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FACTORIZATION PROPERTY OF CONVOLUTIONS OF WHITE NOISE OPERATORS

Un Cig Ji*[,1], Young Yi Kim** and Yoon Jung Park***

*Department of Mathematics, Research Institute of Mathematical Finance, Chungbuk National University, Cheongju 361-763, Korea

**Research Institute for Natural Science, Hanyang University, Seoul 133-791, Korea

***Department of Mathematics, Chungbuk National University, Cheongju 361-763, Korea

e-mails: uncigji@chungbuk.ac.kr, kimyy@chungbuk.ac.kr, yjpark@chungbuk.ac.kr

Dedicated to Professor Kalyan B. Sinha on the occasion of his 70th birthday.

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We first study a general type of convolutions, parameterized by operators, on white noise functionals and study a relation between the convolution and the generalized Fourier-Mehler transform. Secondly, we extend the convolution of generalized white noise functionals to a convolution of white noise operators. Then, as factorization properties of the convolutions, we study a relation between the convolution and quantum generalized Fourier-Mehler transform.

Key words : White noise operator; convolution; quantum generalized Fourier-Mehler transform.

1. Introduction

Convolution products play important roles in many areas, infinite dimensional (harmonic) analysis, signal analysis and quantum mechanics, etc. and there are many kinds of convolution

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product. In particular, in signal analysis, convolution of a system function and an input function means output of signal. Hence as which system is chosen, convolution product has to be changed, e.g., standard convolution product, time-varying convolution [12] and time-scaling convolution for the linear time invariant system, linear time-varying system, time-scaling system. Also, the affine convolution [1] which is a main tool in the time frequency analysis [19] is originated in quantum mechanics.

The white noise theory initiated by Hida in [5] has been extensively developed with wide applications to stochastic calculus, mathematical finance, mathematical physics, etc. In the white noise theory, convolutions have been studied by many authors. In [14], Kuo studied the convolution product of white noise functionals and in [9], the authors studied Yeh convolution of white noise functionals to develop a method to give a rigorous meaning of the Yeh convolution of generalized white noise functionals. Recently, Obata and Ouerdiane in [17] studied a different type convolution of white noise functionals.

On the other hands, in [6], the authors introduced a general convolution of functions on abstract Wiener space, motivated by the study in [20], and studied a relation between the convolution and generalized Fourier-Gauss transform. The convolution product in [6] has been studied in [10] in the white noise setting, focused on the consistency property of the convolution product and, as an application, studied the possibility of extension of the convolution product to generalized white noise functionals. The convolutions in [17], [12] and [1, 19] are special cases of the convoution in [10]. Also, the authors in [8] extended the Kuo's convolution in [14] to the convolution product of white noise operators.

The purpose of this paper is to study a new type of convolution, parameterized by operators, of white noise functionals as a unification of the Kuo's convolution and the convolution studied in [10], and to extend the convolution to a convolution of white noise operators. Then as factorization properties of the convolutions, we study several relations between the convolution of white noise operators and the quantum generalized Fourier-Mehler transform.

This paper is organized as follows: In section 2, we recall basic notions of white noise functionals and white noise operators (see [16, 14]), which are necessary for our study. In Section 3, we introduce a new type of convolution of generalized white noise functionals and then we study a relation between the convolution and generalized Fourier-Mehler transform. In Section 4, we extend the convolution of white noise functionals to a convolution of white noise operators and then, as factorization properties of the convolutions, we study several relations between the convolution and quantum generalized Fourier-Mehler transform.

2. Preliminary

Let H be a (complex) separable Hilbert space with the norm $|\cdot|_0$ and let A be a positive self-adjoint operator on H such that $\left\|A^{-1}\right\|_{\mathrm{OP}} < 1$ and $\left\|A^{-1}\right\|_{\mathrm{HS}}^2 < \infty$. Then by the standard construction from H and A (see [14, 16]), we have a Gelfand triple:

$$E \subset H \subset E^*, \tag{2.1}$$

where E^* is the strong dual space of E. In fact, the topology of E is defined by the Hilbertian norms $\{|\cdot|_p \equiv |A^p \cdot|_0\}_{p\geq 0}$ and then E becomes a countable Hilbert nuclear space. The canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$.

The (Boson) Fock space $\Gamma(H)$ over H is defined by

$$\Gamma(H) = \left\{ \phi = (f_n)_{n=0}^\infty \,\middle|\, f_n \in H^{\widehat{\otimes} n}, \quad \|\phi\|^2 = \sum_{n=0}^\infty n!\, |f_n|_0^2 < \infty \right\},$$

where $H^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of H and $H^{\widehat{\otimes} 0} = \mathbb{C}$. Let $\Gamma(A)$ be the second quantization of the operator A defined by

$$\Gamma(A)\phi = (A^{\otimes n} f_n)_{n=0}^\infty, \qquad \phi = (f_n)_{n=0}^\infty \in \Gamma(H),$$

and then $\Gamma(A)$ is a positive selfadjoint operator in $\Gamma(H)$ with $\left\|\Gamma(A)^{-1}\right\|_{\mathrm{OP}} < 1$ and $\left\|\Gamma(A)^{-1}\right\|_{\mathrm{HS}}^2 < \infty$. By the standard construction from $\Gamma(H)$ and $\Gamma(A)$, we have a Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*.$$

It is known that for each $\Phi \in (E)^*$ there exists a unique sequence $(F_n)_{n=0}^\infty$ with $F_n \in (E^{\otimes n})^*_{\mathrm{sym}}$ such that

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^\infty n!\, \langle F_n, f_n \rangle, \qquad \phi = (f_n)_{n=0}^\infty \in (E),$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the canonical bilinear form on $(E)^* \times (E)$. In this case, Φ is denoted by $\Phi = (F_n)_{n=0}^\infty$.

