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ON SOME SPECIAL TYPES OF FONTAINE SHEAVES AND THEIR PROPERTIES

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In this paper we construct special types of Fontaine sheaves $\mathbb{A}_{\max}$ and $\mathbb{A}_{\max}^{\nabla}$ and we study their properties, most importantly their localizations over small affines. They will be used in sequel work to prove in a different manner a comparison isomorphism theorem of Faltings [7]. We conclude with making several conjectures.

Key words : p-adic periods; Fontaine sheaves; crystalline representation; p-adic cohomology; crystalline cohomology.

1. Introduction

Let $p > 0$ be a prime integer, K a finite, unramified extension of $\mathbb{Q}_p$ with residue field k and $\mathcal{O}_K$ the ring of integers of K. Also fix an algebraic closure $\overline{K}$ of K, and denote by G_K the Galois group of $\overline{K}$ over K. Let X be a smooth, proper and connected scheme over K (hence X has good reduction). We denote by $X_{\overline{K}}$ the geometric generic fiber of X and by $\overline{X}$ the special fiber X_k of X. We write A_{cris} and B_{cris} for the crystalline period rings defined by Fontaine in [8].

The so called *crystalline comparison conjecture* was formulated by Fontaine in [8]:

Conjecture 1.1 — In the notations above for every $n \geq 0$ there is a canonical and functorial isomorphism commuting with all the additional structures (filtrations, G_K-actions and Frobenii):

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong \mathrm{H}^n_{\mathrm{cris}}(\overline{X}/\mathcal{O}_K) \otimes_{\mathcal{O}_K} B_{\mathrm{cris}}.$$

The conjecture was fully proved by Faltings in [7] where he even proved more namely that one can drop the assumption that K is absolutely unramified and allow certain non-trivial coefficients.

Recently, Andreatta and Iovita gave a new proof of the crystalline comparison isomorphism (with non-trivial coefficients) for smooth, proper connected schemes X over K (for K unramified over $\mathbb{Q}_p$) in [1]. They defined sheaves of A_{cris}-algebras $\mathbb{A}_{\text{cris}}^{\nabla}$ and $\mathbb{A}_{\text{cris}}$ on Faltings' site $\mathfrak{X}_{\overline{K}}$ and proved the isomorphisms (compatible with filtrations, G_K-actions and Frobenii):

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \cong \mathrm{H}^n(\mathfrak{X}_{\overline{\mathrm{K}}}, \mathbb{A}_{\text{cris}}^{\nabla}) \otimes_{A_{\text{cris}}} B_{\text{cris}} \cong \mathrm{H}^n_{\text{cris}}(\overline{X}, K) \otimes_K B_{\text{cris}}.$$

By taking G_K-invariants one obtains that $D_{\text{cris}}(\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong \mathrm{H}^n_{\text{cris}}(\overline{X}, K)$. Andreatta and Iovita replaced in [1] the computation of Faltings based on hyper-coverings by small affines with a systematic use of the sheaves $\mathbb{A}_{\text{cris}}^{\nabla}$ and $\mathbb{A}_{\text{cris}}$. Moreover, in proving the isomorphism $\mathrm{H}^n(\mathfrak{X}_{\overline{\mathrm{K}}}, \mathbb{A}_{\text{cris}}^{\nabla}) \otimes_{A_{\text{cris}}} B_{\text{cris}} \cong \mathrm{H}^n_{\text{cris}}(\overline{X}, K) \otimes_K B_{\text{cris}}$ they provided an explicit acyclic resolution of $\mathbb{A}_{\text{cris}}^{\nabla}$. This allowed them to prove this isomorphism for X smooth formal p-adic scheme over $\mathcal{O}_K$ (see [1, Theorem 3.15] for further details). Their Theorem 3.15 could not have been proven with the methods used in [7] where the main technical tool to prove comparison isomorphisms is Poincaré duality.

The ring A_{cris} is both algebraically and topologically complicated and one can use the simpler ring $A_{\max}$ introduced by Colmez in [6] in order to define the notion of crystalline representation.

