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FUGLEDE'S THEOREM

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Dedicated to Professor Kalyan B. Sinha on the occasion of his 70th birthday.

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In this short note, we give an elementary (set-theoretic) proof of Fuglede's theorem that the commutant of a normal operator is *-closed.

Key words : Normal operator; commutant; spectral projections; monotone class theorem.

1. INTRODUCTION

Throughout this note, 'operator' will mean a bounded linear operator (denoted by symbols like A, N, P, T) on a separable Hilbert space $\mathcal{H}$.

***Theorem* 1.1** (Fuglede [1]) — *If an operator T commutes with a normal operator N, then it necessarily commutes with N^*.*

This short note provides a proof of this fact which is 'natural' in the sense that it exactly imitated the most natural proof in case $\mathcal{H}$ is finite dimensional: in that case, the spectral theorem guarantees that N has an expression of the form $N = \sum_{i=1}^{k} \lambda_i P_i$ where P_i is the projection onto $ker(N - \lambda_i)$; since P_i is a polynomial in N, it follows that T commutes with each P_i and hence with $N^* = \sum_i \bar{\lambda}_i P_i$.

We shall use the notation of the functional calculus $f \mapsto f(N)$ for bounded measurable functions defined on $\mathbb{C}$; thus $1_E(N)$ will denote the projection onto the spectral subspace of N corresponding to any E in $\mathcal{B}_{\mathbb{C}}$:= the σ-algebra of Borel sets in $\mathbb{C}$. We shall prove that T commutes with every $1_E(N)$, to conclude that T should commute with $f(N)$ for any bounded measurable function f on $\mathbb{C}$. For $f(z) = 1_{sp(N)}(z)\bar{z}$, this yields the desired result.

Write $\mathcal{M}(E) = ran(1_E(N))$ for the spectral subspace corresponding to an $E \in \mathcal{B}_{\mathbb{C}}$. As $\mathcal{M}(E)^\perp = \mathcal{M}(E')$ (with the 'prime' denoting complement), it will suffice for us to show that T leaves each $\mathcal{M}(E)$ invariant. To this end, let us write

$$\mathcal{F} = \{E \in \mathcal{B}_{\mathbb{C}} : T(\mathcal{M}(E)) \subset \mathcal{M}(E)\}. \tag{1}$$

We proceed through a sequence of simple steps to the desired conclusion. We start with the key observation which is stated and proved for self-adjoint N in [2].

First some notation: write $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$, simply $\mathbb{D} = D(0,1)$ and $\bar{\mathbb{D}}$ for the closed ball $\{z : |z| \leq 1\}$.

Lemma 1.2 — The following conditions on a vector $x \in \mathcal{H}$ are equivalent:

1. $x \in \mathcal{M}(\bar{\mathbb{D}})$.
2. $\|N^n x\| \leq \|x\| \; \forall n \in \mathbb{N}$.
3. $\sup\{\|N^n x\| : n \in \mathbb{N}\} < \infty$.

In particular, $\bar{\mathbb{D}} \in \mathcal{F}$.

PROOF : The implications $1. \Rightarrow 2. \Rightarrow 3.$ are obvious. As for $3. \Rightarrow 1.$, it is enough to see that $x_m := 1_{\{z:|z|\geq 1+\frac{1}{m}\}}(N)x = 0 \; \forall m \in \mathbb{N}$ since $x - \lim_m x_m \in \mathcal{M}(\mathbb{D})$; but this follows from

$$\|N^n x\| \geq \|1_{\{z:|z|\geq 1+\frac{1}{m}\}}(N) N^n x\| = \|N^n x_m\| \geq (1 + \frac{1}{m})^n \|x_m\| \; \forall n \in \mathbb{N} \; .$$

In particular, if $x \in \mathcal{M}(\bar{\mathbb{D}})$ it follows from

$$\|N^n T x\| = \|T N^n x\| \leq \|T\| \|N^n x\|$$

and 3. above that also $Tx \in \mathcal{M}(\bar{\mathbb{D}})$ so that indeed $\bar{\mathbb{D}} \in \mathcal{F}$. □

Corollary 1.3 — $D(z, r) \in \mathcal{F} \; \forall z \in \mathbb{C}, r > 0$.

PROOF : This follows on applying Lemma 1.2 to $\left(\frac{N-z}{r}\right)$. □

***Theorem* 1.4** — *With the foregoing notation, we have*:

1. $\mathcal{F}$ *is closed under countable monotone limits, and is thus a 'monotone class'.*
2. $\mathcal{F}$ *contains all (open or closed) discs.*
3. $\mathcal{F}$ *contains all (open or closed) half-planes.*

4. *$\mathcal{F}$ is closed under countable intersections and countable disjoint unions.*

5. *$\mathcal{F} = \mathcal{B}_{\mathbb{C}}$.*

PROOF :

1. If $E_n \in \mathcal{F}\ \forall n$ and if either $E_n \uparrow E$ or $E_n \downarrow E$, then $1_{E_n}(N) \stackrel{SOT}{\to} 1_E(N)$ so that either $\mathcal{M}(E) = \overline{(\cup \mathcal{M}(E_n))}$ or $\mathcal{M}(E) = \cap \mathcal{M}(E_n)$ whence also $E \in \mathcal{F}$.

2. The assertion regarding closed discs is Corollary 1.3, and the assertion regarding open discs now follows from (1) above.

3. For example, if $a, b \in \mathbb{R}$, then $R_a = \{z \in \mathbb{C} : \Re z > a\} = \cup_{n=1}^{\infty} \{z \in \mathbb{C} | z - (a+n)| < n\} \in \mathcal{F}$ and hence, by (1) above, also $L_b = \{z \in \mathbb{C} : \Re z \leq b\} = - \cap_{n=1}^{\infty} R_{-b-\frac{1}{n}} \in \mathcal{F}$. Similarly, if $c, d \in \mathbb{R}$, we also have $U_c := \{z \in \mathbb{C} : \Im z > c\}, D_d := \{z \in \mathbb{C} : \Im z \leq d\} \in \mathcal{F}$.

4. This is an immediate consequence of the definitions.

5. It follows from items 3. and 4. above that $\mathcal{F}$ contains $(a, b] \times (c, d] = R_a \cap L_b \cap U_c \cap D_d$ and the collection $\mathcal{A}$ of all finite disjoint unions of such rectangles. Since $\mathcal{A}$ is an algebra of sets which generates $\mathcal{B}_{\mathbb{C}}$ as a σ-algebra, and since $\mathcal{F}$ is a monotone class containing $\mathcal{A}$, the desired conclusion is a consequence of the monotone class theorem.

We conclude with the cute observation - see [2] - that by applying Fuglede's theorem to the block operator-matrices $\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$ and $\begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$ we obtain Putnam's generalisation [3]: if N_i is a normal operator on $\mathcal{H}_i, i = 1, 2$, and if $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ satisfies $TN_1 = N_2 T$, then necessarily $TN_1^* = N_2^* T$.

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