Remark 2.1 : Let μ be the Gaussian measure on $E^*_{\mathbb{R}}$ of which the characteristic function is given by

$$\int_{E^*_{\mathbb{R}}} \exp\{i\langle x, \xi\rangle\} d\mu(x) = \exp\left\{ -\frac{1}{2}|\xi|_0^2 \right\}, \qquad \xi \in E_{\mathbb{R}}.$$

Then $(E_{\mathbb{R}}^*, \mu)$ is called the *white noise space* or *Gaussian space*. We denote by $L^2(E_{\mathbb{R}}^*, \mu)$ the complex Hilbert space of all μ-square integrable functions on $E_{\mathbb{R}}^*$. Then the celebrated Wiener-Itô-Segal isomorphism gives the unitary isomorphism between $L^2(E_{\mathbb{R}}^*, \mu)$ and $\Gamma(H)$ which is uniquely determined by the correspondence:

$$L^2(E_{\mathbb{R}}^*, \mu) \ni \phi_\xi(x) = e^{\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle} \quad \longleftrightarrow \quad \phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right) \in \Gamma(H).$$

For each $\xi \in H$, ϕ_ξ is called an *exponential vector* (or *coherent state*) associated with ξ. We note that $\{\phi_\xi \mid \xi \in E\}$ spans a dense subspace of (E) (see [14, 16]) and

$$\phi_\xi(x)\phi_\eta(x) = \phi_{\xi+\eta}(x)e^{\langle \xi, \eta\rangle}. \tag{2.2}$$

Throughout this paper, we denote $\mathcal{L}(X, Y)$ the space of all continuous linear operators from a topological vector space X into another topological vector space Y equipped with the topology of uniform convergence on bounded subsets of X.

From the fact that $\{\phi_\xi \mid \xi \in E\}$ spans a dense subspace of (E), an element $\Phi \in (E)^*$ is uniquely determined by the *S-transform* $S\Phi$ of Φ which is a function defined on E by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi\rangle\rangle, \qquad \xi \in E.$$

More generally, since $\{\phi_{\xi_1} \otimes \cdots \otimes \phi_{\xi_m} : \xi_i \in E, i = 1, 2, \cdots, m\}$ spans a dense subspace of $(E)^{\otimes m}$, every $\Xi \in \mathcal{L}((E)^{\otimes m}, ((E)^{\otimes n})^*)$ is uniquely determined by the function $G : E^{m+n} \to \mathbb{C}$ defined by

$$G(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n) = \langle\langle \Xi(\phi_{\xi_1} \otimes \cdots \otimes \phi_{\xi_m}), \phi_{\eta_1} \otimes \cdots \otimes \phi_{\eta_n}\rangle\rangle \tag{2.3}$$

for $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n \in E$, as the following analytic characterization theorem. In particular, for $\Xi \in \mathcal{L}((E), (E)^*)$, the form given as in (2.3) is denoted by $\widehat{\Xi}$ and called the *symbol* of Ξ.

***Theorem* 2.2** [11] — *A Gâteaux-entire function $G : E^{\otimes m+n} \to \mathbb{C}$ is expressed in the form (2.3) with $\Xi \in \mathcal{L}((E)^{\otimes m}, ((E)^{\otimes n})^*)$ if and only if there exist constant numbers $C \geq 0$, $K \geq 0$ and $p \geq 0$ such that*

$$|G(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n)|^2 \leq Ce^{K(\sum_{j=1}^m |\xi_j|_p^2 + \sum_{k=1}^n |\eta_k|_p^2)}$$

for any $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n \in E$. Moreover $\Xi \in \mathcal{L}((E)^{\otimes m}, (E)^{\otimes n})$ if and only if for any $\epsilon > 0$ and $p \geq 0$ there exist constant numbers $C \geq 0$ and $q \geq 0$ such that

$$|G(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n)|^2 \leq Ce^{\epsilon(\sum_{j=1}^m |\xi_j|_{p+q}^2 + \sum_{k=1}^n |\eta_k|_{-p}^2)}$$

for any $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n \in E$.

Example 2.3 : For each given $K \in \mathcal{L}(E, E^*)$, the function $G : E \times E \to \mathbb{C}$ defined by

$$G(\xi, \eta) = \langle K\xi, \xi\rangle \, e^{\langle \xi, \eta\rangle}, \quad \xi, \eta \in E$$

is a Gâteaux-entire function and satisfies the following growth condition: for any $\epsilon > 0$ and $p \geq 0$ there exist constant numbers $C \geq 0$ and $q \geq 0$ such that

$$|G(\xi, \eta)|^2 \leq Ce^{\epsilon(|\xi|^2_{p+q} + |\eta|^2_{-p})}.$$

Therefore, by Theorem 2.2, there exists a unique operator $\Delta_G(K) \in \mathcal{L}((E), (E))$, which is called the *general Gross Laplacian* (see [2]), such that

$$\widehat{\Delta_G(K)}(\xi, \eta) = G(\xi, \eta) = \langle K\xi, \xi\rangle \, e^{\langle \xi, \eta\rangle}, \quad \xi, \eta \in E.$$

For each given $\Xi \in \mathcal{L}((E), (E)^*)$, by the kernel theorem, there exists a unique $\Phi_\Xi \in (E)^*$ such that

$$\langle\langle \Xi\phi, \varphi\rangle\rangle = \langle\langle \Phi_\Xi, \varphi \otimes \phi\rangle\rangle, \qquad \phi, \varphi \in (E),$$

and the converse is also true. Therefore, there exists a map

$$\mathcal{K} : \mathcal{L}((E), (E)^*) \ni \Xi \to \Phi_\Xi \in (E)^* \otimes (E)^*, \tag{2.4}$$

which gives the following topological isomorphisms:

$$\begin{aligned} \mathcal{L}((E), (E)^*) &\cong (E)^* \otimes (E)^*, \\ \mathcal{L}((E), (E)) &\cong (E) \otimes (E)^*, \\ \mathcal{L}((E)^*, (E)) &\cong (E) \otimes (E). \end{aligned} \tag{2.5}$$

3. Convolutions of Generalized White Noise Functionals

By applying Theorem 2.2, the *Wick product* $\Phi_1 \diamond \Phi_2$ of $\Phi_1, \Phi_2 \in (E)^*$ is characterized by

$$S(\Phi_1 \diamond \Phi_2)(\xi) = S(\Phi_1)(\xi) S(\Phi_2)(\xi), \quad \xi \in E. \tag{3.1}$$

