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NOTES ON NONPURE PIECEWISE-KOSZUL ALGEBRAS

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Let $A = \bigoplus_{i\geq 0} A_i$ be a Noetherian standard graded algebra with finite global dimension, $gr_0(A)$ and $\mathcal{NPK}(A)$ the categories of finitely 0-generated graded A-modules and nonpure piecewise-Koszul modules. It is proved that $gr_0(A) = \mathcal{NPK}(A)$, which implies that all the Noetherian standard graded Artin-Schlter regular algebras are nonpure piecewise-Koszul. Moreover, the H-Galois graded extension of nonpure piecewise-Koszul algebras is discussed, where H is a finite dimensional semisimple and cosemisimple Hopf algebra.

Key words : Koszul algebras and modules; (nonpure) piecewise-Koszul algebras and modules.

1. Introduction

Koszul algebras were first introduced by Priddy in 1970 (ref. [17]), which are a class of quadratic algebras with beautiful homological properties and have many applications in different branches of mathematics. For the long history and full properties of Koszul algebras, we refer to (ref. [2], [8] and [16], etc.). Motivated by the cubic Artin-Schelter regular algebras (ref. [1]), Berger defined the notion of nonquadratic Koszul algebra in 2001 (ref. [3]). Inspired by the quiver theory in representation theory, Green *et al.*, generalized such class of algebras to the nonlocal case (ref. [7]) in 2004 and called them d-Koszul algebras, where $d \geq 2$ is an integer. In order to unify the notions of Koszul algebra and d-Koszul algebra, Lü *et al.*, introduced the notion of piecewise-Koszul algebra (ref. [13]) in 2007, which is determined by a pair of parameters (p, d), one shows its periodicity, and the other one is related to the degree of jump. It agrees with the classical Koszul algebra when the period equals

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to the jumping degree, and goes back to the d-Koszul algebra when the period $p = 2$. In particular, piecewise-Koszul algebras provide a negative answer to a question of Green and Marcos' related to δ-Koszul algebras (ref. [6], [11]).

It should be noted that Koszul objects, d-Koszul objects and piecewise-Koszul objects all admit a "pure" resolution. A natural question is that can we break this restrict? As a try, Bian *et al.*, introduced the notions of generalized d-Koszul algebra and generalized d-Koszul module in 2010 ([4]), which answers a question rased in [7]. Motivated by [4], the author of the present paper defined the so-called nonpure piecewise-Koszul objects recently (ref. [10]) and studied some basic properties of such objects.

This present paper is a continuous work of [10]. More precisely, let $A = \bigoplus_{i\geq 0} A_i$ be a Noetherian standard graded algebra with finite global dimension. We prove that such an algebra is nonpure piecewise-Koszul, and all the finitely 0-generated graded A-modules are nonpure piecewise-Koszul modules. The following is one of the main results:

***Theorem* 1.1** — *Let $A = \bigoplus_{i\geq 0} A_i$ be a Noetherian standard graded algebra with finite global dimension, $gr_0(A)$ and $\mathcal{NPK}(A)$ denote the categories of 0-generated graded A-modules and nonpure piecewise-Koszul modules. Then*

(1) A is a nonpure piecewise-Koszul algebra;

(2) $gr_0(A) = \mathcal{NPK}(A)$.

As a direct corollary of Theorem 1.1, we can get

Corollary 1.2 — All the Noetherian standard graded Artin-Schelter regular algebras are nonpure piecewise-Koszul.