The general aim of this work is to construct the families of sheaves $(\mathbb{A}_{\max,n}^{\nabla})_{n\geq 1}$ and $(\mathbb{A}'^{\nabla}_{\max,n})_{n\geq 1}$ in §2 and the family of sheaves $(\mathbb{A}_{\max,n})_{n\geq 1}$ in §3 on Faltings' topology $\mathfrak{X}_{\overline{K}}$ on a smooth proper model of X over $\mathcal{O}_K$ and to study their properties, especially their localizations over small affines in order to simplify and may be extend the work of Andreatta-Iovita. The definition of Faltings' topos is recalled in §2.2. Our main goal is to construct different types of Fontaine sheaves, to prove a comparison isomorphism theorem (the "max" version) and to generalize their results to the case when the ramification degree of K is larger then 1. For this we use Faltings' topology $\mathfrak{X}_{\overline{K}}$ associated to X and a smooth, proper model of it and construct for the moment the specified new Fontaine sheaves of rings on this topology and study their properties.

Let us briefly recall the definition of the site $\mathfrak{X}_{\overline{K}}$. The objects of the underlying category are pairs $(\mathcal{U}, \mathcal{W})$ where $\mathcal{U} \to X$ is an étale morphism and $\mathcal{W} \to \mathcal{U}_{\overline{K}}$ is a finite étale morphism.

Let $\mathcal{U}$=Spec($R_{\mathcal{U}}$) be a "small" affine open of the étale site X^{et} of X (a small affine is an object such that $R_{\mathcal{U}} \otimes_{\mathcal{O}_K} k$ is geometrically irreducible over k and there are parameters $T_1, T_2, ..., T_d \in R_{\mathcal{U}}^{\times}$ such that the map $R_0 := \mathcal{O}_K\{T_1^{\pm 1}, T_2^{\pm 1}, ..., T_d^{\pm 1}\} \subset R_{\mathcal{U}}$ is formally étale). Fix now an algebraic closure Ω of the fraction field of $X_{\overline{K}}$. Denote by $\overline{R}_{\mathcal{U}} \subset \Omega$ the union of all $R_{\mathcal{U}} \otimes_{\mathcal{O}_K} \overline{K}$-subalgebras S of Ω, such that S is normal and $R_{\mathcal{U}} \otimes_{\mathcal{O}_K} \overline{K} \subset S[1/p]$ is finite and étale.

Let $A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})$ be the ring defined by Brinon in [5] (we recall its definition in §2.5).

Let $W_n = W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ be the ring of length n Witt vectors with values in $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ (see §2.1 for details) and $\mathbb{W}_n$ be a certain sheaf of $\mathcal{O}_{\overline{K}}$-algebras with ring operations defined by Witt polynomials (see §2 for details)).

Let $\mathbb{A}_{\max}^{\nabla}$ be the sheaf in $\text{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$ defined by the family $\{\mathbb{A}_{\max,n}^{\nabla}\}_n$ with transition maps induced by $\{r_{n+1}\}_n$, where $\mathbb{A}_{\max,n}^{\nabla}$ is the sheaf $A_{\max,n} \otimes_{W_n} \mathbb{W}_n$ on $\mathfrak{X}_{\overline{K}}$ and $r_{n+1} : \mathbb{W}_{n+1} \to \mathbb{W}_n$ are sheaf homomorphisms defined by the natural projection composed with Frobenius. Similarly, $\mathbb{A}_{\max}'^{\nabla}$ is the sheaf defined by the family $\{\mathbb{A}_{\max,n}'^{\nabla}\}_n$ with transition maps induced by $\{r_{n+1}\}_n$, where $\mathbb{A}_{\max,n}'^{\nabla}$ is the sheaf $A_{\max}/p^n A_{\max} \otimes_{W_n} \mathbb{W}_n$ (see §2.3 for details).

In §2.5 we construct maps $g_n' : A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})/p^n A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}_{\max,n}'^{\nabla}(\overline{R}_{\mathcal{U}})$ which are essentially the composition of the reduction modulo p^n-map and a certain map induced by the projection on the $n+1$th component (see §2.5 for details). In §2.5 we also construct maps $q_{\overline{K}}' := \{q_{n,\overline{K}}'\}_n : \mathbb{A}_{\max}'^{\nabla} \to \mathbb{A}_{\max}^{\nabla}$ where $q_{n,\overline{K}}' : \mathbb{A}_{\max,n}'^{\nabla} \to \mathbb{A}_{\max,n}^{\nabla}$ are associated to the map of presheaves induced by $q_n' : A_{\max}/p^n A_{\max} \to A_{\max,n}$ and by Frobenius $\mathbb{W}_n(\mathcal{U}, \mathcal{W}) \to \mathbb{W}_n(\mathcal{U}, \mathcal{W})$ (q_n' being induced by the natural projection $W_{n+1} \to W_n$).