Let $U, V \in \mathcal{L}(E^*, E^*)$. For each given $\mathbf{F} \in (E)^*$, we define a new type of convolution $*_{U,V;\mathbf{F}}$ of generalized white noise functionals by

$$\Phi *_{U,V;\mathbf{F}} \Psi = \Gamma(U)\Phi \diamond \Gamma(V)\Psi \diamond \mathbf{F}, \qquad \Phi, \Psi \in (E)^*. \tag{3.2}$$

Example 3.1 :

(1) For any $\Phi, \Psi \in (E)^*$ and the vacuum vector ϕ_0, we have

$$\Phi *_{U,V;\phi_0} \Psi = \Gamma(U)\Phi \diamond \Gamma(V)\Psi \diamond \phi_0 = \Gamma(U)\Phi \diamond \Gamma(V)\Psi = \Phi *^{l}_{U^*,V^*} \Psi,$$

where U^* is the adjoint operator of the given linear operator U with respect to the canonical bilinear form $\langle \cdot, \cdot \rangle$ and the convolution $*^{l}_{U^*,V^*}$ has been studied in [10].

(2) For any $\Phi, \Psi \in (E)^*$, we have

$$\Phi *_{U,V;\mathbf{F}} \Psi = \Gamma(U)\Phi \diamond \Gamma(V)\Psi \diamond \mathbf{F} = \Gamma(U)\Phi *_{\mathbf{F}} \Gamma(V)\Psi,$$

where the convolution $*_{\mathbf{F}}$ has been studied in [8].

(3) For any $\Phi, \Psi \in (E)^*$, we have

$$\Phi *_{\frac{1}{\sqrt{2}}I,-\frac{1}{\sqrt{2}}I;\phi_0} \Psi = \Gamma\left(\frac{1}{\sqrt{2}}I\right)\Phi \diamond \Gamma\left(-\frac{1}{\sqrt{2}}I\right)\Psi.$$

The convolution $*_{\frac{1}{\sqrt{2}}I,-\frac{1}{\sqrt{2}}I;\phi_0}$ has been studied in [20] (see also [9, 10]) and it is called the *Yeh convolution*.

(4) For any $\Phi, \Psi \in (E)^*$, we have

$$\Phi *_{I,I;\phi_0} \Psi = \Phi \diamond \Psi$$

of which the convolution has been studied by Obata and Ouerdiane in [17].

For any $A \in \mathcal{L}(E, E^*)$ and $B \in \mathcal{L}(E, E)$, by Theorem 2.2, there exists a unique operator $\mathcal{G}_{A,B} \in \mathcal{L}((E), (E))$ such that

$$\mathcal{G}_{A,B}\phi_\xi = \phi_{B\xi} \exp\left\{\frac{\langle A\xi, \xi\rangle}{2}\right\}, \quad \xi \in E. \tag{3.3}$$

From (3.3), we can easily see that

$$\mathcal{G}_{A,B} = \Gamma(B)e^{\frac{1}{2}\Delta_G(A)}. \tag{3.4}$$

The operator $\mathcal{G}_{A,B} \in \mathcal{L}((E), (E))$ is called the *generalized Fourier-Gauss transform*. The adjoint operator $\mathcal{G}^*_{A,B}$ of the generalized Fourier-Gauss transform $\mathcal{G}_{A,B}$ is denoted by $\mathcal{F}_{A,B}$ and called the *generalized Fourier-Mehler transform*. Then by the duality of (3.4), we have

$$\mathcal{F}_{A,B} = e^{\frac{1}{2}\Delta^*_G(A)}\Gamma(B^*).$$

For more detailed study, we refer to [2].

The following theorem gives a relation between the convolution $*_{U,V;\mathbf{F}}$ and the generalized Fourier-Mehler transform.

***Theorem* 3.2** — *Let $A_i \in \mathcal{L}(E, E^*)$, $B_i \in \mathcal{L}(E, E)$, $i = 1, 2, 3$, $U, U', V, V' \in \mathcal{L}(E^*, E^*)$ and $\mathbf{F}, \mathbf{G} \in (E)^*$. Then for any $\Phi, \Psi \in (E)^*$,*

$$\mathcal{F}_{A_1,B_1}(\Phi *_{U,V;\mathbf{F}} \Psi) = (\mathcal{F}_{A_2,B_2}\Phi) *_{U',V';\mathbf{G}} (\mathcal{F}_{A_3,B_3}\Psi) \tag{3.5}$$

if and only if $B_1^* U = U' B_2^*$, $B_1^* V = V' B_3^*$ and

$$\mathbf{G} = e^{\frac{1}{2}\Delta_G^*(A_1 - U'A_2U'^* - V'A_3V'^*)}\Gamma(B_1^*)\mathbf{F}. \tag{3.6}$$

Proof : For any $\xi \in E$, we obtain that

$$\begin{aligned} S(\mathcal{F}_{A_1,B_1}(\Phi *_{U,V;\mathbf{F}} \Psi))(\xi) &= \left\langle\!\left\langle \Gamma(U)\Phi \diamond \Gamma(V)\Psi \diamond \mathbf{F},\, \Gamma(B_1)e^{\frac{1}{2}\Delta_G(A_1)}\phi_\xi \right\rangle\!\right\rangle \\ &= \langle\langle \Gamma(B_1^*U)\Phi,\, \phi_\xi\rangle\rangle \, \langle\langle \Gamma(B_1^*V)\Psi,\, \phi_\xi\rangle\rangle \\ &\quad \times \left\langle\!\left\langle e^{\frac{1}{2}\Delta_G^*(A_1)}\Gamma(B_1^*)\mathbf{F},\, \phi_\xi \right\rangle\!\right\rangle \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & S\left(\mathcal{F}_{A_2,B_2}\Phi *_{U',V';\mathbf{G}} \mathcal{F}_{A_3,B_3}\Psi\right)(\xi) \\ &= \left\langle\!\left\langle \Phi,\, \mathcal{G}_{A_2,B_2}\Gamma(U'^*)\phi_\xi \right\rangle\!\right\rangle \left\langle\!\left\langle \Psi,\, \mathcal{G}_{A_3,B_3}\Gamma(V'^*)\phi_\xi \right\rangle\!\right\rangle \langle\langle \mathbf{G},\, \phi_\xi\rangle\rangle \\ &= \left\langle\!\left\langle \Gamma(U'B_2^*)\Phi,\, \phi_\xi \right\rangle\!\right\rangle \left\langle\!\left\langle \Gamma(V'B_3^*)\Psi,\, \phi_\xi \right\rangle\!\right\rangle \langle\langle \mathbf{G},\, \phi_\xi\rangle\rangle \, e^{\frac{1}{2}\left\langle (U'A_2U'^* + V'A_3V'^*)\xi,\, \xi\right\rangle}. \end{aligned} \tag{3.8}$$