Another purpose of this paper is to study the H-Galois graded extension of nonpure piecewise-Koszul algebras, where H is a finite dimensional semisimple and cosemisimple Hopf algebra. In particular, we prove

***Theorem* 1.3** — *Let H be a finite dimensional semisimple and cosemisimple Hopf algebra, $A = \bigoplus_{n\geq 0} A_n$ be a graded right H-module algebra such that A_i is finite dimensional for all $i \geq 0$, and let $B = A^{coH}$, the coinvariant subalgebra of A. Suppose that A/B is an H-Galois graded extension. Then B is a nonpure piecewise-Koszul algebra if and only if A is a nonpure piecewise-Koszul algebra.*

Throughout, $\Bbbk$ denotes a fixed field, $\mathbb{N} = \{0, 1, 2, \cdots\}$ denotes the set of natural numbers. A positively graded $\Bbbk$-algebra $A = \bigoplus_{i\geq 0} A_i$ is called standard provided that

- $A_0 = \Bbbk \times \cdots \times \Bbbk$, a finite product of $\Bbbk$;
- $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$;
- $\dim_{\Bbbk} A_i < \infty$ for all $i \geq 0$.

It is easy to see that the graded Jacobson radical of such a graded algebra A is $\bigoplus_{i\geq 1} A_i$, which will be denoted by J.

Now we will recall the so-called nonpure piecewise-Koszul objects to end this section.

Definition 1.4 (ref. [10]) — Let A be a standard graded algebra and $M = \bigoplus_{i\geq 0} M_i$ a finitely generated graded A-module. Let

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

be a graded projective resolution of M. Then M is called a *nonpure piecewise-Koszul module* if for all $n \geq 0$, Q_n is generated in degrees in $\Delta_p^d(n)$, where d and p are fixed integers with $d \geq p \geq 2$, $k \in \mathbb{N}$ and the set function Δ_p^d is defined as:

$$\Delta_p^d(n) = \begin{cases} \{kd\}, & n = pk, \\ \{kd+1,\ kd+2,\ \cdots,\ kd+d-p+1\}, & n = pk+1, \\ \cdots & \cdots \\ \{kd+p-2,\ kd+p-1,\ \cdots,\ kd+d-2\}, & n = pk+p-2, \\ \{kd+p-1,\ kd+p,\ \cdots,\ kd+d-1\}, & n = pk+p-1. \end{cases}$$

In particular, the standard graded algebra A will be called a *nonpure piecewise-Koszul algebra* if the trivial A-module A_0 is a nonpure piecewise-Koszul module.

2. Proof of Theorem 1.1

We begin with several lemmas.

Lemma 2.1 — Let A be a standard graded algebra and X a graded A_0-module. Then $A \otimes_{A_0} X$ is a graded projective A-module.

Proof : It is easy to see that $A \otimes_{A_0} X$ is a graded A-module under the grading

$$(A \otimes_{A_0} X)_i = \sum_{s+t=i} A_s \otimes_{A_0} X_t.$$

Now apply the functor $\mathrm{Hom}_A(A \otimes_{A_0} X, -)$ to the epimorphism $M \longrightarrow N \longrightarrow 0$ of graded

A-modules, by the adjoint theorem, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(A\otimes_{A_0} X, M) & \xrightarrow{\alpha} & \mathrm{Hom}_A(A\otimes_{A_0} X, N) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_{A_0}(X, \mathrm{Hom}_A(A, M)) & \xrightarrow{\beta} & \mathrm{Hom}_{A_0}(X, \mathrm{Hom}_A(A, N)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_{A_0}(X, M) & \xrightarrow{\gamma} & \mathrm{Hom}_{A_0}(X, N). \end{array}$$

Note that A_0 is semisimple, then γ is an epimorphism, which implies that both β and α are epimorphisms. Thus $A \otimes_{A_0} X$ is a graded projective A-module. □

Lemma 2.2 — Let A be a standard graded algebra, M a finitely generated graded A-module. Suppose that there are finitely graded projective modules P_1 and P_2 such that $P_1 \to M \to 0$ and $P_2 \to M \to 0$ are graded projective covers. Then $P_1 \cong P_2$ as graded A-modules.

Proof : See (Proposition 2.4(6), [8]). □

Lemma 2.3 — Let A be a standard graded algebra, M a bounded below graded A-module. Then there is a bounded below graded projective A-module P, such that $P \to M \to 0$ is a graded projective cover. Moreover, the degrees of the minimal homogeneous generators of P and M are the same.

