Indian J. Pure Appl. Math., **46**(4): 495-515, August 2015

DOI: 10.1007/s13226-015-0149-0

NONCOMMUTATIVE DIFFERENTIAL CALCULUS ON A QUADRATIC ALGEBRA

Partha Sarathi Chakraborty and Satyajit Guin

The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600 113, *India*
e-mail: parthac@imsc.res.in, gsatyajit@imsc.res.in

Dedicated to Prof. Kalyan B. Sinha on occasion of his 70th birthday.

(*Received* 22 *September* 2014; *after final revision* 11 *November* 2014;
accepted 12 *November* 2014)

We consider the algebra $k[X^2, XY, Y^2]$ where characteristic of the field k is zero. We compute a differential calculus, introduced earlier by the authors, by associating an algebraic spectral triple with this algebra. This algebra can also be viewed as the coordinate ring of the singular variety $UV - W^2$ and hence, is a quadratic algebra. We associate two canonical algebraic spectral triples with this algebra and its quadratic dual, and compute the associated Connes' calculus. We observe that the resulting Connes' calculi are also quadratic algebras, and they turn out to be quadratic dual to each other.

Key words : Connes' calculus; Connes-type calculus; algebraic spectral triple; quadratic algebra; DGA.

1. Introduction

In noncommutative geometry, associated to a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ Connes defined a differential graded algebra (dga) $\Omega_D^\bullet(\mathcal{A})$ in [3]. Here $\mathcal{A}$ is an associative algebra represented on the Hilbert space $\mathcal{H}$ and D is an unbounded, self-adjoint operator with compact resolvent acting on $\mathcal{H}$, often called the Dirac operator. A dga specifies a differential structure on an associative algebra, for which it is called a differential calculus. The dga $\Omega_D^\bullet$ defined by Connes is useful in many contexts (see Ch. 6 in [3]). There are other instances also where differential calculus has been used in various noncommutative contexts for e.g. [10, 9, 1, 7] and references therein. To see applications of differential calculus in noncommutative geometry one can look at [4, 6]. While investigating the natural question of the behaviour of $\Omega_D^\bullet$ under the tensor product of spectral triples, Kastler-Testard could not reach a conclusive answer in [5]. This question is then reinvestigated by the authors in [2]. Main outcome of [2] is that $\Omega_D^\bullet$ behaves nicely under the multiplication of spectral triples provided one restricts attention to a suitable subclass of spectral triples. Note that to define $\Omega_D^\bullet$ one does not use self-adjointness and compactness of the resolvent of D. Authors casted Connes' definition in a slightly more general algebraic framework in [2] which we recall now.

We consider the quadruple $(\mathcal{A}, \mathbb{V}, D, \gamma)$ where $\mathcal{A}$ is an associative, unital algebra over a field k, represented on the vector space $\mathbb{V}$, $D \in \mathcal{E}nd(\mathbb{V})$, and $\gamma \in \mathcal{E}nd(\mathbb{V})$ is a $\mathbb{Z}_2$-grading operator which commutes with $\mathcal{A}$ and anticommutes with D. We call it an *even algebraic spectral triple*. It is proved in [2] that the collection $\widetilde{\mathcal{S}pec}$ of even algebraic spectral triples is a monoidal category and the dga $\Omega_D^\bullet$ gives a covariant functor $\mathcal{F} : \widetilde{\mathcal{S}pec} \longrightarrow DGA$. Here DGA denotes the category of dgas over field k . Then a suitable monoidal subcategory $\widetilde{\mathcal{S}pec}_{sub}$ was identified and main result of [2] is that $\mathcal{F}$ is a monoidal functor when restricted to $\widetilde{\mathcal{S}pec}_{sub}$. It is necessary to validate the nontriviality of the functor $\mathcal{F}$ when restricted to $\widetilde{\mathcal{S}pec}_{sub}$. For that purpose authors constructed a faithful covariant functor $\mathcal{G} : \widetilde{\mathcal{S}pec} \longrightarrow \widetilde{\mathcal{S}pec}_{sub}$ and computed $\mathcal{F} \circ \mathcal{G}$ for two canonical algebraic spectral triples associated with compact manifolds and the noncommutative torus. Note that for each algebraic spectral triple $\mathcal{F} \circ \mathcal{G}$ gives a dga i,e. a differential calculus, and we call it the Connes-type calculus.

Goal of this article is two fold. On the one hand, we want to compute the Connes-type calculus for a "purely algebraic" algebraic spectral triple which will strengthen the fact that this calculus is actually computable. For that purpose we consider the algebra $k[X^2, XY, Y^2]$ with $ch(k) = 0$ and associate an algebraic spectral triple with it. We compute the Connes-type calculus $\mathcal{F} \circ \mathcal{G}$ and observe that it is not trivial. On the other hand, note that $k[X^2, XY, Y^2]$ can be identified with $\mathcal{A} := k[U, V, W]/\langle UV - W^2\rangle$ which is a quadratic algebra. We identify its quadratic dual $\mathcal{A}_{du} := k\{\alpha, \beta, \gamma\}/\langle \alpha^2, \beta^2, \alpha\gamma + \gamma\alpha, \beta\gamma + \gamma\beta, \alpha\beta + \beta\alpha + \gamma^2\rangle$. Then we associate two canonical algebraic spectral triples $(\mathcal{A}, \mathbb{V}, D)$ and $(\mathcal{A}_{du}, \mathbb{V}_{du}, \delta)$ with $\mathcal{A}$ and $\mathcal{A}_{du}$ respectively. We observe that the Connes' calculi $\Omega_D^\bullet(\mathcal{A})$ and $\Omega_\delta^\bullet(\mathcal{A}_{du})$ are also quadratic (graded)algebras. Moreover, $\Omega_D^\bullet(\mathcal{A})$ and $\Omega_\delta^\bullet(\mathcal{A}_{du})$ are quadratic dual to each other. These phenomena leads us to a series of open questions to be investigated, which we discuss at the end.

Organization of this paper is as follows. In section 2 we recall the definition of Connes' calculus, algebraic spectral triple and Connes-type calculus from [2]. Section 3 is devoted to the computation of Connes-type calculus for the algebra $k[X^2, XY, Y^2]$. Final section carries the quadratic duality between the Connes' calculi for the algebra $\mathcal{A} = k[U,V,W]/\langle UV - W^2\rangle$ and its quadratic dual $\mathcal{A}_{du}$.

2. Preliminaries on The Connes' Calculus and The Connes-type Calculus

In this section we define the Connes' calculus and recall few essential results from [2].

Definition 2.1 [2] — An algebraic *spectral triple* $(\mathcal{A}, \mathbb{V}, D)$ over an associative k-algebra $\mathcal{A}$ consists of the following things :

1. a representation π of $\mathcal{A}$ on the k-vector space $\mathbb{V}$;
2. a linear operator D acting on $\mathbb{V}$.

It is said to be an *even algebraic spectral triple* if there exists a $\mathbb{Z}_2$-grading $\gamma \in \mathcal{E}nd(\mathbb{V})$ such that γ commutes with each element of $\mathcal{A}$ and anticommutes with D. This is denoted by $(\mathcal{A}, \mathbb{V}, D, \gamma)$. It will be assumed that $\mathcal{A}$ is unital and the unit $1 \in \mathcal{A}$ acts as the identity operator on $\mathbb{V}$. Associated to every algebraic spectral triple $(\mathcal{A}, \mathbb{V}, D)$ we have the following differential graded algebra due to Connes.

Definition 2.2 — Let $\Omega^\bullet(\mathcal{A}) = \bigoplus_{k=0}^{\infty} \Omega^k(\mathcal{A})$ be the reduced universal differential graded algebra over $\mathcal{A}$. Here $\Omega^k(\mathcal{A}) := \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes k}$, $\bar{\mathcal{A}} = \mathcal{A}/\mathbb{K}$. The graded product is given by

$$\begin{aligned}
&\left(\sum_k a_{0k} \otimes \overline{a_{1k}} \otimes \ldots \otimes \overline{a_{mk}}\right) . \left(\sum_{k'} b_{0k'} \otimes \overline{b_{1k'}} \otimes \ldots \otimes \overline{b_{nk'}}\right) \\
:= &\sum_{k,k'} a_{0k} \otimes (\otimes_{j=1}^{m-1} \overline{a_{jk}}) \otimes \overline{a_{mk} b_{0k'}} \otimes (\otimes_{i=1}^{n} \overline{b_{ik'}}) \\
&\quad + \sum_{i=1}^{m-1} (-1)^i a_{0k} \otimes \overline{a_{1k}} \otimes \ldots \otimes \overline{a_{m-i,k} a_{m-i+1,k}} \otimes \ldots \otimes \overline{a_{mk}} \otimes (\otimes_{i=0}^{n} \overline{b_{ik'}}) \\
&\quad + (-1)^m a_{0k} a_{1k} \otimes (\otimes_{j=2}^{m} \overline{a_{jk}}) \otimes (\otimes_{i=0}^{n} \overline{b_{ik'}}) .
\end{aligned}$$

for $\sum_k a_{0k} \otimes \overline{a_{1k}} \otimes \ldots \otimes \overline{a_{mk}} \in \Omega^m(\mathcal{A})$ and $\sum_{k'} b_{0k'} \otimes \overline{b_{1k'}} \otimes \ldots \otimes \overline{b_{nk'}} \in \Omega^n(\mathcal{A})$. There is a differential d acting on $\Omega^\bullet(\mathcal{A})$ given by

$$d(a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_k}) := 1 \otimes \overline{a_0} \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_k} , \quad \forall\, a_j \in \mathcal{A}$$

and it satisfies the following relations

1. $d^2\omega = 0 \ \forall\, \omega \in \Omega^\bullet(\mathcal{A})$,

2. $d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^{deg(\omega_1)}\omega_1 d\omega_2\,,\ \forall$ homogeneous $\omega_1 \in \Omega^\bullet(\mathcal{A})$.