Therefore, by comparing (3.7) and (3.8) we see that (3.5) holds if and only if $B_1^*U = U'^*B_2$, $B_1^*V = V'^*B_3$ and for any $\xi \in E$,

$$\begin{aligned} \langle\langle \mathbf{G},\, \phi_\xi\rangle\rangle &= \left\langle\!\left\langle e^{\frac{1}{2}\Delta_G^*(A_1)}\Gamma(B_1^*)\mathbf{F},\, \phi_\xi \right\rangle\!\right\rangle e^{-\frac{1}{2}\left\langle (U'A_2U'^* + V'A_3V'^*)\xi,\, \xi\right\rangle} \\ &= \left\langle\!\left\langle e^{\frac{1}{2}\Delta_G^*(A_1 - U'A_2U'^* - V'A_3V'^*)}\Gamma(B_1^*)\mathbf{F},\, \phi_\xi \right\rangle\!\right\rangle, \end{aligned}$$

which is equivalent to (3.6). ■

Corollary 3.3 — Let $A_i \in \mathcal{L}(E, E^*)$, $B_i \in \mathcal{L}(E, E)$, $i = 1, 2$, $U, U', V, V' \in \mathcal{L}(E^*, E^*)$ and $\mathbf{F}, \mathbf{G} \in (E)^*$. Then for any $\Phi, \Psi \in (E)^*$,

$$\mathcal{F}_{A_1,B_1}(\Phi *_{U,V;\mathbf{F}} \Psi) = (\mathcal{F}_{A_2,B_2}\Phi) *_{U',V';\mathbf{G}} (\mathcal{F}_{A_2,B_2}\Psi)$$

if and only if $B_1^* U = U' B_2^*$, $B_1^* V = V' B_2^*$ and

$$\mathbf{G} = e^{\frac{1}{2}\Delta_G^*(A_1 - U'A_2U'^* - V'A_2V'^*)}\Gamma(B_1^*)\mathbf{F}.$$

PROOF : The proof is immediate from Theorem 3.2. ■

Corollary 3.4 — Let $A_i \in \mathcal{L}(E, E^*)$, $B_i \in \mathcal{L}(E, E)$, $i = 1, 2, 3$, $U, V, \in \mathcal{L}(E^*, E^*)$ and $\mathbf{F} \in (E)^*$. Then for any $\Phi, \Psi \in (E)^*$,

$$\mathcal{F}_{A_1,B_1}(\Phi *_{U,V;\mathbf{F}} \Psi) = (\mathcal{F}_{A_2,B_2}\Phi) *_{U,V;\mathbf{F}} (\mathcal{F}_{A_3,B_3}\Psi)$$

if and only if

$$B_1 = I, \quad U = UB_2^*, V = VB_3^*, \qquad A_1 = UA_2U^* + VA_3V^*.$$

PROOF : The proof is immediate from Theorem 3.2. ■

Corollary 3.5 — Let $A \in \mathcal{L}(E, E^*)$, $B \in \mathcal{L}(E, E)$, $U, U', V, V' \in \mathcal{L}(E^*, E^*)$ and $\mathbf{F}, \mathbf{G} \in (E)^*$. Then for any $\Phi, \Psi \in (E)^*$,

$$\mathcal{F}_{A,B}(\Phi *_{U,V;\mathbf{F}} \Psi) = (\mathcal{F}_{A,B}\Phi) *_{U',V';\mathbf{G}} (\mathcal{F}_{A,B}\Psi)$$

if and only if $B^* U = U' B^*$, $B^* V = V' B^*$ and

$$\mathbf{G} = e^{\frac{1}{2}\Delta_G^*(A - U'AU'^* - V'AV'^*)}\Gamma(B^*)\mathbf{F}.$$

PROOF : The proof is immediate from Theorem 3.2. ■

Example 3.6 : From Theorem 3.2 and the above corollaries, the followings are immediate.

(1) For any $\Phi, \Psi \in (E)^*$, we have

$$\mathcal{F}_{A_1,B_1}(\Phi *_{U,V;\phi_0} \Psi) = (\mathcal{F}_{A_2,B_2}\Phi) *_{U',V';\phi_0} (\mathcal{F}_{A_3,B_3}\Psi)$$

if and only if $B_1^* U = U' B_2^*$, $B_1^* V = V' B_3^*$ and $A_1 = U'A_2U'^* + V'A_3V'^*$.

(2) For any $\Phi, \Psi \in (E)^*$, we have

$$\mathcal{F}_{A_1,B_1}(\Phi *_{I,I;\mathbf{F}} \Psi) = (\mathcal{F}_{A_2,B_2}\Phi) *_{I,I;\mathbf{G}} (\mathcal{F}_{A_3,B_3}\Psi)$$

if and only if $B_1 = B_2 = B_3$ and

$$\mathbf{G} = e^{\frac{1}{2}\Delta_G^*(A_1 - A_2 - A_3)}\Gamma(B_1^*)\mathbf{F}.$$

(3) For any $\Phi, \Psi \in (E)^*$, we have

$$\mathcal{F}_{A_1,B_1}(\Phi *_{\frac{1}{\sqrt{2}}I,-\frac{1}{\sqrt{2}}I;\phi_0} \Psi) = (\mathcal{F}_{A_2,B_2}\Phi) *_{\frac{1}{\sqrt{2}}I,-\frac{1}{\sqrt{2}}I;\phi_0} (\mathcal{F}_{A_3,B_3}\Psi)$$

if and only if $B_1 = B_2 = B_3$ and $2A_1 = A_2 + A_3$.