Proof : Note that M is a bounded below graded A-module, thus one can assume that $M = M_0 \oplus M_1 \oplus M_2 \oplus \cdots$. Recall that A is a standard graded algebra, then M can be rewritten as

$$M = M_0 \oplus (A_1M_0 + G_1) \oplus (A_2M_0 + A_1G_1 + G_2) \oplus \cdots ,$$

where $M_0,\ G_1,\ G_2,\ \cdots$ are the A_0-spaces spanned by the minimal homogeneous generators of M with $G_i \subseteq M_i,\ (i \geq 1)$.

Let $P := A \otimes_{A_0} (M_0 \oplus G_1 \oplus G_2 \oplus \cdots) = A \otimes_{A_0} M/JM$. By Lemma 2.1, P is a graded projective A-module and the degrees of its minimal homogeneous generators are the same as those of M.

Define

$$f : P \to M,$$

$$\sum a_i \otimes g_j \mapsto \sum a_i g_j.$$

Obviously, f is an epimorphism of graded A-modules. Observe also that

$$\begin{aligned}\ker f &= \left\{\sum a_i \otimes g_j \in A \otimes_{A_0} M/JM | \sum a_i g_j = 0\right\} \\ &\subseteq \left\{\sum a_i \otimes g_j \in A \otimes_{A_0} M/JM | a_i \in J\right\} \\ &= J \otimes_{A_0} M/JM \\ &= JP,\end{aligned}$$

which implies that $P \to M \to 0$ is a graded projective cover. □

Lemma 2.4 — Let A be a standard graded algebra and M a bounded below graded A-module. Then M admits a graded projective resolution

$$\cdots \longrightarrow Q_n \xrightarrow{d_n} \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} M \longrightarrow 0$$

such that for all $i \geq 0$, $\ker d_i \subseteq JP_i$. Moreover, such a resolution is unique up to isomorphisms.

Proof : By Lemma 2.3, there exists a bounded below graded projective A-module Q_0, such that $Q_0 \xrightarrow{d_0} M \longrightarrow 0$ is an epimorphism with $\ker d_0 \subseteq JP_0$. Now consider the graded A-module $\ker d_0$, which is also bounded below since Q_0 is bounded below. By Lemma 2.3 again, there exists a bounded below graded A-module Q_1, such that $Q_1 \xrightarrow{\overline{d_1}} \ker d_0 \longrightarrow 0$ is a graded projective cover. Let d_1 be the composition of $Q_1 \twoheadrightarrow \ker d_0 \hookrightarrow Q_0$, then we have the following exact sequence

$$Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} M \longrightarrow 0,$$

such that for $i = 0, 1$, $\ker d_i \subseteq JP_i$. Repeat the above procedure, we obtain that M has a graded projective resolution

$$\cdots \longrightarrow Q_n \xrightarrow{d_n} \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} M \longrightarrow 0$$

such that for all $i \geq 0$, $\ker d_i \subseteq JP_i$. We complete the proof since the uniqueness of such a resolution is immediate from Lemma 2.2. □

Lemma 2.5 — Let $A = \bigoplus_{i \geq 0} A_i$ be a Noetherian standard graded algebra with finite global dimension and M be any finitely 0-generated graded A-module. Then

(1) M is a nonpure piecewise-Koszul module;

(2) A is a nonpure piecewise-Koszul algebra.

PROOF : It suffices to prove (1).

By the hypothesis, M is a bounded below graded A-module. By Lemma 2.4, M has a unique (up to isomorphism) graded projective resolution

$$0 \longrightarrow Q_n \xrightarrow{d_n} \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} M \longrightarrow 0,$$

such that for all $i \geq 0$, $\ker d_i \subseteq JP_i$. By Lemma 2.3, Q_0 is generated in degree 0 since M is generated in degree 0. Note that M is finitely generated and A is Noetherian, then Q_0 and $\ker d_0$ are both finitely generated. Without loss of generality, we assume that $\ker d_0$ is generated in degrees $\{l_0^1,\ l_1^1,\ l_2^1,\ \cdots,\ l_{s_1}^1\}$, where $l_0^1 < l_1^1 < l_2^1 < \cdots < l_{s_1}^1$. Note that