We can represent $\Omega^\bullet(\mathcal{A})$ on $\mathbb{V}$ by the following

$$\pi(a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_k}) := a_0[D, a_1]\ldots[D, a_k]\ ;\ a_j \in \mathcal{A}\,.$$

Let $J_0^{(k)} = \{\omega \in \Omega^k : \pi(\omega) = 0\}$ and $J' = \bigoplus J_0^{(k)}$. But J' fails to be a differential ideal. We consider $J^\bullet = \bigoplus J^{(k)}$ where $J^{(k)} = J_0^{(k)} + dJ_0^{(k-1)}$. Then $J^\bullet$ becomes a differential graded two-sided ideal and hence the quotient $\Omega_D^\bullet = \Omega^\bullet/J^\bullet$ becomes a differential graded algebra. The representation π gives the following isomorphism,

$$\Omega_D^k \cong \pi(\Omega^k)/\pi(dJ_0^{k-1}) \quad \forall\, k \geq 1\,. \tag{2.1}$$

The abstract differential d induces a differential $\widetilde{d}$ on the complex $\Omega_D^\bullet(\mathcal{A})$ so that we get a chain complex $(\Omega_D^\bullet(\mathcal{A}), \widetilde{d}\,)$ and a chain map $\pi_D : \Omega^\bullet(\mathcal{A}) \to \Omega_D^\bullet(\mathcal{A})$ such that the following diagram

$$\begin{array}{ccc} \Omega^\bullet(\mathcal{A}) & \xrightarrow{\pi_D} & \Omega_D^\bullet(\mathcal{A}) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \widetilde{d}} \\ \Omega^{\bullet+1}(\mathcal{A}) & \xrightarrow[\pi_D]{} & \Omega_D^{\bullet+1}(\mathcal{A}) \end{array}$$

commutes. This makes $\Omega_D^\bullet$ a differential graded algebra and we call it the Connes' calculus.

It is observed in [2] that the collection of *even algebraic spectral triples* $(\mathcal{A}, \mathbb{V}, D, \gamma)$ form a category $\widetilde{\mathcal{S}pec}$ (see Definition 2.4 in [2]) and it is a monoidal category (Proposition 2.6 in [2]). Recall the following results from [2].

Lemma 2.3 [2] — There is a covariant functor $\mathcal{F} : \widetilde{\mathcal{S}pec} \longrightarrow DGA$ given by $(\mathcal{A}, \mathbb{V}, D, \gamma) \longmapsto \Omega_D^\bullet(\mathcal{A})$, where DGA denotes the category of differential graded algebras over a field k.

Following the same notation as in [2], let $\widetilde{\mathcal{S}pec}_{sub}$ be the subcategory of $\widetilde{\mathcal{S}pec}$ objects of which are $(\mathcal{A}, \mathbb{V}, D, \gamma)$ with $\gamma \in \pi(\mathcal{A})$. This is a monoidal subcategory of $\widetilde{\mathcal{S}pec}$. Starting with $(\mathcal{A}, \mathbb{V}, D, \gamma) \in \widetilde{\mathcal{S}pec}$ one can consider the algebra $(\mathcal{A} \oplus \mathcal{A}\,, \star)$ where the multiplication $\star$ is given by

$$(a, b) \star (\bar{a}, \bar{b}) = (a\bar{a} + b\bar{b}\,,\ a\bar{b} + b\bar{a}),$$

and

$$\widetilde{\pi} : (a, b) \longmapsto \pi(a) + \gamma\pi(b) \in \mathcal{E}nd(\mathbb{V})$$

represents $(\mathcal{A} \oplus \mathcal{A}, \star)$ on the vector space $\mathbb{V}$. It shown in [2] that $((\mathcal{A} \oplus \mathcal{A}, \star), \mathbb{V}, D, \gamma) \in \widetilde{\mathcal{S}pec_{sub}}$ and we have the following result.

Proposition 2.4 [2] — There is a faithful covariant functor $\mathcal{G} : \widetilde{\mathcal{S}pec} \longrightarrow \widetilde{\mathcal{S}pec_{sub}}$ which sends $(\mathcal{A}, \mathbb{V}, D, \gamma)$ to $((\mathcal{A} \oplus \mathcal{A}, \star), \mathbb{V}, D, \gamma)$.

Recall the functor $\mathcal{F}$ in Lemma 2.3. The main result in [2] is the following.

***Theorem* 2.5** [2] — *Restricted to the monoidal subcategory* $\widetilde{\mathcal{S}pec_{sub}}$ *of* $\widetilde{\mathcal{S}pec}$, *the covariant functor* $\mathcal{F} : \widetilde{\mathcal{S}pec_{sub}} \longrightarrow DGA$ *is a monoidal functor.*

However, it must be shown that the functor $\mathcal{F}$ restricted to $\widetilde{\mathcal{S}pec_{sub}}$ does not become trivial. To prove nontriviality of $\mathcal{F}$, authors have computed $\mathcal{F} \circ \mathcal{G} : \widetilde{\mathcal{S}pec} \longrightarrow DGA$ for two canonical algebraic spectral triples associated with compact manifolds and the noncommutative torus in [2].

Definition 2.6 — For an even algebraic spectral triple $(\mathcal{A}, \mathbb{V}, D, \gamma)$, the calculus $\mathcal{F} \circ \mathcal{G}(\mathcal{A}, \mathbb{V}, D, \gamma)$ is called the Connes-type calculus.

3. Computation for the Algebra $k[X^2, XY, Y^2]$

The calculus $\mathcal{F} \circ \mathcal{G}$ was computed for the compact manifold and the noncommutative torus in [2]. In this section we do it for a "purely algebraic" *even algebraic spectral triple* and observe that this is not trivial in this case also.

Let k be a field of characteristic zero and consider the k-algebra $k[X^2, XY, Y^2]$. Our candidate for the even algebraic spectral triple is $\left(\mathcal{A} := k[X^2, XY, Y^2]\,,\, \mathbb{V} := k[X, Y]\,,\, \widetilde{D}\,,\, \gamma\right)$, where $\widetilde{D}$ and γ are given by

$$\widetilde{D}(X^m Y^n) \;:=\; \alpha m X^{m-1} Y^n + \beta n X^m Y^{n-1}\,;\; \alpha, \beta \in k\,; \tag{3.2}$$

$$\gamma\, \xi(X, Y) \;:=\; \xi(-X, -Y)\,;\; \forall\, \xi \in k[X, Y]\,. \tag{3.3}$$

Here $\mathcal{A}$ is represented on the k-vector space $k[X, Y]$ via the multiplication operator $a \mapsto M_a$. Since $\gamma^2 = Id$, we have $k[X, Y] = k[X, Y]_{even} \oplus k[X, Y]_{odd}$, where

$$k[X, Y]_{even} := \; span\{X^m Y^n : m + n \textit{ is even}\}\,,$$

$$k[X, Y]_{odd} := \; span\{X^m Y^n : m + n \textit{ is odd}\}\,.$$

Observe that $\widetilde{D}$ takes $k[X, Y]_{even}$ to $k[X, Y]_{odd}$ and vice versa, and hence induces maps $D_{\pm}$ such that

$$\widetilde{D} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

where both $D_- : k[X,Y]_{odd} \longrightarrow k[X,Y]_{even}$ and $D_+ : k[X,Y]_{even} \longrightarrow k[X,Y]_{odd}$ are given by equation 3.2. Moreover, in view of equation (3.3), we can write

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Observe that $\mathcal{A} = k[X,Y]_{even}$ as k-vector space.

Notation : $\widetilde{\mathcal{A}} := \mathcal{G}\left((\mathcal{A}, \mathbb{V}, \widetilde{D}, \gamma)\right)$ throughout the section.

Since the representation of $\mathcal{A}$ on $\mathbb{V}$ is faithful, we get $J_0^0(\widetilde{\mathcal{A}}) = \{0\}$. Observe that $[\widetilde{D}, M_{X^mY^n}] = M_{\widetilde{D}(X^mY^n)}$, the multiplication operator on $\mathbb{V}$. Let $\mathcal{M}_1^{\pm} := span\{M_a D_\pm M_b : k[X,Y]_\pm \longrightarrow k[X,Y]_\mp : a,b \in \mathcal{A}\}$, where $k[X,Y]_+ = k[X,Y]_{even}$ and $k[X,Y]_- = k[X,Y]_{odd}$. Henceforth, for notational brevity, we always denote the multiplication operator M_ξ by ξ.