The main result of §2 is the following theorem, which is the analogous result for the sheaf $\mathbb{A}_{\text{cris}}^{\nabla}$ of [1, Proposition 2.28].

***Theorem* 1.2** — *a) For every $n \in \mathbb{N}^*$ the map*

$$g_n' : A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})/p^n A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}_{\max,n}'^{\nabla}(\overline{R}_{\mathcal{U}}) \text{ is injective;}$$

b) The map $\mathbb{A}_{\max}'^{\nabla}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})$ defined by $q_{\overline{K}}'$ is an isomorphism.

Corollary 1.3 — The induced map $A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}}) \longrightarrow \mathbb{A}_{\max}^{\nabla}(\overline{R}_{\mathcal{U}}) = \varprojlim \mathbb{A}_{\max,n}^{\nabla}(\overline{R}_{\mathcal{U}})$ is an isomorphism.

Let $A_{\max}(\overline{R}_{\mathcal{U}})$ be the algebra defined in [4, Remark 8.3.5] and recalled by us in §3.1.

In §3 we prove the following theorem, which is the analogous result for the sheaf $\mathbb{A}_{\text{cris}}$ of

[1, Theorem 2.32] and [1, Proposition 2.35].

Theorem 1.4 — *There exists a unique continuous sheaf* $\mathbb{A}_{\max}$ *on* $\mathfrak{X}_{\overline{K}}$ *of* $\mathbb{A}^{\nabla}_{\max}$*-algebras such that for every small affine* $\mathcal{U} = Spec(R_{\mathcal{U}})$ *of* X^{et} *we have a canonical isomorphism as* $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$*-algebras:* $\mathbb{A}_{\max}(\overline{R}_{\mathcal{U}}) \cong A_{\max}(\overline{R}_{\mathcal{U}})$.

The properties of the continuous sheaf $\mathbb{A}_{\max}$ are summarized in the following theorem, which is the analogous result for the sheaf $\mathbb{A}_{\text{cris}}$ of [1, Proposition 2.37] and [1, Proposition 2.38].

Theorem 1.5 — *(1) The sheaf* $\mathbb{A}_{\max}$ *has a decreasing filtration by sheaves of ideals* $\text{Fil}^r\mathbb{A}_{\max} := (Ker(\theta))^r$*, for all* $r \geq 0$.

(2) There is a unique connection $\nabla := \{\nabla_n\}_{n\geq 1} : \mathbb{A}_{\max} \longrightarrow \mathbb{A}_{\max} \otimes_{\mathcal{O}_{\hat{X}}} \Omega^1_{\hat{X}/\mathcal{O}_K}$ *such that*

(a) $\nabla|_{\mathbb{A}^{\nabla}_{\max}} = 0$

(b) for every $n \geq 0$ *and every small affine* $\mathcal{U}$ *of* X *with parameters* $T_1, T_2, ..., T_d$ *and for every pair* $(\mathcal{V}, \mathcal{W})$ *in* $\mathfrak{U}_{\overline{K},n}$*, for the elements* $y_1, y_2, ..., y_d \in \mathbb{A}_{\max,n}(\mathcal{V}, \mathcal{W})$ *constructed in section 3.1, one has* $\nabla_n(y_i) = 1 \otimes dT_i \in \mathbb{A}_{\max,n}(\mathcal{V}, \mathcal{W}) \otimes_{R_{\mathcal{V}}} \widehat{\Omega}^1_{R_{\mathcal{V}}/\mathcal{O}_K}$.

(3) The connection described at (2) has the property that it is integrable and $\mathbb{A}^{\nabla}_{\max} = \left(\mathbb{A}_{\max}\right)^{\nabla=0}$.