$$Q_1 \xrightarrow{d_1} \ker d_0 \longrightarrow 0$$

is a graded projective cover of $\ker d_0$, then Q_1 is finitely generated. By Lemma 2.3, Q_1 can be generated in degrees $\{l_0^1,\ l_1^1,\ l_2^1,\ \cdots,\ l_{s_1}^1\}$, where $l_0^1 < l_1^1 < l_2^1 < \cdots < l_{s_1}^1$. Similarly, $\ker d_1$ is finitely generated and suppose that $\ker d_1$ is generated in degrees $\{l_0^2,\ l_1^2,\ l_2^2,\ \cdots,\ l_{s_2}^2\}$, where $l_0^2 < l_1^2 < l_2^2 < \cdots < l_{s_2}^2$. Recall that $\ker d_1 \subseteq JQ_1$, then $l_0^2 \geq l_0^1 + 1$. Because

$$Q_2 \xrightarrow{d_1} \ker d_1 \longrightarrow 0$$

is a graded projective cover of $\ker d_1$, then Q_2 is finitely generated. By Lemma 2.3, Q_2 can be generated in degrees $\{l_0^2,\ l_1^2,\ l_2^2,\ \cdots,\ l_{s_2}^2\}$, where $l_0^2 < l_1^2 < l_2^2 < \cdots < l_{s_2}^2$.

Repeat the above procedure, one can get that Q_i is generated in degrees $\{l_0^i,\ l_1^i,\ l_2^i,\ \cdots,\ l_{s_i}^i\}$, where $l_0^i < l_1^i < l_2^i < \cdots < l_{s_i}^i$, $l_0^i \geq l_0^{i-1} + 1$ $(i = 3,\ 4,\ \cdots,\ n)$.

Now put $p = n + 1$, $d = \max\{n + l_{s_1}^1,\ n + l_{s_2}^2 - 1,\ \cdots,\ l_{s_n}^n + 1\}$. Then for all $0 \leq i \leq n$, Q_i is generated in degrees in $\Delta_p^d(i)$. Therefore, M is a nonpure piecewise-Koszul module. □

Note that any nonpure piecewise-Koszul module is generated in degree 0 because $\Delta_p^d(0) = 0$ and Lemma 2.3, thus Theorem 1.1 has been proved by combining Lemma 2.5. Corollary 1.2 is an immediate consequence of Lemma 2.5 since Artin-Schelter regular algebras are of finite global dimension.

3. PROOF OF THEOREM 1.3

We begin with

Lemma 3.1 (ref. [10]) — Let A be a standard graded algebra and $\mathrm{Ext}_A^*(A_0, A_0)$ be its Yoneda algebra. Then A is a nonpure piecewise-Koszul algebra if and only if for all $i \geq 0$, $\mathrm{Ext}_A^i(A_0, A_0) = \mathrm{Ext}_A^i(A_0, A_0)_{-j} = \mathrm{Ext}_{Gr(A)}^i(A_0, A_0[j]) = 0$ unless $j \in \Delta_p^d(i)$.

Lemma 3.2 (ref. [9]) — Let H be a finite dimensional semisimple and cosemisimple Hopf algebra and A/B be an H-Galois graded extension. If $A = \bigoplus_{i\geq 0} A_i$ is a standard graded algebra, then A_0/B_0 is an H-Galois extension.

Lemma 3.3 (ref. [9]) — Let H be a finite dimensional semisimple and cosemisimple Hopf algebra, $A = \bigoplus_{n\geq 0} A_n$ be a graded right H-module algebra and $B = A^{coH}$, the coinvariant subalgebra of A. Suppose that A/B is an H-Galois graded extension. Then we have an isomorphism of bigraded algebras

$$\mathrm{Ext}^*_B(A_0, A_0) \cong \mathrm{Ext}^*_A(A_0, A_0)\#H,$$

where the bigradeding of $\mathrm{Ext}^*_A(A_0, A_0)\#H$ is induced from that of $\mathrm{Ext}^*_A(A_0, A_0)$.