(4) For any $\Phi, \Psi \in (E)^*$, we have

$$\mathcal{F}_{A_1,B_1}(\Phi *_{I,I;\phi_0} \Psi) = (\mathcal{F}_{A_2,B_2}\Phi) *_{I,I;\phi_0} (\mathcal{F}_{A_3,B_3}\Psi)$$

if and only if $B_1 = B_2 = B_3$ and $A_1 = A_2 + A_3$.

4. Convolutions of White Noise Operators

In this section, we extend the convolution studied in Section 3 to a convolution of white noise operators and study a relation between the convolution and the quantum generalized Fourier-Mehler transform.

By applying Theorem 2.2, for each given $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$, the *Wick product* $\Xi_1 \diamond \Xi_2$ of $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$ is well-defined by

$$\widehat{\Xi_1 \diamond \Xi_2}(\xi, \eta) = \widehat{\Xi_1}(\xi, \eta)\widehat{\Xi_2}(\xi, \eta)e^{-\langle \xi, \eta\rangle}, \quad \xi, \eta \in E, \tag{4.1}$$

see [4].

For the notational convenience, we put

$$\mathfrak{T} = \{\mathcal{U} = (U_1, U_2) : U_1, U_2 \in \mathcal{L}(E^*, E^*)\}.$$

For any $\mathcal{U} = (U_1, U_2) \in \mathfrak{T}$ and $\Xi \in \mathcal{L}((E), (E)^*)$, the symbol of $\Gamma(U_1)\Xi\Gamma(U_2^*)$ is given by

$$(\Gamma(U_1)\Xi\Gamma(U_2^*))\widehat{\ }\ (\xi, \eta) = \left\langle\left\langle \Xi_1\phi_{U_2^*\xi},\, \phi_{U_1^*\eta}\right\rangle\right\rangle,$$

and so for any $p \geq 0$, there exists $C_p \geq 0$ such that

$$\begin{aligned} |\langle\langle \Xi\Gamma(U_2^*)\phi_\xi,\, \Gamma(U_1^*)\phi_\eta\rangle\rangle| &\leq \|\Xi\Gamma(U_2^*)\phi_\xi\|_{-p} \|\Gamma(U_1^*)\phi_\eta\|_p \\ &\leq C_p \left\|\phi_{U_2^*\xi}\right\|_p \left\|\phi_{U_1^*\eta}\right\|_p. \end{aligned}$$

Hence by Theorem 2.2, $\Gamma(U_1)\Xi\Gamma(U_2^*) \in \mathcal{L}((E), (E)^*)$.

Motivated by the convolution of white noise functionals, for each $\mathcal{U} = (U_1, U_2), \mathcal{V} = (V_1, V_2) \in \mathfrak{T}$, and fixed $\Upsilon \in \mathcal{L}((E), (E)^*)$, we define a *convolution* $\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2$ of two white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$ by

$$\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2 = \Gamma(U_1)\Xi_1\Gamma(U_2^*) \diamond \Gamma(V_1)\Xi_2\Gamma(V_2^*) \diamond \Upsilon. \tag{4.2}$$

Remark 4.1 : If $U_i = V_i = I,\ i = 1, 2$ in (4.2), then we have

$$\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2 = \Xi_1 \diamond \Xi_2 \diamond \Upsilon, \qquad \Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*),$$

which implies that the convolution $\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2$ coincides with the convolution $\Xi_1 *_\Upsilon \Xi_2$ studied in [8].

Remark 4.2 : For each $\phi, \psi \in (E)$, by considering (E) as a subspace of $L^2(E^*_{\mathbb{R}}, \mu)$ under the Wiener-Itô-Segal isomorphism, the (pointwise) multiplication $\phi\psi$ of ϕ and ψ is well-defined as an element in (E). Moreover, for each $\phi \in (E)$, the multiplication operator $M_\phi : (E) \ni \psi \mapsto \phi\psi \in (E)$ is a continuous linear operator, i.e., $M_\phi \in \mathcal{L}((E), (E))$ (see [16]). Therefore, for each $\Phi \in (E)^*$ and $\psi \in (E)$, the multiplication $\Phi\psi$ of Φ and ψ are well-defined as an element of $(E)^*$ by

$$\langle\langle \Phi\psi, \phi \rangle\rangle = \langle\langle \Phi, \psi\phi \rangle\rangle, \qquad \phi \in (E).$$

Moreover, for each $\Phi \in (E)$, the multiplication operator

$$M_\Phi : (E) \ni \psi \mapsto \Phi\psi \in (E)^*$$

is a continuous linear operator, i.e., $M_\Phi \in \mathcal{L}((E), (E)^*)$.

***Theorem* 4.3** — *Let $U, V \in \mathcal{L}(E^*, E^*)$ and $\mathbf{F} \in (E)^*$. Then for any $\Phi, \Psi \in (E)^*$, we have*

$$M_{\Phi *_{U,V;\mathbf{F}} \Psi} = M_\Phi *_{\mathcal{U},\mathcal{V};M_{\mathbf{F}} \diamond \Gamma(3I - UU^* - VV^*)} M_\Psi, \tag{4.3}$$

where $\mathcal{U} = (U_1, U_2), \mathcal{V} = (V_1, V_2) \in \mathfrak{T}$ with $U_1 = U_2 = U$, $V_1 = V_2 = V$.