(4) We have $\nabla(\text{Fil}^r\mathbb{A}_{\max}) \subset \text{Fil}^{r-1}\mathbb{A}_{\max} \otimes_{\mathcal{O}_{\hat{X}}} \Omega^1_{\hat{X}/\mathcal{O}_K}$ *for every* $r \geq 1$*, i.e.* ∇ *satisfies the Griffith transversality property.*

Our proof of Theorem 1.2 is similar to the proof of [1, Proposition 2.28] but we work with different families of rings namely $\{A_{\max,n}\}_n$ instead of $\{A_{\text{cris},n}\}_n$ as [1] hence our results can be viewed as an extension of theirs to the "$A_{\max}$" case (since $A_{\text{cris}} \subset A_{\max}$); we prove a priori several important results such as the fact that the ring $A^{\nabla}_{\max}$ is p-torsion free (see Proposition 2.10) and the comparison between a certain family of rings $\{A'_{\max,n}\}_n$ and $\{A_{\max,n}\}_n$ (see Proposition 2.4).

We also prove the existence of the sheaf $\mathbb{A}_{\max}$ (see the construction of a certain projective system of torsion sheaves $\{\mathbb{A}_{\max,\mathfrak{U},n}\}_{n\geq 0}$ in section 3.1). We point out that in [1], Andreatta and Iovita proved the existence of the $\mathcal{O}_K$-DP envelope $\mathbb{A}_{\text{cris},n,\overline{K}}$ of $\mathbb{W}_{X,n,\overline{K}}$ with respect to a certain sheaf of ideals $\tau_{X,n,\overline{K}}$ that we won't use (for the definition of the sheaf $\mathbb{W}_{X,n,\overline{K}}$ we refer the reader to §3.1). In §4 we also give a method of defining our sheaves for the special case when K is ramified over $\mathbb{Q}_p$ and X over $\mathcal{O}_K$ is a smooth, proper and connected scheme, such that there exists a scheme X_0 defined over $\mathcal{O}_{K_0}$ (K_0 being the maximal absolutely unramified subfield of K and $\mathcal{O}_{K_0}$ its ring of integers), such that $X \cong X_0 \times_{\mathcal{O}_{K_0}} \mathcal{O}_K$.

In §4 we also make several conjectures. As a potential application of our work one may try to prove the comparison isomorphism

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\max} \cong \mathrm{H}^n(\mathfrak{X}_{\overline{\mathrm{K}}}, \mathbb{A}^{\nabla}_{\max}) \otimes_{A_{\max}} B_{\max} \cong \mathrm{H}^n_{\mathrm{cris}}(\overline{X}, K_0) \otimes_{K_0} B_{\max}.$$

The ring $B_{\max} = A_{\max}[1/t]$ has a better topology than $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$, the advantages of working with it instead of B_{cris} being clearly emphasized in [6]. We expect that the study of the comparison isomorphism theorems for the formal schemes X which are defined over $\mathcal{O}_K$ (and not only for those which are base change of formal schemes defined over $\mathcal{O}_{K_0}$) can be done with $A_{\max}$.

We hope that our rings will be used in sequel work to define a Riemann-Hilbert correspondence between p-adic locally constant sheaves on X and F-isocrystals on the special fiber of the fixed smooth model of X over $\mathcal{O}_K$ (see §4 for details).

This paper is based on the unpublished part of the author's PhD thesis [9].

2. The sheaf $\mathbb{A}^{\nabla}_{\max}$

In this section we define a new type of Fontaine sheaf, $\mathbb{A}^{\nabla}_{\max}$, we prove some properties of it and we study its localization over small affines, the main result being that $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{A}^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$, where $A^{\nabla}_{\max}$ is the ring defined by Brinon in [5] and recalled in this section.

Let $p > 0$ be a prime integer, K a finite, unramified extension of $\mathbb{Q}_p$ with residue field k and $\mathcal{O}_K$ the ring of integers of K.

2.1 Mod p^n *versions of the ring* $A_{\max}$

The goal of this subsection is to introduce and review the basic properties of the rings $A_{\max,n}$, $A'_{\max,n}$ and $A_{\max}$. We point out that in [10] these rings (and only them) were also introduced along with an algorithm which relates the families of rings $A_{\max,n}$ and $A'_{\max,n}$. However, in this subsection we prove Proposition 2.4 in a different and easier manner than in [10]. Put

$$\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}) = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}.$$