Lemma 3.1 — $\pi(\Omega^1(\widetilde{\mathcal{A}})) = \mathcal{M}_1^- \oplus \mathcal{M}_1^+$.

Proof : Since

$$\left[\widetilde{D}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right] = \begin{pmatrix} 0 & D_-b - aD_- \\ D_+a - bD_+ & 0 \end{pmatrix},$$

elements of $\pi(\Omega^1(\widetilde{\mathcal{A}}))$ are linear combinations of 2×2 matrices of the form $\begin{pmatrix} 0 & \xi D_-\eta \\ \xi' D_+ \eta' & 0 \end{pmatrix}$ with $\xi, \eta, \xi', \eta' \in \mathcal{A}$. This proves that $\pi(\Omega^1(\widetilde{\mathcal{A}})) \subseteq \mathcal{M}_1^- \oplus \mathcal{M}_1^+$. To see equality observe that

$$\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} \left[\widetilde{D}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} -b' & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & aD_-b \\ a'D_+b' & 0 \end{pmatrix}$$

for all $a, b, a', b' \in \mathcal{A}$. □

Since both D_+ and D_- are given by equation (3.2), henceforth we simply write D when no confusion arises regarding domain of D_+ and D_-. For similar reason we write $\mathcal{M}_1$ instead of $\mathcal{M}_1^{\pm}$. Now define

$$\begin{aligned} \widetilde{\Phi} : \mathcal{M}_1 &\longrightarrow k[X,Y]_{odd} \oplus \mathcal{A} \\ aDb &\longmapsto (aD(b)\,,\, ab) \end{aligned}$$

Lemma 3.2 — The map

$$\Phi := \widetilde{\Phi} \oplus \widetilde{\Phi} : \mathcal{M}_1 \oplus \mathcal{M}_1 \longrightarrow k[X,Y]_{odd} \oplus \mathcal{A} \oplus k[X,Y]_{odd} \oplus \mathcal{A}$$

gives a k-linear bijection.

PROOF : Since $aDb = a[D,b] + abD$ and $[D,b] = M_{D(b)}$, we have $\sum a_i D b_i = 0$ implies $\sum a_i D(b_i) = 0$ and $\sum a_i b_i D = 0$. Since $D(X) = \alpha$ we have $\sum a_i b_i = 0$. This shows that Φ is well-defined. Clearly Φ is injective and $\mathcal{R}(\widetilde{\Phi}) \subseteq k[X,Y]_{odd} \oplus \mathcal{A}$. To see surjectivity first observe that $\widetilde{\Phi} : aD1 \longmapsto (0\,,a)$ for any $a \in \mathcal{A}$. Now choose any $X^m Y^n \in k[X,Y]_{odd}$. Then

Case 1 : m even and n odd.

$$X^m DY^{n+1} \xrightarrow{\widetilde{\Phi}} (\beta(n+1)X^m Y^n, X^m Y^{n+1})\,.$$

Case 2 : m odd and n even.

$$X^{m-1}DXY^{n+1} \xrightarrow{\widetilde{\Phi}} (\alpha X^{m-1}Y^{n+1} + \beta(n+1)X^m Y^n, X^m Y^{n+1})\,,$$

and use the previous case to subtract the term $\alpha X^{m-1}Y^{n+1}$.
Since $\Phi := \widetilde{\Phi} \oplus \widetilde{\Phi}$ by definition, we are done. □

Lemma 3.3 — The following action

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .(X^m Y^n\,,\, a\,,\, X^r Y^s\,,\, b) := (fX^m Y^n\,,\, fa\,,\, gX^r Y^s\,,\, gb)$$

$$(X^m Y^n\,,\, a\,,\, X^r Y^s\,,\, b).\begin{pmatrix} f' & 0 \\ 0 & g' \end{pmatrix} := (aD(g') + X^m Y^n g'\,,\, ag'\,,\, bD(f') + X^r Y^s f'\,,\, bf')$$

gives an $\widetilde{\mathcal{A}}$-bimodule structure on the k-vector space $k[X,Y]_{odd} \oplus \mathcal{A} \oplus k[X,Y]_{odd} \oplus \mathcal{A}$.

PROOF : Define

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .\,(X^m Y^n\,,\, a\,,\, X^r Y^s\,,\, b) := \Phi\left(\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .\Phi^{-1}(X^m Y^n\,,\, a\,,\, X^r Y^s\,,\, b)\right),$$

for $f, g \in \mathcal{A}$, where Φ is in Lemma 3.2. It is clearly a left module structure induced by that on $\Omega^1_D(\widetilde{\mathcal{A}})$. Now one can check that

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .(X^m Y^n\,,\, a\,,\, 0\,,\, 0) = (fX^m Y^n, fa, 0, 0)\,,$$

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .(0\,,\, 0\,,\, X^r Y^s\,,\, b) = (0, 0, gX^r Y^s, gb)\,.$$

Similarly for the right module structure we define

$$(X^mY^n\,,\,a\,,\,X^rY^s\,,\,b).\begin{pmatrix} f' & 0\\ 0 & g'\end{pmatrix} := \Phi\left(\Phi^{-1}(X^mY^n\,,\,a\,,\,X^rY^s\,,\,b).\begin{pmatrix} f' & 0\\ 0 & g'\end{pmatrix}\right)$$

for $f', g' \in \mathcal{A}$, and one can check that it is equal to the following

$$(X^mY^n\,,\,a\,,\,0\,,\,0).\begin{pmatrix} f' & 0\\ 0 & g'\end{pmatrix} = (aD(g') + X^mY^n g', ag', 0, 0)\,,$$

$$(0\,,\,0\,,\,X^rY^s\,,\,b).\begin{pmatrix} f' & 0\\ 0 & g'\end{pmatrix} = (0, 0, bD(f') + X^rY^s f', bf')\,.\square$$

Lemma 3.4 — $\pi(\Omega^2(\widetilde{\mathcal{A}})) = \mathcal{M}_2 \oplus \mathcal{M}_2$, where $\mathcal{M}_2 := span\{aDbDc : k[X,Y] \longrightarrow k[X,Y]\colon a, b, c \in \mathcal{A}\}$.

Proof : Elements of $\pi(\Omega^2(\widetilde{\mathcal{A}}))$ are linear combinations of matrices of the form $\begin{pmatrix} \xi D\eta D\zeta & 0\\ 0 & \xi' D\eta' D\zeta'\end{pmatrix}$ with $\xi, \eta, \zeta, \xi', \eta', \zeta' \in \mathcal{A}$. This proves that $\pi(\Omega^2(\widetilde{\mathcal{A}}) \subseteq \mathcal{M}_2 \oplus \mathcal{M}_2$. To see equality observe that

$$\begin{pmatrix} a & 0\\ 0 & a'\end{pmatrix}\left[\widetilde{D}, \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix} -b' & 0\\ 0 & b\end{pmatrix}\left[\widetilde{D}, \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix} -c & 0\\ 0 & c'\end{pmatrix} = \begin{pmatrix} aDbDc & 0\\ 0 & a'Db'Dc'\end{pmatrix}$$

for all $a, b, c, a', b', c' \in \mathcal{A}$. $\square$

Define

$$\begin{aligned} \widetilde{\Phi} : \mathcal{M}_2 &\longrightarrow k[X,Y]_{even} \oplus k[X,Y]_{odd} \oplus \mathcal{A}\\ aDbDc &\longmapsto (aD(b)D(c) + abD^2(c), aD(b)c + 2abD(c), abc)\,. \end{aligned}$$

Lemma 3.5 — The map

$$\Phi := \widetilde{\Phi} \oplus \widetilde{\Phi} : \mathcal{M}_2 \oplus \mathcal{M}_2 \longrightarrow k[X,Y] \oplus \mathcal{A} \oplus k[X,Y] \oplus \mathcal{A}$$

gives a k-linear bijection.