Proof : For each $\xi, \eta \in E$, by applying (3.1) and (4.1), we obtain that

$$\begin{aligned}
\left\langle\left\langle \left(M_{\Phi *_{U,V;\mathbf{F}} \Psi}\right) \phi_\xi,\, \phi_\eta \right\rangle\right\rangle &= \langle\langle \Phi *_{U,V;\mathbf{F}} \Psi,\, \phi_\eta\phi_\xi \rangle\rangle \\
&= \langle\langle \Phi *_{U,V;\mathbf{F}} \Psi,\, \phi_{\xi+\eta} \rangle\rangle\, e^{\langle \xi,\, \eta \rangle} \\
&= \langle\langle \Gamma(U)\Phi \diamond \Gamma(V)\Psi \diamond \mathbf{F},\, \phi_{\xi+\eta} \rangle\rangle\, e^{\langle \xi,\, \eta \rangle} \\
&= \langle\langle \Gamma(U)\Phi,\, \phi_{\xi+\eta} \rangle\rangle\, \langle\langle \Gamma(V)\Psi,\, \phi_{\xi+\eta} \rangle\rangle\, \langle\langle \mathbf{F},\, \phi_{\xi+\eta} \rangle\rangle\, e^{\langle \xi,\, \eta \rangle} \\
&= \langle\langle \Phi,\, \phi_{U^*\eta}\phi_{U^*\xi} \rangle\rangle\, \langle\langle \Psi,\, \phi_{V^*\eta}\phi_{V^*\xi} \rangle\rangle\, \langle\langle \mathbf{F},\, \phi_\eta\phi_\xi \rangle\rangle\, e^{\langle (-UU^* - VV^*)\xi,\, \eta \rangle} \\
&= \langle\langle M_\Phi \phi_{U^*\xi},\, \phi_{U^*\eta} \rangle\rangle\, \langle\langle M_\Psi \phi_{V^*\xi},\, \phi_{V^*\eta} \rangle\rangle\, \langle\langle M_{\mathbf{F}} \phi_\xi,\, \phi_\eta \rangle\rangle\, e^{\langle (-UU^* - VV^*)\xi,\, \eta \rangle} \\
&= \langle\langle \Gamma(U) M_\Phi \Gamma(U^*) \phi_\xi,\, \phi_\eta \rangle\rangle\, \langle\langle \Gamma(V) M_\Psi \Gamma(V^*) \phi_\xi,\, \phi_\eta \rangle\rangle \\
&\qquad \times \langle\langle M_{\mathbf{F}} \phi_\xi,\, \phi_\eta \rangle\rangle\, \langle\langle \Gamma(-UU^* - VV^*) \phi_\xi,\, \phi_\eta \rangle\rangle\,,
\end{aligned}$$

where we used the fact that $\phi_\xi\phi_\eta = \phi_{\xi+\eta} e^{\langle \xi,\, \eta \rangle}$ (see (2.2)), which implies (4.3). ■

Let $A, B \in \mathcal{L}(E, E^*)$ and $C, D \in \mathcal{L}(E, E)$. By applying Theorem 2.2, there exists a unique operator $\widetilde{\mathcal{G}}^Q_{A,B;C,D} \in \mathcal{L}((E) \otimes (E), (E) \otimes (E))$ such that

$$\widetilde{\mathcal{G}}^Q_{A,B;C,D}(\phi_{\xi_1} \otimes \phi_{\xi_2}) = e^{\langle A\xi_1, \xi_1\rangle + \langle B\xi_2, \xi_2\rangle} \phi_{C\xi_1} \otimes \phi_{D\xi_2}, \qquad \xi_1, \xi_2 \in E.$$

For the operator $\widetilde{\mathcal{G}}^Q_{A,B;C,D} \in \mathcal{L}((E) \otimes (E), (E) \otimes (E))$ and the adjoint operator $\widetilde{\mathcal{F}}^Q_{A,B;C,D} \in \mathcal{L}((E)^* \otimes (E)^*, (E)^* \otimes (E)^*)$, we put

$$\begin{aligned}
\mathcal{G}^Q_{A,B;C,D} &= \mathcal{K}^{-1} \widetilde{\mathcal{G}}^Q_{A,B;C,D} \mathcal{K} \in \mathcal{L}(\mathcal{L}((E)^*, (E)), \mathcal{L}((E)^*, (E))), \\
\mathcal{F}^Q_{A,B;C,D} &= \mathcal{K}^{-1} \widetilde{\mathcal{F}}^Q_{A,B;C,D} \mathcal{K} \in \mathcal{L}(\mathcal{L}((E), (E)^*), \mathcal{L}((E), (E)^*)),
\end{aligned}$$

where $\mathcal{K}$ is the topological isomorphism in (2.4). Then $\mathcal{G}^Q_{A,B;C,D}$ and $\mathcal{F}^Q_{A,B;C,D}$ are called the *quantum generalized Fourier-Gauss transform* and *quantum generalized Fourier-Mehler transform*, respectively. Moreover, for each $\Xi \in \mathcal{L}((E), (E)^*)$, we have

$$\mathcal{F}^Q_{A,B;C,D}(\Xi) = \mathcal{F}_{A,C} \Xi \mathcal{G}_{B,D}. \tag{4.4}$$

The quantum generalized Fourier-Gauss transform and quantum generalized Fourier-Mehler transform, parameterized by scalars, have been studied in [7, 8].

***Theorem* 4.4** — *Let $A_i, B_i \in \mathcal{L}(E, E^*)$, $C_i, D_i \in \mathcal{L}(E, E)$, $i = 1, 2, 3$, $\mathcal{U} = (U_1, U_2)$, $\mathcal{U}' = (U_1', U_2'), \mathcal{V} = (V_1, V_2), \mathcal{V}' = (V_1', V_2') \in \mathfrak{T}$ and $\Upsilon, \Upsilon' \in \mathcal{L}((E), (E)^*)$. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,*

$$\mathcal{F}^Q_{A_1,B_1;C_1,D_1}(\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2) = (\mathcal{F}^Q_{A_2,B_2;C_2,D_2}\Xi_1) *_{\mathcal{U}',\mathcal{V}';\Upsilon'} (\mathcal{F}^Q_{A_3,B_3;C_3,D_3}\Xi_2) \tag{4.5}$$

if and only if

$$C_1^* U_1 = U_1' C_2^*, \quad U_2^* D_1 = D_2 U_2'^*, \quad C_1^* V_1 = V_1' C_3^*, \quad V_2^* D_1 = D_3 V_2'^* \tag{4.6}$$

and

$$\Upsilon' = \Gamma(C_1{}^*)\Upsilon\Gamma(D_1) \diamond e^{\Delta^*_G(A)}\Gamma(3I - 2C_1{}^*D_1)e^{\Delta_G(B)} \tag{4.7}$$

with $A = \frac{1}{2}(A_1 - U_1' A_2 U_1'^* - V_1' A_3 V_1'^*)$ and $B = \frac{1}{2}(B_1 - U_2' B_2 U_2'^* - V_2' B_3 V_2'^*)$.