We use the notation $(x_n)_{n\geq 0} = (x_0, x_1, \cdots) \in \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$ where $x_{n+1}^p = x_n$ to denote its elements. The reduction modulo p induces a bijection

$$\varprojlim_{x\mapsto x^p} \widehat{\mathcal{O}_{\overline{K}}} \simeq \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}) \text{ given by } x^{(n)} \mapsto x^{(n)} \pmod p$$

where we use the notation $(x^{(n)})_{n\geq 0} \in \varprojlim_{x\mapsto x^p} \widehat{\mathcal{O}_{\overline{K}}}$. Note that its inverse map is given by

$$x^{(n)} = \lim_{m\to\infty} \hat{x}_{n+m}^{p^m}$$

where $\hat{x}_i \in \widehat{\mathcal{O}_{\overline{K}}}$ is an arbitrary lift of $x_i \in \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$. Then $\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$ is a perfect valuation ring of characteristic p, its valuation v being given by $v(x) = v_p(x^{(0)}),\ x = (x_n)_{n\geq 0} \in \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$. We identify $\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$ with $\varprojlim_{x\mapsto x^p} \widehat{\mathcal{O}_{\overline{K}}}$ using the above bijection.

Let $A_{\text{inf}}^+ = W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$ be the ring of Witt vectors of $\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$. We define the element

$$\xi := [\tilde{p}] - p = (\tilde{p}, 0, 0, \cdots) - (0, 1, 0, \cdots) \in A_{\text{inf}}^+ = W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$$

where $\tilde{p} \in \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$ is such that $\tilde{p}^{(0)} = p$. Let $W_n = W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ be the length n Witt vectors of $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$. One defines a ring homomorphism

$$\theta_n : W_n \to \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \text{ given by } (s_0, s_1, \cdots) \mapsto \sum_{i=0}^{n-1} p^i \tilde{s}_i^{p^{n-1-i}}$$

where $\tilde{s}_i \in \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ is any lift of $s_i \in \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$. We also define the projection of $W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$ on the first n-components

$$\pi_n : A_{\text{inf}}^+ = W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \to W_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \text{ given by } (s_0, s_1, \cdots) \mapsto (s_0, s_1, \cdots, s_{n-1})$$

and the projection of $\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$ on the $(n+1)$th component $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$

$$\bar{q}_n : \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}) \to \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} \text{ given by } (x_m)_{m\geq 0} \mapsto x_n.$$

Remark that the homomorphism π_n induces an isomorphism $A_{\text{inf}}^+/p^nA_{\text{inf}}^+ \simeq W_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$. It is clear that π_n is surjective. Since $p^n = (\underbrace{0, 0, \cdots, 0}_{n}, 1, 0, \cdots) \in A_{\text{inf}}^+$ we have

$$\ker(\pi_n) = \{(s_0, s_1, \cdots) \in A_{\text{inf}}^+ : s_0 = s_1 = \cdots = s_{n-1} = 0\} = p^nA_{\text{inf}}^+.$$

It follows that $A_{\text{inf}}^+/p^nA_{\text{inf}}^+ \simeq W_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$. This is also clear as $\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$ is perfect.

One also has the following:

Proposition 2.1 [10, Proposition 2] — The kernel of the projection $\bar{q}_n : \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}) = \varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} \mapsto \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ is the ideal of $\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$ generated by $\tilde{p}^{p^n}$.

The homomorphism $\bar{q}_n$ induces a surjective ring homomorphism:

$$q_n : W_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \to W_n = W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \text{ given by } (s_0, s_1, \cdots, s_{n-1}) \mapsto (s_0^{(n)} \pmod p, s_1^{(n)} \pmod p, \cdots, s_{n-1}^{(n)} \pmod p).$$

Proposition 2.2 [10, Proposition 3]— The kernel of the ring homomorphism $q_n \circ \pi_n$ is the ideal in $W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$ generated by $\{[\tilde{p}]^{p^n}, V([\tilde{p}]^{p^n}), V^2([\tilde{p}]^{p^n}), ..., V^{n-1}([\tilde{p}]^{p^n})\}$.

Definition 2.3 — Let A be a p-adically complete $\mathcal{O}_K$-algebra. Then we define

$$A\{T\} := \varprojlim A[T]/p^n A[T]$$

where T is a variable.