Proof : Note that

$$aDbDc = M_{aD(b)D(c)+ab[D^2,c]} + M_{aD(b)c} \circ D + M_{abc} \circ D^2\,.$$

But $[D^2, c] = M_{D^2(c)} \oplus M_{2D(c)} \circ D$. Hence,

$$aDbDc = M_{aD(b)D(c)+abD^2(c)} \bigoplus M_{aD(b)c+2abD(c)} \circ D \bigoplus M_{abc} \circ D^2\,.$$

This shows that Φ is well-defined. Clearly Φ is injective and

$$\mathcal{R}(\widetilde{\Phi}) \subseteq k[X,Y]_{even} \bigoplus k[X,Y]_{odd} \bigoplus \mathcal{A} \cong k[X,Y] \bigoplus \mathcal{A}.$$

To see surjectivity first observe that $\widetilde{\Phi} : aD1D1 \longmapsto (0,0,a)$. Now choose any $X^mY^n \in k[X,Y]_{odd}$. Then

Case 1 : m even and n odd.

$$X^mDY^{n+1}D1 \xrightarrow{\widetilde{\Phi}} (0, \beta(n+1)X^mY^n, X^mY^{n+1}).$$

Case 2 : m odd and n even.

$$X^{m-1}DXY^{n+1}D1 \xrightarrow{\widetilde{\Phi}} (0, \alpha X^{m-1}Y^{n+1} + \beta(n+1)X^mY^n, X^mY^{n+1}),$$

and use the previous case. This says that $k[X,Y]_{odd} \oplus \mathcal{A} \subseteq \mathcal{R}(\widetilde{\Phi})$. Finally, choose arbitrary $X^mY^n \in k[X,Y]_{even}$. Then

Case 1 : $m = n = 0$.

$$DXYD1 \xrightarrow{\widetilde{\Phi}} (0, \alpha Y + \beta X, XY),$$

$$D1DXY \xrightarrow{\widetilde{\Phi}} (2\alpha\beta, 2\alpha Y + 2\beta X, XY).$$

Case 2 : m even and n even.

$$X^mDY^2DY^n \xrightarrow{\widetilde{\Phi}} ((n^2+n)\beta^2X^mY^n, 2(n+1)\beta X^mY^{n+1}, X^mY^{n+2}).$$

Case 3 : m odd and n odd.

$$X^mYDY^{n-1}DY^2 \xrightarrow{\widetilde{\Phi}} (2\beta^2nX^mY^n, (n+3)\beta X^mY^{n+1}, X^mY^{n+2}).$$

Since $\Phi = \widetilde{\Phi} \oplus \widetilde{\Phi}$ by definition, we are done. □

Lemma 3.6 — $\pi(dJ_0^1(\widetilde{\mathcal{A}})) \cong k[X,Y] \oplus k[X,Y]$.

Proof : Elements of $\pi(dJ_0^1(\widetilde{\mathcal{A}}))$ looks like

$$\sum[D, pa+qb][D, pe+qf] \text{ such that } \sum(pa+qb)[D, pe+qf] = 0,$$

where $p = (1+\gamma)/2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q = (1-\gamma)/2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are the projections onto the eigenspaces of γ. Expanding the commutators and simplifying we get that arbitrary element of $\pi(dJ_0^1(\widetilde{\mathcal{A}}))$ looks like

$$\sum \begin{pmatrix} abD^2 - aD^2b & 0 \\ 0 & a'b'D^2 - a'D^2b' \end{pmatrix} \quad s.t. \quad \begin{cases} \sum aDb' = \sum abD \\ \sum a'Db = \sum a'b'D \end{cases}. \tag{3.4}$$

Clearly $\Phi\left(\pi(dJ_0^1(\widetilde{\mathcal{A}}))\right) \subseteq k[X,Y]\oplus k[X,Y]$ (Φ of Lemma 3.5). To fulfill our claim we first show that arbitrary $(aD^2(b), 2aD(b), a'D^2(b'), 2a'D(b'))$, such that the condition of equation (3.4) holds, can generate $k[X,Y]\oplus k[X,Y]$ where $a,b,a',b'\in\mathcal{A}$. Since the condition of equation (3.4) is symmetric in a,a' and b,b', let $a'=b'=0$. Note the following facts :

(1) m,n even.

$$X^mY^2D1DY^n - X^mD1DY^{n+2} \xrightarrow{\widetilde{\Phi}} (-2\beta^2(2n+1)X^mY^n, -4\beta X^mY^{n+1})\,.$$

(2) m,n odd and $n\neq 1$.

$$X^mYD1DY^{n+1} - X^mY^nD1DY^2 \xrightarrow{\widetilde{\Phi}} ((n^2+n-2)\beta^2X^mY^n, 2(n-1)\beta X^mY^{n+1})\,.$$

(3) m odd and $n=1$.

$$X^mYD1DY^2 - X^mY^3D1D1 \xrightarrow{\widetilde{\Phi}} (2\beta^2X^mY, 4\beta X^mY^2)\,.$$

(4) m even, n odd and $n\geq 3$.

$$X^mD1DY^{n+1} + X^mY^4D1DY^{n-3} - 2X^mY^2D1DY^{n-1} \xrightarrow{\widetilde{\Phi}} (8\beta^2X^mY^{n-1}, 0)\,.$$

(5) m odd, n even and $n\geq 4$.

$$X^mYD1DY^n + X^mY^5D1DY^{n-4} - 2X^mY^3D1DY^{n-2} \xrightarrow{\widetilde{\Phi}} (8\beta^2X^mY^{n-1}, 0)\,.$$

(6) m odd, n even and $n\geq 4$.

$$X^{m-1}D1DY^n + X^{m-1}Y^4D1DY^{n-4} - 2X^{m-1}Y^2D1DY^{n-2} \xrightarrow{\widetilde{\Phi}} (8\beta^2X^{m-1}Y^{n-2}, 0)\,.$$

(7) $m\geq 4$ even.

$$D1DX^m + X^4D1DX^{m-4} - 2X^2D1DX^{m-2} \xrightarrow{\widetilde{\Phi}} (8\alpha^2X^{m-2}, 0)\,.$$

(8) m even.

$$X^2D1DX^m - D1DX^{m+2} \xrightarrow{\widetilde{\Phi}} (-2\alpha^2(2m+1)X^m, -4\alpha X^{m+1})\,.$$

(9) $m\geq 1$ odd.

$$X^{m-1}D1DX^3Y - X^{m+1}D1DXY \xrightarrow{\widetilde{\Phi}} (6\alpha^2X^mY + 4\alpha\beta X^{m+1}, 4\alpha X^{m+1}Y)\,.$$

(10) $$\alpha(Y^2D1D1 - D1DY^2) + 2\beta(D1DXY - XYD1D1) \xrightarrow{\widetilde{\Phi}} (0\,,\, 4\beta^2X)\,.$$

Check that in all these cases condition of equation (3.4) is satisfied. Now choose arbitrary $X^mY^n \in k[X,Y]$. Then either $m+n$ is even or odd. We have the following cases.

Case 1 : $m = n = 0$: This follows from (8) – (10).

Case 2 : m even and $n \geq 2$ even : This follows from (6).

Case 3 : $m \geq 2$ even and $n = 0$: This follows from (7).

Case 4 : m odd and $n \geq 3$ odd : This follows from (5).

Case 5 : m odd and $n = 1$: This follows from (9) – (1) – (7).

These five cases capture the subspace $k[X,Y]_{even}$ of $k[X,Y]$. Finally, to capture $k[X,Y]_{odd}$ we have the following cases.

Case 1 : $m = 1\,,\, n = 0$: This follows from (10).

Case 2 : $m \geq 3$ odd and $n = 0$: This follows from (8) – (7).

Case 3 : m even and $n \geq 3$ odd : This follows from (1) – (4).

Case 4 : m even and $n = 1$: This follows from (1) – (7).

Case 5 : m odd and $n \geq 4$ even : This follows from (2).

Case 6 : m odd and $n = 2$: This follows from (3).

Thus we have $k[X,Y]_{even} \oplus k[X,Y]_{odd} \subseteq \mathcal{R}(\widetilde{\Phi})$ i,e. $k[X,Y] \subseteq \mathcal{R}(\widetilde{\Phi})$. Similarly, one can do with $a = b = 0$ and this justifies our claim. □

Proposition 3.7 — $\Omega^2_D(\widetilde{\mathcal{A}}) \cong \mathcal{A} \oplus \mathcal{A}$.

PROOF : Combine Lemma 3.5 and 3.6. □

Lemma 3.8 — The action

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .(a\,,b). \begin{pmatrix} f' & 0 \\ 0 & g' \end{pmatrix} := (fag', gbf')$$

gives an $\widetilde{\mathcal{A}}$-bimodule structure on the k-vector space $\mathcal{A} \oplus \mathcal{A}$.