Now we are ready to prove Theorem 1.3.

PROOF : By the assumption, B_0 is a finite dimensional semisimple algebra. By Lemma 3.2, A_0/B_0 is an H-Galois extension since A/B is an H-Galois graded extension. Now note that $A_0\#H$ and B_0, A_0 and $(A_0\#H)\#H^*$ are both Morita equivalent, and H is a finite dimensional semisimple and cosemisimple Hopf algebra, we have that B_0 is semisimple if and only if A_0 is semisimple. Further, as a right B_0-module, $A_0 = B_0 \oplus S$ for some finite dimensional B_0-module S.

($\Rightarrow$) By assumption, B is a nonpure piecewise-Koszul algebra, by Lemma 3.1, which is equivalent to that $\mathrm{Ext}^i_B(B_0, B_0) = \mathrm{Ext}^i_B(B_0, B_0)_{-j}$ and $j \in \Delta^d_p(i)$ for all $i \geq 0$. Note that S is a direct summand of a finite sum of B_0, which implies that $\mathrm{Ext}^i_B(B_0, S) = \mathrm{Ext}^i_B(B_0, S)_{-j}$, $\mathrm{Ext}^i_B(S, B_0) = \mathrm{Ext}^i_B(S, B_0)_{-j}$ and $\mathrm{Ext}^i_B(S, S) = \mathrm{Ext}^i_B(S, S)_{-j}$ for all $i \geq 0$, where $j \in \Delta^d_p(i)$. Also observe that we have the following isomorphism

$$\mathrm{Ext}^i_B(A_0, A_0) = \mathrm{Ext}^i_B(B_0, B_0) \oplus \mathrm{Ext}^i_B(B_0, S) \oplus \mathrm{Ext}^i_B(S, B_0) \oplus \mathrm{Ext}^i_B(S, S)$$

for all $i \geq 0$, which implies that $\mathrm{Ext}^i_B(A_0, A_0) = \mathrm{Ext}^i_B(A_0, A_0)_{-j}$ for all $i \geq 0$, where $j \in \Delta^d_p(i)$. By Lemma 3.3, we have

$$\mathrm{Ext}^i_A(A_0, A_0)\#H = (\mathrm{Ext}^i_A(A_0, A_0)\#H)_{-j}$$

for all $i \geq 0$, where $j \in \Delta^d_p(i)$. By the definition of the bigrading of $\mathrm{Ext}^i_A(A_0, A_0)\#H$, we obtain that $\mathrm{Ext}^i_A(A_0, A_0) = \mathrm{Ext}^i_A(A_0, A_0)_{-j}$ for all $i \geq 0$, where $j \in \Delta^d_p(i)$. By Lemma 3.1, we get that A is a nonpure piecewise-Koszul algebra.

($\Leftarrow$) Suppose that A is a nonpure piecewise-Koszul algebra, by Lemma 3.1, which is equivalent to

$$\mathrm{Ext}_A^i(A_0, A_0) = \mathrm{Ext}_A^i(A_0, A_0)_{-j}$$

for all $i \geq 0$, where $j \in \Delta_p^d(i)$. By Lemma 3.3, we have $\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(A_0, A_0)_{-j}$ for all $i \geq 0$, where $j \in \Delta_p^d(i)$. Note that $A_0 = B_0 \oplus S$ and S is a direct summand of a finite sum of B_0, which imply that

$$\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(B_0, B_0) \oplus \mathrm{Ext}_B^i(B_0, S) \oplus \mathrm{Ext}_B^i(S, B_0) \oplus \mathrm{Ext}_B^i(S, S)$$

for all $i \geq 0$, which of course implies that $\mathrm{Ext}_B^i(B_0, B_0) = \mathrm{Ext}_B^i(B_0, B_0)_{-j}$ for all $i \geq 0$, where $j \in \Delta_p^d(i)$. By Lemma 3.1, we get that B is a nonpure piecewise-Koszul algebra. $\square$

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