Proof : For any $\xi, \eta \in E$, we obtain that

$$\begin{aligned}
&\left\langle\!\left\langle \mathcal{F}^Q_{A_1,B_1;C_1,D_1}(\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2)\phi_\xi,\, \phi_\eta \right\rangle\!\right\rangle \\
&\quad = \langle\langle (\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2)\mathcal{G}_{B_1,D_1}\phi_\xi,\, \mathcal{G}_{A_1,C_1}\phi_\eta \rangle\rangle
\end{aligned}$$

$$= \langle\langle(\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2)\phi_{D_1\xi},\, \phi_{C_1\eta}\rangle\rangle\, e^{\frac{1}{2}\langle B_1\xi,\,\xi\rangle+\frac{1}{2}\langle A_1\eta,\,\eta\rangle}$$
$$= \langle\langle\Gamma(C_1^*U_1)\Xi_1\Gamma(U_2^*D_1)\phi_\xi,\, \phi_\eta\rangle\rangle\, \langle\langle\Gamma(C_1^*V_1)\Xi_2\Gamma(V_2^*D_1)\phi_\xi,\, \phi_\eta\rangle\rangle$$
$$\times \langle\langle\Gamma(C_1^*)\Upsilon\Gamma(D_1)\phi_\xi,\, \phi_\eta\rangle\rangle\, e^{\frac{1}{2}\langle B_1\xi,\,\xi\rangle+\frac{1}{2}\langle A_1\eta,\,\eta\rangle-2\langle D_1\xi,\,C_1\eta\rangle}. \tag{4.8}$$

We also obtain that

$$\left\langle\left\langle[(\mathcal{F}^Q_{A_2,B_2;C_2,D_2}\Xi_1) *_{\mathcal{U}',\mathcal{V}';\Upsilon'} (\mathcal{F}^Q_{A_3,B_3;C_3,D_3}\Xi_2)]\phi_\xi,\, \phi_\eta\right\rangle\right\rangle$$
$$= \left\langle\left\langle\mathcal{F}^Q_{A_2,B_2;C_2,D_2}\Xi_1\Gamma(U_2'^*)\phi_\xi,\, \Gamma(U_1'^*)\phi_\eta\right\rangle\right\rangle$$
$$\times \left\langle\left\langle\mathcal{F}^Q_{A_3,B_3;C_3,D_3}\Xi_2\Gamma(V_2'^*)\phi_\xi,\, \Gamma(V_1'^*)\phi_\eta\right\rangle\right\rangle \left\langle\left\langle\Upsilon'\phi_\xi,\, \phi_\eta\right\rangle\right\rangle e^{-2\langle\xi,\,\eta\rangle}$$
$$= \left\langle\left\langle\Xi_1\mathcal{G}_{B_2,D_2}\Gamma(U_2'^*)\phi_\xi,\, \mathcal{G}_{A_2,C_2}\Gamma(U_1'^*)\phi_\eta\right\rangle\right\rangle$$
$$\times \left\langle\left\langle\Xi_2\mathcal{G}_{B_3,D_3}\Gamma(V_2'^*)\phi_\xi,\, \mathcal{G}_{A_3,C_3}\Gamma(V_1'^*)\phi_\eta\right\rangle\right\rangle \left\langle\left\langle\Upsilon'\phi_\xi,\, \phi_\eta\right\rangle\right\rangle e^{-2\langle\xi,\,\eta\rangle}$$
$$= \left\langle\left\langle\Gamma(U_1'C_2^*)\Xi_1\Gamma(D_2U_2'^*)\phi_\xi,\, \phi_\eta\right\rangle\right\rangle \left\langle\left\langle\Gamma(V_1'C_3^*)\Xi_2\Gamma(D_3V_2'^*)\phi_\xi,\, \phi_\eta\right\rangle\right\rangle$$
$$\times \left\langle\left\langle\Upsilon'\phi_\xi,\, \phi_\eta\right\rangle\right\rangle e^{\frac{1}{2}\left\langle(U_2'B_2U_2'^*+V_2'B_3V_2'^*)\xi,\,\xi\right\rangle+\frac{1}{2}\left\langle(U_1'A_2U_1'^*+V_1'A_3V_1'^*)\eta,\,\eta\right\rangle-2\langle\xi,\,\eta\rangle}. \tag{4.9}$$

Therefore, by comparing (4.8) and (4.9), we see that (4.5) holds if and only if (4.6) holds and for any $\xi, \eta \in E$,

$$\left\langle\left\langle\Upsilon'\phi_\xi,\, \phi_\eta\right\rangle\right\rangle = \langle\langle\Gamma(C_1^*)\Upsilon\Gamma(D_1)\phi_\xi,\, \phi_\eta\rangle\rangle$$
$$\times e^{\frac{1}{2}\left\langle(B_1-U_2'B_2U_2'^*-V_2'B_3V_2'^*)\xi,\,\xi\right\rangle+\frac{1}{2}\left\langle(A_1-U_1'A_2U_1'^*-V_1'A_3V_1'^*)\eta,\,\eta\right\rangle+2\left\langle(I-C_1^*D_1)\xi,\,\eta\right\rangle}$$
$$= \langle\langle\Gamma(C_1^*)\Upsilon\Gamma(D_1)\phi_\xi,\, \phi_\eta\rangle\rangle$$
$$\times \left\langle\left\langle\Gamma(3I-2C_1^*D_1)e^{\frac{1}{2}\Delta_G(B)}\phi_\xi,\, e^{\frac{1}{2}\Delta_G(A)}\phi_\eta\right\rangle\right\rangle e^{-\langle\xi,\,\eta\rangle},$$

where $A = \frac{1}{2}(A_1 - U_1'A_2U_1'^* - V_1'A_3V_1'^*)$ and $B = \frac{1}{2}(B_1 - U_2'B_2U_2'^* - V_2'B_3V_2'^*)$, which is equivalent to (4.7). ■

The following corollaries are immediate from Theorem 4.4.