We define the rings

$$A_{\max,n} := W_n[\delta]/(p\delta - \xi_n), \quad A_{\max} := \varprojlim_n A_{\max,n} \tag{1}$$

where $\xi_n = [\tilde{p}^{1/p^{n-1}}] - p \in W_n$ and the transition maps in the projective system are induced by $F \circ \mathrm{pr}_n$ (with F the Frobenius and $\mathrm{pr}_n : W_{n+1} \to W_n$ the projection to the first n components) and $\delta \mapsto \delta$. Note that $(F \circ \mathrm{pr}_n)(\xi_{n+1}) = \xi_n$. We then have that

$$\begin{aligned} A_{\max} &= W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))\{\xi/p\} = A^+_{\inf}\{\delta\}/(p\delta - \xi) \\ &= \{\sum_{i\geq 0} a_i(\xi/p)^i : a_i \in W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})),\ a_i \to 0 \text{ as } i \to \infty\} \end{aligned} \tag{2}$$

where we use the p-adic topology on $W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$. We also define

$$A'_{\max,n} := W_n[\delta]/(p\delta - \mathrm{pr}_n(\xi_{n+1})) \tag{3}$$

where $\mathrm{pr}_n(\xi_{n+1}) = \mathrm{pr}_n([\tilde{p}^{1/p^n}] - p) \in W_n$. Then we observe that for i, $0 \leq i \leq n-1$,

$$\begin{aligned} V^i([\tilde{p}]^{p^n}) &= p^i([\tilde{p}]^{p^n})^{p^{-i}} = p^i[\tilde{p}]^{p^{n-i}} = p^i(\xi + p)^{p^{n-i}} \\ &= p^i(p(\delta+1))^{p^{n-i}} \equiv p^{i+p^{n-i}}\delta^{p^{n-i}} \equiv 0 \pmod{p^n A_{\max}}, \end{aligned}$$

where for the first equality one uses the Witt coordinate $(r_0, r_1, ...) = \sum p^n[r_n^{p^{-n}}]$(or one computes it directly). Using Proposition 2.2, we obtain that

$$\ker(q_n \circ \pi_n) \cdot A_{\max} \subseteq p^n A_{\max}. \tag{4}$$

The map $q_n \circ \pi_n : W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \to W_n$ define a homomorphism

$$r'_n : A_{\max} = W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))\{\delta\}/(p\delta - \xi) \to A'_{\max,n} = W_n[\delta]/(p\delta - \mathrm{pr}_n(\xi_{n+1}))$$

$$\delta \mapsto \delta$$

This map is well-defined since $q_n \circ \pi_n : W(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \to W_n$ sends ξ to $\mathrm{pr}_n(\xi_{n+1})$.

Proposition 2.4 — For any integer $n \geq 1$, the map r'_n induces a ring isomorphism $A_{\max}/p^n A_{\max} \simeq A'_{\max,n}$.

PROOF : Note firstly that $\ker(r'_n) = \ker(q_n \circ \pi_n) \cdot A_{\max}$. Since $(q_n \circ \pi_n)(p^n) = 0$ and r'_n is a ring homomorphism, we obtain that $p^n A_{\max} \subseteq \ker(q_n \circ \pi_n) \cdot A_{\max}$. Using (4) we get

$$p^n A_{\max} = \ker(q_n \circ \pi_n) \cdot A_{\max} = \ker(r'_n).$$

The result follows because r'_n is surjective. □

Note that, via the isomorphism $A_{\max}/p^n A_{\max} \cong A'_{\max,n}$, we have a surjective map of rings:

$$q'_n : A_{\max}/p^n A_{\max} \to A_{\max,n}$$

induced by Frobenius on W_n and sending $\mathrm{pr}_n(\xi_{n+1}) \to \xi_n$, and that we also have a map:

$$u_n : A_{\max,n+1} \to A_{\max}/p^n A_{\max}$$

induced by the natural projection $W_{n+1} \to W_n$ and sending $\xi_{n+1} \to \mathrm{pr}_n(\xi_{n+1})$.

2.2 *Faltings' topology*

The goal of this section is to review Faltings' topology for the reader's convenience, this being a category of sheaves on a certain site $\mathfrak{X}_M$, constructed and studied in [1]. Let now X be a scheme of finite type over $\mathcal{O}_K$ and also let M be an algebraic extension of K. One denotes by X^{et} the small étale site on X and by X_M^{fet} the finite étale site of $X_M = X \times_{Spec(\mathcal{O}_K)} Spec(M)$. Further, one denotes by $\mathrm{Sh}(X^{\mathrm{et}})$ and $\mathrm{Sh}(X_M^{\mathrm{fet}})$ the categories of sheaves of abelian groups of these two sites, respectively. We refer to [1, Section 2] for the detailed discussion.

