B'(z)|r(s(z))}\right||B'(z)r(s(z))|.$$

Which further implies

$$|(r^*(s(z)))'| \geq |(r(s(z)))'|.$$

Hence Theorem 2 yields

$$|(r(s(z)))'| \leq \frac{1}{2}|B'(z)|||r(s)||.$$

Lemma 3 is thereby allowing us to write

$$|r'(s(z))| \leq \frac{1}{2mm'}|B'(z)|||r(s(z))||.$$

This proves the theorem for $r(s(z)) \neq 0$. Since the above inequality is trivially true for $r(s(z)) = 0$.

Therefore we conclude that the theorem is true for all $z \in T$.

PROOF OF THEOREM 4 : Let $r(s(z)) \neq 0$. Since $z \in T$, therefore we have by use of lemma 6

$$\left|\frac{(r(s(z)))'}{r(s(z))}\right| \geq Re\left(z\frac{(r(s(z)))'}{r(s(z))}\right) \geq \frac{1}{2}\left\{|B'(z)| - m(n-n')\right\}.$$

Which yields by the use of inequality (1.1)

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| - m(n-n')\right\}|r(s(z))| \qquad (5.10).$$

This proves Theorem 4 for $r(s(z)) \neq 0$. Since inequality (5.10) is trivially true for $r(s(z)) = 0$. Therefore Theorem 4 holds for all $z \in T$.

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