PROOF : Define

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .(a\,,b) := \Phi\left(\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .\Phi^{-1}(a\,,b)\right),$$

for $f, g \in \mathcal{A}$, where Φ is in Lemma 3.5. This is clearly a left module structure induced by that on $\Omega^2_D(\widetilde{\mathcal{A}})$. Now one can check that

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .(a\,,b) = (fa\,,gb)\,.$$

Similarly for the right module structure, we define

$$(a\,,b).\begin{pmatrix} f' & 0 \\ 0 & g' \end{pmatrix} := \Phi\left(\Phi^{-1}(a\,,b).\begin{pmatrix} f' & 0 \\ 0 & g' \end{pmatrix}\right)$$

for $f', g' \in \mathcal{A}$, and observe that it is equal to (ag', bf'). □

***Theorem* 3.9** — *For the algebra $\mathcal{A} = k[X^2, XY, Y^2]$, the Connes-type calculus is given by the following*

$$\Omega^n_D(\widetilde{\mathcal{A}}) \cong \begin{cases} k[X,Y] \oplus k[X,Y]\,;\ if\ n = 1, \\ \mathcal{A} \oplus \mathcal{A}\,;\ if\ n\ even, \\ k[X,Y]_{odd} \oplus k[X,Y]_{odd}\,;\ if\ n \geq 3\ odd. \end{cases}$$

Proof : The $n = 1$ case follows from Lemma 3.2 together with the observation that $\mathcal{A} = k[X,Y]_{even}$. Lemma 3.3 shows that this is an $\widetilde{\mathcal{A}}$-bimodule isomorphism. The $n = 2$ case follows from Proposition 3.7 and Lemma 3.8 shows that this is also an $\widetilde{\mathcal{A}}$-bimodule isomorphism. Note that

$$\begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{cases} \tag{3.5}$$

These matrices play a crucial role to compute $\Omega^n_D(\widetilde{\mathcal{A}})$ for all $n \geq 3$. Let

$$(k[X,Y]_{even} \oplus k[X,Y]_{odd})^r := \underbrace{(k[X,Y]_{even} \oplus k[X,Y]_{odd}) \bigoplus \ldots\ldots \bigoplus (k[X,Y]_{even} \oplus k[X,Y]_{odd})}_{r\ times}.$$

Assume that

$$\pi(\Omega^n(\widetilde{\mathcal{A}})) = \begin{cases} ((k[X,Y]_{even} \oplus k[X,Y]_{odd})^r \bigoplus \mathcal{A})^2\ ;\ if\ n = 2r\ even, \\ (k[X,Y]_{odd} \bigoplus (k[X,Y]_{even} \oplus k[X,Y]_{odd})^r \bigoplus \mathcal{A})^2\ ;\ if\ n = 2r+1\ odd. \end{cases} \tag{3.6}$$

Lemma 3.2 and 3.5 give the $n = 1$ and $n = 2$ cases respectively. Now for $n \geq 3$ use induction, and equation (3.5), together with the fact that

$$\Omega^n(\widetilde{\mathcal{A}}) = \underbrace{\Omega^1(\widetilde{\mathcal{A}}) \otimes_{\widetilde{\mathcal{A}}} \ldots\ldots \otimes_{\widetilde{\mathcal{A}}} \Omega^1(\widetilde{\mathcal{A}})}_{n \ times}.$$

Recall from Lemma 3.6 that

$$\begin{aligned} \pi(dJ_0^1(\widetilde{\mathcal{A}})) &\cong k[X,Y] \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k[X,Y] \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &\cong k[X,Y] \oplus k[X,Y] \end{aligned}$$

whereas, Lemma 3.2 says that

$$\begin{aligned} \pi(\Omega^1(\widetilde{\mathcal{A}})) &\cong (k[X,Y]_{odd} \oplus \mathcal{A}) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (k[X,Y]_{odd} \oplus \mathcal{A}) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &\cong k[X,Y] \oplus k[X,Y] \end{aligned}$$

because $\mathcal{A} = k[X,Y]_{even}$. This proves that $\pi(dJ_0^1(\widetilde{\mathcal{A}})) \cong \pi(\Omega^1(\widetilde{\mathcal{A}}))$. Now we claim that $\pi(dJ_0^n(\widetilde{\mathcal{A}})) \cong \pi(\Omega^n(\widetilde{\mathcal{A}}))$ for all $n \geq 2$. Recall Lemma 2.11 from [2], which says that for any even algebraic spectral triple $(\mathcal{A}, \mathbb{V}, D, \gamma)$, with $\gamma \in \pi(\mathcal{A})$, we have $[D^2, a] \in \pi(dJ_0^1(\mathcal{A}))$. It is then easy to prove that

$$\pi(dJ_0^n) = \sum_{i=0}^{n-1} \pi\left(\Omega^i \otimes_{\mathcal{A}} J^2 \otimes_{\mathcal{A}} \Omega^{n-1-i}\right) \quad \forall\, n \geq 2\,,$$

by writing down any arbitrary element of $\pi(dJ_0^n)$ and then passing D through the commutators from left to right. Since in our case $\pi(J^2(\widetilde{\mathcal{A}})) = \pi(dJ_0^1(\widetilde{\mathcal{A}}) = \pi(\Omega^1(\widetilde{\mathcal{A}}))$, we are done. Hence,

$$\Omega_D^n(\widetilde{\mathcal{A}}) \cong \pi(\Omega^n(\widetilde{\mathcal{A}}))/\pi(\Omega^{n-1}(\widetilde{\mathcal{A}})).$$

Finally, equation (3.6) gives us

$$\Omega_D^n(\widetilde{\mathcal{A}}) \cong \begin{cases} k[X,Y]_{odd} \oplus k[X,Y]_{odd}\,;\, if\ n\ odd, \\ \mathcal{A} \oplus \mathcal{A}\,;\, if\ n\ even, \end{cases}$$

for all $n \geq 3$, and this completes the proof. □

4. Duality of The Connes' Calculus

Recall the definition of a quadratic algebra and its quadratic dual [8]. The k-algebra $k[X^2, XY, Y^2]$ studied in the last section has another viewpoint. This is the co-ordinate ring of the singular variety $UV - W^2$, hence a quadratic algebra. One can observe this by sending $U \mapsto X^2, V \mapsto$

Y^2 and $W \mapsto XY$. In this section we observe two things. Firstly, the Connes' calculus for this algebra and its quadratic dual are also quadratic algebras, and secondly these two Connes' calculi are also quadratic dual to each other. Here also $\mathrm{ch}(k) = 0$.

Proposition 4.1 — The quadratic dual of the quadratic k-algebra $\mathcal{A} := k[X, Y, Z]/\langle XY - Z^2\rangle$ is the noncommutative k-algebra $\mathcal{A}_{du} := k\{\alpha, \beta, \gamma\}/\langle \alpha^2, \beta^2, \alpha\gamma + \gamma\alpha, \beta\gamma + \gamma\beta, \alpha\beta + \beta\alpha + \gamma^2\rangle$.

Proof : Let $\mathbb{V} = kX \oplus kY \oplus kZ$ be the k-vector space of dimension 3. Consider the following subspace of $\mathbb{V} \otimes \mathbb{V}$

$$\mathbb{I} = span\{X \otimes Y - Z \otimes Z\,,\, X \otimes Y - Y \otimes X\,,\, X \otimes Z - Z \otimes X\,,\, Y \otimes Z - Z \otimes Y\} \subseteq \mathbb{V} \otimes \mathbb{V}.$$

Let $\langle \mathbb{I} \rangle$ be the ideal in $T(\mathbb{V})$, the tensor algebra of $\mathbb{V}$, generated by $\mathbb{I}$. Then the quadratic algebra $T(\mathbb{V})/\langle \mathbb{I}\rangle$ is the algebra $\mathcal{A}$. We denote the basis $\{X, Y, Z\}$ of $\mathbb{V}$ by $\{e_1, e_2, e_3\}$, whereas $\{e_1^*, e_2^*, e_3^*\}$ denotes the dual basis of $\mathbb{V}^*$, and let $\langle\, .\, ,\, .\, \rangle$ be the pairing between $\mathbb{V}$ and $\mathbb{V}^*$. Then $\{e_{ij} := e_i \otimes e_j :\ 1 \leq i, j \leq 3\}$ is the basis of $\mathbb{V} \otimes \mathbb{V}$. With this notation,

$$\mathbb{I} = span\{e_{12} - e_{33}\,,\, e_{12} - e_{21}\,,\, e_{13} - e_{31}\,,\, e_{23} - e_{32}\} \subseteq \mathbb{V} \otimes \mathbb{V}.$$

Let $\mathbb{I}^\perp \subseteq \mathbb{V}^* \otimes \mathbb{V}^*$ be the orthogonal complement to $\mathbb{I}$ with respect to the natural pairing

$$\langle v_1 \otimes v_2\,,\, v_1^* \otimes v_2^* \rangle = \langle v_1, v_1^*\rangle \langle v_2, v_2^*\rangle$$

between $\mathbb{V} \otimes \mathbb{V}$ and $\mathbb{V}^* \otimes \mathbb{V}^*$. For any $\xi = \sum \alpha_{ij} e_{ij}^* \in \mathbb{I}^\perp$ (here $e_{ij}^* = e_i^* \otimes e_j^*$) we have $\langle \sum \alpha_{ij} e_{ij}^*\,,\, \eta\rangle = 0$ for $\eta \in \{e_{12} - e_{33}\,,\, e_{12} - e_{21}\,,\, e_{13} - e_{31}\,,\, e_{23} - e_{32}\}$. This gives us the following

$$\alpha_{12} = \alpha_{33} = \alpha_{21}\ ;\ \alpha_{13} = \alpha_{31}\ ;\ \alpha_{23} = \alpha_{32}\,.$$

Hence, $\xi = \alpha_{12}(e_{12}^* + e_{21}^* + e_{33}^*) + \alpha_{13}(e_{13}^* + e_{31}^*) + \alpha_{23}(e_{23}^* + e_{32}^*) + \alpha_{11} e_{11}^* + \alpha_{22} e_{22}^*$. Letting $e_1^* = \alpha, e_2^* = \beta, e_3^* = \gamma$ we get

$$\mathbb{I}^\perp = span\{\alpha \otimes \beta + \beta \otimes \alpha + \gamma \otimes \gamma\,,\, \alpha \otimes \gamma + \gamma \otimes \alpha\,,\, \beta \otimes \gamma + \gamma \otimes \beta\,,\, \alpha \otimes \alpha\,,\, \beta \otimes \beta\} \subseteq \mathbb{V}^* \otimes \mathbb{V}^*.$$

Hence, the quadratic dual of the quadratic k-algebra $\mathcal{A} := k[X, Y, Z]/\langle XY - Z^2\rangle$ is given by

$$\begin{aligned} \mathcal{A}_{du} &= \frac{T(\mathbb{V}^*)}{\langle \mathbb{I}^\perp \rangle} \\ &= \frac{k\{\alpha, \beta, \gamma\}}{\langle \alpha^2, \beta^2, \alpha\gamma + \gamma\alpha, \beta\gamma + \gamma\beta, \alpha\beta + \beta\alpha + \gamma^2\rangle} \end{aligned}$$

and this is a noncommutative k-algebra. $\square$

Now we associate the following canonical algebraic spectral triples with $\mathcal{A}$ and $\mathcal{A}_{du}$ respectively.