Corollary 4.5 — Let $A_i, B_i \in \mathcal{L}(E, E^*)$, $C_i, D_i \in \mathcal{L}(E, E)$, $i = 1, 2$, $\mathcal{U}, \mathcal{U}', \mathcal{V}, \mathcal{V}' \in \mathfrak{T}$ and $\Upsilon, \Upsilon' \in \mathcal{L}((E), (E)^*)$. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,

$$\mathcal{F}^Q_{A_1,B_1;C_1,D_1}(\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2) = (\mathcal{F}^Q_{A_2,B_2;C_2,D_2}\Xi_1) *_{\mathcal{U}',\mathcal{V}';\Upsilon'} (\mathcal{F}^Q_{A_2,B_2;C_2,D_2}\Xi_2)$$

if and only if

$$C_1^*U_1 = U_1'C_2^*, \quad U_2^*D_1 = D_2U_2'^*, \quad C_1^*V_1 = V_1'C_2^*, \quad V_2^*D_1 = D_2V_2'^*$$

and

$$\Upsilon' = \Gamma(C_1{}^*)\Upsilon\Gamma(D_1) \diamond e^{\Delta_G^*(A)}\Gamma(3I - 2C_1{}^*D_1)e^{\Delta_G(B)}$$

with $A = \frac{1}{2}(A_1 - U_1' A_2 U_1'^* - V_1' A_2 V_1'^*)$ and $B = \frac{1}{2}(B_1 - U_2' B_2 U_2'^* - V_2' B_2 V_2'^*)$.

Corollary 4.6 — Let $A_i, B_i \in \mathcal{L}(E, E^*)$, $C_i, D_i \in \mathcal{L}(E, E)$, $i = 1, 2, 3$, $\mathcal{U} = (U_1, U_2))$, $\mathcal{V} = (V_1, V_2) \in \mathfrak{T}$ and $\Upsilon \in \mathcal{L}((E), (E)^*)$. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,

$$\mathcal{F}^Q_{A_1,B_1;C_1,D_1}(\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2) = (\mathcal{F}^Q_{A_2,B_2;C_2,D_2}\Xi_1) *_{\mathcal{U},\mathcal{V};\Upsilon} (\mathcal{F}^Q_{A_3,B_3;C_3,D_3}\Xi_2)$$

if and only if $C_1 = D_1 = I$

$$U_1 = U_1 {C_2}^*, \quad U_2 = {U_2}^* D_2, \quad V_1 = V_1 {C_3}^*, \quad V_2 = {V_2}^* D_3$$

and

$$A_1 = U_1 A_2 {U_1}^* + V_1 A_3 {V_1}^*, \quad B_1 = U_2 B_2 {U_2}^* + V_2 B_3 {V_2}^*.$$

Corollary 4.7 — Let $A, B \in \mathcal{L}(E, E^*)$, $C, D \in \mathcal{L}(E, E)$, $\mathcal{U} = (U_1, U_2), \mathcal{U}' = (U_1', U_2'), \mathcal{V} = (V_1, V_2), \mathcal{V}' = (V_1', V_2') \in \mathfrak{T}$ and $\Upsilon, \Upsilon' \in \mathcal{L}((E), (E)^*)$. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,

$$\mathcal{F}^Q_{A,B;C,D}(\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2) = (\mathcal{F}^Q_{A,B;C,D}\Xi_1) *_{\mathcal{U}',\mathcal{V}';\Upsilon'} (\mathcal{F}^Q_{A,B;C,D}\Xi_2)$$

if and only if

$$C^* U_1 = U_1' C^*, \quad U_2^* D = D U_2'^*, \quad C^* V_1 = V_1' C^*, \quad V_2^* D = D V_2'^*$$

and

$$\Upsilon' = \Gamma(C^*)\Upsilon\Gamma(D) \diamond e^{\Delta^*_G(A')}\Gamma(3I - 2{C_1}^* D_1) e^{\Delta_G(B')}$$

with $A' = \frac{1}{2}(A - U_1' A U_1'^* - V_1' A V_1'^*)$ and $B' = \frac{1}{2}(B - U_2' B U_2'^* - V_2' B V_2'^*)$.

Example 4.8 : For each $\mathcal{U} = (I, I), \mathcal{V} = (I, I) \in \mathfrak{T}$ and $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$, we have

$$\mathcal{F}^Q_{A_1,B_1;C_1,D_1}(\Xi_1 *_{\mathcal{U},\mathcal{V};\Upsilon} \Xi_2) = (\mathcal{F}^Q_{A_2,B_2;C_2,D_2}\Xi_1) *_{\mathcal{U},\mathcal{V};\Upsilon'} (\mathcal{F}^Q_{A_3,B_3;C_3,D_3}\Xi_2)$$

if and only if

$$C_1 = C_2 = C_3, \quad D_1 = D_2 = D_3,$$

and

$$\Upsilon' = \Gamma({C_1}^*)\Upsilon\Gamma(D_1) \diamond e^{\Delta^*_G(A')}\Gamma(3I - 2{C_1}^* D_1) e^{\Delta_G(B')}$$

$$\text{with } A' = \frac{1}{2}(A_1 - A_2 - A_3) \text{ and } B' = \frac{1}{2}(B_1 - B_2 - B_3).$$

The convolution $*_{\mathcal{U},\mathcal{V};\Upsilon}$ for $\mathcal{U} = (I, I), \mathcal{V} = (I, I) \in \mathfrak{T}$ has been studied in [8].

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