(I) Consider a derivation $D : k[X,Y,Z] \longrightarrow k[X,Y,Z]$ which takes $1 \mapsto 0, X \mapsto X, Y \mapsto Y$ and $Z \mapsto Z$. Observe that D preserves the ideal $\mathcal{I} := \langle XY - Z^2 \rangle \subseteq k[X,Y,Z]$, and hence induces an well-defined derivation on $\mathcal{A} := k[X,Y,Z]/\mathcal{I}$. Let $\mathbb{V}$ denotes the k-vector space $\mathcal{A}$. The algebra $\mathcal{A}$ acts on $\mathbb{V}$ via the multiplication operator $a \mapsto M_a$. The tuple $(\mathcal{A}, \mathbb{V}, D)$ forms an algebraic spectral triple.

(II) Consider a derivation $\delta : k\{\alpha,\beta,\gamma\} \longrightarrow k\{\alpha,\beta,\gamma\}$ which takes $1 \mapsto 0, \alpha \mapsto \alpha, \beta \mapsto \beta$ and $\gamma \mapsto \gamma$. Observe that δ preserves the ideal $\mathcal{J} := \langle \alpha^2, \beta^2, \alpha\gamma+\gamma\alpha, \beta\gamma+\gamma\beta, \alpha\beta+\beta\alpha+\gamma^2 \rangle \subseteq k\{\alpha,\beta,\gamma\}$, and hence induces an well-defined derivation on $\mathcal{A}_{du} = k\{\alpha,\beta,\gamma\}/\mathcal{J}$. Let $\mathbb{V}_{du}$ denotes the k-vector space $\mathcal{A}_{du}$. The algebra $\mathcal{A}_{du}$ acts on $\mathbb{V}_{du}$ via the multiplication operator $a \mapsto M_a$. The tuple $(\mathcal{A}_{du}, \mathbb{V}_{du}, \delta\,)$ forms an algebraic spectral triple.

We are intended to compute the Connes' calculi $\Omega_D^\bullet(\mathcal{A})$ and $\Omega_\delta^\bullet(\mathcal{A}_{du})$, and we will see that both $\Omega_D^\bullet(\mathcal{A})$ and $\Omega_\delta^\bullet(\mathcal{A}_{du})$ are also quadratic k-algebras. Furthermore, it turns out that $\Omega_D^\bullet(\mathcal{A})$ and $\Omega_\delta^\bullet(\mathcal{A}_{du})$ are also quadratic dual to each other.

We first consider the algebraic spectral triple $(\mathcal{A}, \mathbb{V}, D)$. It is easy to see that arbitrary element $a \in \mathcal{A}$ is a k-linear span of elements of the form $[U^iV^jW]$ with $i,j \geq 0$. Since D is a derivation, $[D, M_a] = M_{D(a)}$ for all $a \in \mathcal{A}$. Henceforth, for notational brevity, we will always denote the multiplication operator M_ξ by ξ throughout this section.

Lemma 4.2 — $\Omega_D^1(\mathcal{A}) = \mathcal{A}/k$.

Proof Observe that $J_0^0(\mathcal{A}) = \{0\}$. Hence, $\Omega_D^1 \cong \pi(\Omega^1)$ by the isomorphism in (2.1). Notice that $D([U^iV^jW]) = (i+j+1)[U^iV^jW]$. Hence $\Omega_D^1 \subseteq \mathcal{A}/k$. To see equality observe that $D(\frac{1}{i+j+1}[U^iV^jW]) = [U^iV^jW]$. $\square$

Lemma 4.3 — $\Omega_D^2(\mathcal{A}) = \{0\}$.

Proof : Note that $\pi(\Omega^2) = span\{aD(b)D(c) \colon a,b,c \in \mathcal{A}\} \subseteq \mathcal{A}$.

Case 1 : It is easy to see that $[U^iV^jW] \in \pi(\Omega^2)$ except for $i = j = 0$. Consider U^iW for $i \geq 1$. Then $\omega = [U^i]d([W]) - \frac{1}{i+1}d([U^iW]) \in J_0^1(\mathcal{A})$ and $\pi(d\omega) = i[U^iW] \in \pi(dJ_0^1)$. Similarly $[V^jW] \in \pi(dJ_0^1)$. Since $\pi(dJ_0^1)$ is an $\mathcal{A}$-bimodule, we get $[U^iV^jW] \in \pi(dJ_0^1)$ except for $i = j = 0$.

Case 2 : Consider $\omega' = [W]d([W]) - \frac{1}{2}d([W^2]) \in J_0^1(\mathcal{A})$. Then $\pi(d\omega') = [UW] \in \pi(dJ_0^1)$. Since $\pi(dJ_0^1)$ is an $\mathcal{A}$-bimodule, we get $[U^iV^j] \in \pi(dJ_0^1)$ for all $i, j \geq 1$.

These two cases show that $\pi(\Omega^2) \subseteq \pi(dJ_0^1)$, and hence $\Omega_D^2 = \{0\}$ by the isomorphism in (2.1). □

Lemma 4.4 — $\Omega_D^n(\mathcal{A}) = \{0\},\ \forall\, n \geq 3$.

Proof : Note that for all $n \geq 2$

$$\Omega^n = \underbrace{\Omega^1 \otimes_{\mathcal{A}} \ldots\ldots \otimes_{\mathcal{A}} \Omega^1}_{n \ times}.$$

Now for all $n \geq 3$ we have

$$\begin{aligned}
\pi(\Omega^n) &= \pi(\Omega^{n-2} \otimes \Omega^2) \\
&= \pi(\Omega^{n-2})\pi(\Omega^2) \\
&\subseteq \pi(\Omega^{n-2})\pi(dJ_0^1) \qquad \textit{by Lemma}\, 4.3 \\
&= \pi(\Omega^{n-2})\pi(J^2) \\
&\subseteq \pi(\Omega^{n-2}.J^2) \\
&\subseteq \pi(J^n) \\
&= \pi(dJ_0^{n-1}).
\end{aligned}$$

Here the last inclusion follows from the fact that $J^\bullet$ is a graded ideal in $\Omega^\bullet$. Hence $\Omega_D^n(\mathcal{A}) = \{0\}, \forall\, n \geq 3$ by the isomorphism in (2.1). □

Proposition 4.5 — For the algebraic spectral triple $(\mathcal{A}, \mathbb{V}, D)$ we have

1. $\Omega_D^0(\mathcal{A}) = \mathcal{A}\,$;
2. $\Omega_D^1(\mathcal{A}) = \mathcal{A}/k\,$;
3. $\Omega_D^n(\mathcal{A}) = \{0\},\ \forall\, n \geq 2$.

Proof : Observe that $J_0^0(\mathcal{A}) = \{0\}$. This gives case (1). Now combine Lemma (4.2, 4.3, 4.4). □

Hence, for the algebraic spectral triple $(\mathcal{A}, \mathbb{V}, D)$ the Connes' calculus is given by $\Omega_D^\bullet(\mathcal{A}) = \mathcal{A} \oplus \mathcal{A}/k$. The algebra structure on $\Omega_D^\bullet$ is specified by

$$(a, 0)(b, 0) = (ab, 0)\ ;\ (0, a)(0, b) = (0, 0)\ ;\ (a, 0)(0, b) = (0, ab)$$

for all $a, b \in \mathcal{A}$, i,e. $(a,b)(x,y) = (ax, bx + ay)$ in $\Omega_D^\bullet$.

Now we will concentrate on the algebraic spectral triple $(\mathcal{A}_{du}, \mathbb{V}_{du}, \delta)$.

Lemma 4.6 — $\mathcal{J} := \{[\beta^m\alpha^n\gamma^k] \colon m, n \in \{0,1\}\,;\, k \in \mathbb{N}\cup\{0\}\}$ generates $\mathcal{A}_{du}$ as k-algebra i,e. any element $a \in \mathcal{A}_{du}$ can be written as finite linear combination of elements from $\mathcal{J}$.

Proof : If $\alpha\beta + \beta\alpha + \gamma^2 = 0$ and $\alpha\gamma + \gamma\alpha = 0$ then $\gamma^2\alpha = \alpha\gamma^2$ and hence for all $m \geq 2$,

$$\begin{aligned} \alpha^m\beta &= -\alpha^{m-1}(\beta\alpha + \gamma^2) \\ &= \alpha^{m-2}(\beta\alpha + \gamma^2)\alpha - \alpha^{m-1}\gamma^2 \\ &= \alpha^{m-2}\beta\alpha^2. \end{aligned}$$

Hence, if m is even then $\alpha^m\beta = \beta\alpha^m$ and if m is odd then $\alpha^m\beta = -\beta\alpha^m - \alpha^{m-1}\gamma^2$. When we impose the relation $\beta\gamma + \gamma\beta = 0$ then we get $\beta^n\alpha = \alpha\beta^n$ for even n. Hence we have

$$\alpha^m\beta = \beta\alpha^m\,,\ \forall\ m\ \textit{even}, \tag{4.7}$$

$$\beta^n\alpha = \alpha\beta^n\,,\ \forall\ n\ \textit{even}, \tag{4.8}$$

$$\alpha^m\beta = -\beta\alpha^m - \alpha^{m-1}\gamma^2\,,\ \forall\ m\ \textit{odd}. \tag{4.9}$$

Equations (4.7) and (4.8) imply that

$$\alpha^m\beta^n = \beta^n\alpha^m \text{ except for both } m \text{ and } n \text{ odd.}$$

For m odd and $n \geq 3$ odd,

$$\begin{aligned} \alpha^m\beta^n &= \beta^{n-1}\alpha^m\beta \\ &= \beta^{n-1}(-\beta\alpha^m - \alpha^{m-1}\gamma^2) \quad \textit{by } (4.9) \\ &= -\beta^n\alpha^m - \beta^{n-1}\alpha^{m-1}\gamma^2. \end{aligned}$$

Hence, we have

$$\alpha^m\beta^n = \begin{cases} -\beta^n\alpha^m - \beta^{n-1}\alpha^{m-1}\gamma^2\,;\ \textit{both } m,n \textit{ odd} \\ \beta^n\alpha^m\,;\ \textit{otherwise.} \end{cases}$$

Now impose the relations $\alpha^2 = \beta^2 = 0$ to observe that $\mathcal{J}$ generates $\mathcal{A}_{du}$ as k-algebra. □

Lemma 4.7 — $\Omega^1_\delta(\mathcal{A}_{du}) = \mathcal{A}_{du}/k$.

Proof : Observe that $J^0_0(\mathcal{A}_{du}) = \{0\}$. Hence, $\Omega^1_\delta \cong \pi(\Omega^1)$ by the isomorphism in (2.1). Notice that $\delta([\beta^m\alpha^n\gamma^k]) = (m+n+k)[\beta^m\alpha^n\gamma^k]$. Hence $\Omega^1_\delta \subseteq \mathcal{A}_{du}/k$. To see equality observe that $\delta(\frac{1}{m+n+k}[\beta^m\alpha^n\gamma^k]) = [\beta^m\alpha^n\gamma^k]$. □

Lemma 4.8 — $\Omega^2_\delta(\mathcal{A}_{du}) = \{0\}$.

PROOF : Clearly $\pi(\Omega^2) \subseteq \mathcal{A}_{du}$ and only the k-linear subspace of $\mathcal{A}_{du}$ generated by $\{1, \alpha, \beta, \gamma\}$ intersects $\pi(\Omega^2)$ trivially. We prove that (i)$\beta\gamma^k, k \geq 1$ (ii)$\alpha\gamma^k, k \geq 1$ (iii)$\beta\alpha\gamma^k, k \geq 0$ and (iv)$\gamma^k, k \geq 2$ all lie in $\pi(dJ_0^1)$ i,e. $\pi(\Omega^2) \subseteq \pi(dJ_0^1)$.

(i,ii) Let $\omega = 2\beta d(\gamma) - d(\beta\gamma) \in \Omega^1$. Then $\pi(d\omega) = 2\beta\gamma \in \pi(dJ_0^1)$. Similarly replacing β with α and using the fact that $\pi(dJ_0^1)$ is an $\mathcal{A}_{du}$-bimodule we get (i) and (ii).

(iii) Let $k = 0$ and $\omega = \beta d(\alpha) + \frac{1}{2}d(\alpha\beta) + \frac{1}{2}d(\gamma^2) \in \Omega^1$. Then $\pi(d\omega) = \beta\alpha \in \pi(dJ_0^1)$. For $k \geq 1$, choose $\omega = \frac{k+1}{k}\beta\alpha d(\gamma^k) - \beta d(\alpha\gamma^k) \in \Omega^1$. Then $\pi(d\omega) = (k+1)\beta\alpha\gamma^k \in \pi(dJ_0^1)$.

(iv) Let $\omega = \alpha d(\beta) + \beta d(\alpha) + \frac{1}{2}d(\gamma^2) \in \Omega^1$. Then $\pi(d\omega) = -\gamma^2 \in \pi(dJ_0^1)$ and hence $\gamma^k \in \pi(dJ_0^1), \forall\, k \geq 2$.
Since $\Omega^2_\delta \cong \pi(\Omega^2)/\pi(dJ_0^1)$ (see 2.1) we are done. □

Proposition 4.9 — For the algebraic spectral triple $(\mathcal{A}_{du}, \mathbb{V}_{du}, \delta)$ we have

1. $\Omega^0_\delta(\mathcal{A}_{du}) = \mathcal{A}_{du}$;
2. $\Omega^1_\delta(\mathcal{A}) = \mathcal{A}_{du}/k$;
3. $\Omega^n_\delta(\mathcal{A}_{du}) = \{0\}, \ \forall\, n \geq 2$.

PROOF : Observe that $J_0^0(\mathcal{A}_{du}) = \{0\}$. This gives case (1). In view of Lemma 4.8, Lemma 4.4 is true when we replace D by δ. Now combine Lemma (4.7, 4.8, 4.4). □

Hence, for the algebraic spectral triple $(\mathcal{A}_{du}, \mathbb{V}_{du}, \delta)$ the Connes' calculus is given by $\Omega^\bullet_\delta(\mathcal{A}_{du}) = \mathcal{A}_{du} \oplus \mathcal{A}_{du}/k$. The algebra structure on $\Omega^\bullet_\delta$ is specified by

$$(a, 0)(b, 0) = (ab, 0)\ ;\ (0, a)(0, b) = (0, 0)\ ;\ (a, 0)(0, b) = (0, ab)$$

for all $a, b \in \mathcal{A}_{du}$, i,e. $(a, b)(x, y) = (ax, bx + ay)$ in $\Omega^\bullet_\delta$.

***Theorem* 4.10** — *Consider the quadratic k-algebra* $\mathcal{A} := k[X, Y, Z]/\langle XY - Z^2\rangle$ *and its quadratic dual* $\mathcal{A}_{du} := k\{\alpha, \beta, \gamma\}/\langle \alpha^2, \beta^2, \alpha\gamma + \gamma\alpha, \beta\gamma + \gamma\beta, \alpha\beta + \beta\alpha + \gamma^2\rangle$. *Then, both the Connes' calculi* $\Omega^\bullet_D(\mathcal{A})$ *and* $\Omega^\bullet_\delta(\mathcal{A}_{du})$ *are also quadratic k-algebras. Moreover, these two Connes' calculi are quadratic dual to each other.*

PROOF : To observe that both the Connes' calculi are quadratic algebras, note that $\Omega^\bullet_D(\mathcal{A}) = \mathcal{A} \oplus \mathcal{A}/k$ is generated by $\{([X], 0), ([Y], 0), ([Z], 0), (0, [X']), (0, [Y']), (0, [Z'])\}$. Now $([X], 0) \in \mathcal{A} \oplus \mathcal{A}/k$ can be identified with $[X'] \in \Omega^0_D$ and $(0, [Y']) \in \mathcal{A} \oplus \mathcal{A}/k$ is identified with $[D,[Y']] \in \Omega^1_D$. Then in $\Omega^\bullet_D$, $([X],0)(0,[Y']) = [X'][D,[Y']] \in \Omega^1_D$, which is equal to $[X'Y'] = [Z']^2$ in $\Omega^1_D \cong \mathcal{A}/k$. Thus $([X],0)(0,[Y']) = (0,[Z']^2) \in \Omega^\bullet_D$. Similarly,

$$\begin{aligned}(0,[Y'])([X],0) &= [D,[Y']][X'] \\ &= [D,[Y'][X']] - [Y'][D,[X']] \\ &= D([Y'X']) - [Y'][X'] \\ &= 2[Y'X'] - [Y'X'] \\ &= [Z']^2 \\ &= (0,[Z']^2).\end{aligned}$$

in $\Omega^\bullet_D$. Let

$$V := span\{([X],0),([Y],0),([Z],0),(0,[X']),(0,[Y']),(0,[Z'])\}.$$

The relations among the generators of $\Omega^\bullet_D(\mathcal{A})$ is given by the following subspace of $T(V)$

$$\begin{aligned}\mathbb{I} \;:=\; span\{ & ([X],0)\otimes([Y],0) - ([Z],0)\otimes([Z],0)\,,\ ([X],0)\otimes([Y],0) - ([Y],0)\otimes([X],0)\,, \\ & ([X],0)\otimes([Z],0) - ([Z],0)\otimes([X],0)\,,\ ([Y],0)\otimes([Z],0) - ([Z],0)\otimes([Y],0)\,, \\ & ([X],0)\otimes(0,[Y']) - ([Z],0)\otimes(0,[Z'])\,,\ ([X],0)\otimes(0,[Y']) - (0,[Y'])\otimes([X],0)\,, \\ & ([X],0)\otimes(0,[Z']) - (0,[Z'])\otimes([X],0)\,,\ ([Y],0)\otimes(0,[Z']) - (0,[Z'])\otimes([Y],0)\,, \\ & (0,[\eta'_1])\otimes(0,[\eta'_2])\,\}\end{aligned}$$

for $\eta'_1,\eta'_2 \in \{X',Y',Z'\}$. The multiplication rule among the generators of $\Omega^\bullet_D(\mathcal{A})$ is described by the following table.

	$([X],0)$	$([Y],0)$	$([Z],0)$	$(0,[X'])$	$(0,[Y'])$	$(0,[Z'])$
$([X],0)$	$([X^2],0)$	$([XY],0)$	$([XZ],0)$	$(0,[X'^2])$	$(0,[Z'^2])$	$(0,[X'Z'])$
$([Y],0)$	$([XY],0)$	$([Y^2],0)$	$([YZ],0)$	$(0,[Z'^2])$	$(0,[Y'^2])$	$(0,[Y'Z'])$
$([Z],0)$	$([XZ],0)$	$([YZ],0)$	$([Z^2],0)$	$(0,[X'Z'])$	$(0,[Y'Z'])$	$(0,[Z'^2])$
$(0,[X'])$	$(0,[X'^2])$	$(0,[Z'^2])$	$(0,[X'Z'])$	0	0	0
$(0,[Y'])$	$(0,[Z'^2])$	$(0,[Y'^2])$	$(0,[Y'Z'])$	0	0	0
$(0,[Z'])$	$(0,[X'Z'])$	$(0,[Y'Z'])$	$(0,[Z']^2)$	0	0	0

Since $\mathbb{I} \subseteq V\otimes V$ we see that $\Omega^\bullet_D(\mathcal{A})$ is a quadratic algebra. Similarly, $\Omega^\bullet_\delta(\mathcal{A}_{du}) = \mathcal{A}_{du} \oplus \mathcal{A}_{du}/k$ is generated by $\{([\alpha],0),([\beta],0),([\gamma],0),(0,[\alpha']),(0,[\beta']),(0,[\gamma'])\}$. Let

$$V^* = span\{([\alpha],0),([\beta],0),([\gamma],0),(0,[\alpha']),(0,[\beta']),(0,[\gamma'])\}.$$

The relations among the generators of $\Omega^\bullet_\delta(\mathcal{A}_{du})$ is given by the following subspace of $T(V^*)$

$$\begin{aligned}\widetilde{\mathbb{I}} \;:=\; span\{ & ([\alpha],0)\otimes([\alpha],0)\,,\,([\beta],0)\otimes([\beta],0)\,,\,([\alpha],0)\otimes([\gamma],0)+([\gamma],0)\otimes([\alpha],0)\,,\\ & ([\beta],0)\otimes([\gamma],0)+([\gamma],0)\otimes([\beta],0)\,,\,([\alpha],0)\otimes(0,[\alpha'])\,,\,([\beta],0)\otimes(0,[\beta'])\,,\\ & ([\alpha],0)\otimes([\beta],0)+([\beta],0)\otimes([\alpha],0)+([\gamma],0)\otimes([\gamma],0)\,,\,(0,[\xi])\otimes(0,[\eta])\,,\\ & ([\alpha],0)\otimes(0,[\gamma'])+(0,[\gamma'])\otimes([\alpha],0)\,,\,([\beta],0)\otimes(0,[\gamma'])+(0,[\gamma'])\otimes([\beta],0)\,,\\ & ([\alpha],0)\otimes(0,[\beta'])+(0,[\beta'])\otimes([\alpha],0)+([\gamma],0)\otimes(0,[\gamma'])\,\}\,.\end{aligned}$$

The multiplication rule among the generators of $\Omega^\bullet_\delta(\mathcal{A}_{du})$ is described by the following table.

	$([\alpha],0)$	$([\beta],0)$	$([\gamma],0)$	$(0,[\alpha'])$	$(0,[\beta'])$	$(0,[\gamma'])$
$([\alpha],0)$	0	$([\alpha\beta],0)$	$([\alpha\gamma],0)$	0	$(0,[\alpha'\beta'])$	$(0,[\alpha'\gamma'])$
$([\beta],0)$	$([\beta\alpha],0)$	0	$([\beta\gamma],0)$	$(0,[\beta'\alpha'])$	0	$(0,[\beta'\gamma'])$
$([\gamma],0)$	$(-[\alpha\gamma],0)$	$(-[\beta\gamma],0)$	$(-[\alpha\beta+\beta\alpha],0)$	$(0,-[\alpha'\gamma'])$	$(0,-[\beta'\gamma'])$	$(0,-[\alpha'\beta'+\beta'\alpha'])$
$(0,[\alpha'])$	0	$(0,[\alpha'\beta'])$	$(0,[\alpha'\gamma'])$	0	0	0
$(0,[\beta'])$	$(0,[\beta'\alpha'])$	0	$(0,[\beta'\gamma'])$	0	0	0
$(0,[\gamma'])$	$(0,-[\alpha'\gamma'])$	$(0,-[\beta'\gamma'])$	$(0,-[\alpha'\beta'+\beta'\alpha'])$	0	0	0

Since $\widetilde{\mathbb{I}} \subseteq V^*\otimes V^*$ we see that $\Omega^\bullet_\delta(\mathcal{A}_{du})$ is a quadratic algebra. Check that $\widetilde{\mathbb{I}} = \mathbb{I}^\perp$ in $V^*\otimes V^*$ with respect to the natural pairing between $V\otimes V$ and $V^*\otimes V^*$, i,e. $\widetilde{\mathbb{I}}$ is the orthogonal complement to $\mathbb{I}$. This proves the duality between the Connes' calculi $\Omega^\bullet_D(\mathcal{A})$ and $\Omega^\bullet_\delta(\mathcal{A}_{du})$. □

Conclusion and Remark

In this article we investigate a particular quadratic algebra. This investigation leads us to the following series of open questions.

- Is the Connes' calculus of a quadratic algebra always a quadratic (graded)algebra?
- If not, can one classify the quadratic ideals $\mathcal{J}$ in $T(\mathbb{V})$ such that the Connes' calculus of the quadratic algebra $T(\mathbb{V})/\mathcal{J}$ is a quadratic (graded)algebra?
- Suppose the Connes' calculus of a quadratic algebra $\mathcal{A}$ becomes a quadratic (graded)algebra. Is it then always the case that the Connes' calculi of $\mathcal{A}$ and its quadratic dual $\mathcal{A}_{du}$ will also be dual to each other?
- If not, can one classify all such quadratic algebras for which this happens?

References

1. T. Brzeziński and S. Majid, A class of bicovariant differential calculi on Hopf algebras, *Lett. Math. Phys.*, **26**(1) (1992), 67-78.
2. P. S. Chakraborty and S. Guin, Multiplicativity of Connes' calculus, http://arxiv.org/abs/1402.5735, arXiv:1402.5735 [math.QA] (submitted).
3. A. Connes, *Noncommutative Geometry*, Academic Press, Inc., San Diego, CA, 1994.
4. M. Dubois-Violette, Lectures on graded differential algebras and noncommutative geometry, in: Y. Maeda, et al. (Eds.), *Noncommutative Differential Geometry and its Applications to Physics*, Kluwer Academic Publishers, 2001, pp. 245-306. Shonan, Japan, 1999.
5. D. Kastler and D. Testard, Quantum forms of tensor products, *Comm. Math. Phys.*, **155**(1) (1993), 135-142.
6. J. Madore, *An introduction to noncommutative differential geometry and its physical applications*, Second edition, London Mathematical Society Lecture Note Series, **257**, Cambridge University Press, Cambridge, 1999.
7. S. Majid, Classification of bicovariant differential calculi, *J. Geom. Phys.*, **25**(1-2) (1998), 119-140.
8. A. Polishchuk and L. Positselski, *Quadratic algebras*, University Lecture Series, **37**, American Mathematical Society, Providence, RI, 2005.
9. P. Podleś, Differential calculus on quantum spheres, *Lett. Math. Phys.*, **18**(2) (1989), 107-119.
10. S. L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), *Comm. Math. Phys.*, **122**(1) (1989), 125-170.