Definition 2.5 ([1, Definition 2.1]) — Let E_{X_M} be the category defined as follows:

1) the objects consist of pairs $(g : U \to X, f : W \to U_M)$ where g is an étale morphism and f is a finite étale morphism. One further denotes by (U, W) this object to simplify the notations;

2) a morphism $(U', W') \to (U, W)$ in E_{X_M} is a pair (α, β), where $\alpha : U' \to U$ is a

morphism over X and $\beta : W' \to W$ is a morphism commuting with $\alpha \otimes_{\mathcal{O}_K} Id_M$.

Definition 2.6 ([1, Definition 2.3]) — Let $\{(U_i, W_i) \to (U, W)\}_{i\in I}$ be a family of morphisms in E_{X_M}. We say it is of type α respectively β if either:
$\alpha)\{U_i \to U\}_{i\in I}$ is a covering in X^{et} and $W_i \cong W \times_U U_i$ for every $i \in I$, the morphism $W \to U$ used in the fibre product being the composition $W \to U_M \to U$,

or

$\beta)$ $U_i \cong U$ for all $i \in I$ and $\{W_i \to W\}_{i\in I}$ is a covering in X_M^{fet}.

One further endows E_{X_M} with the topology generated by the covering families described in definition 2.6 and one denotes by $\mathfrak{X}_M$ the associated site. One calls $\mathfrak{X}_M$ the locally Galois site associated to (X, M).

Definition 2.7 ([1, Definition 2.4]) — A family $\{(U_{ij}, W_{ij}) \to (U, W)\}_{i\in I, j\in J}$ is called a strict covering family if:

i) For each $i \in I$ there exists an étale morphism $U_i \to X$ such that one has $U_i \cong U_{ij}$ over X for all $j \in J$;

ii) $\{U_i \to U\}_{i\in I}$ is a covering in X^{et};

iii) For each $i \in I$ the family $\{W_{ij} \to W \times_U U_i\}_{j\in J}$ is a covering in X_M^{fet}.

Each strict covering family is a covering family (see [1, Remark 2.5]).

Let now (U, W) be an object of E_{X_M}. Andreatta and Iovita defined in [1, Definition 2.10] the presheaf $\mathcal{O}_{\mathfrak{X}_M}$ on E_{X_M}, by requiring that $\mathcal{O}_{\mathfrak{X}_M}(U, W)$ consists of the normalization of $\Gamma(U, \mathcal{O}_U)$ in $\Gamma(W, \mathcal{O}_W)$. They also proved [1, Proposition 2.11] that the presheaf $\mathcal{O}_{\mathfrak{X}_M}$ is a sheaf.

2.3 *The construction of* $\mathbb{A}_{\max}^{\nabla}$

Let $\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$ be the sheaf of rings on $\mathfrak{X}_{\overline{K}}$ defined by requiring that for every object (U, W) in $\mathfrak{X}_{\overline{K}}$, the ring $\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(U, W)$ is the normalization of $\Gamma(U, \mathcal{O}_U)$ in $\Gamma(W, \mathcal{O}_W)$. Note that $\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$ is a sheaf of $\mathcal{O}_{\overline{K}}$-algebras.

Let $\hat{\bar{\mathcal{O}}}_{\mathfrak{X}_{\overline{K}}} := \varprojlim_n \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p^n\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}} \in \mathrm{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$. Also, let $\underline{\mathcal{R}}(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})$ be the sheaf of rings in $\mathrm{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$ defined by the inverse system $\{\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}\}$, the transition maps being given by Frobenius.

For every $s \in \mathbb{N}$ we define now the sheaf of rings $\mathbb{A}^{+}_{\mathrm{inf},s,\overline{K}} := \varprojlim \mathbb{W}_{s,\overline{K}}$ where $\mathbb{W}_{s,\overline{K}} :=$

$\mathbb{W}_s(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})$ is the sheaf $(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})^s$ with ring operations defined by Witt polynomials and the transition maps in the projective limit are defined by Frobenius.

We further define the sheaf of rings $\mathbb{A}^+_{\mathrm{inf},\overline{K}} := \varprojlim \mathbb{W}_{n,\overline{K}}$ in $\mathrm{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$, where the transition maps in the projective limit are defined as the composite of the projection $\mathbb{W}_{n+1,\overline{K}} \to \mathbb{W}_{n,\overline{K}}$ and the Frobenius on $\mathbb{W}_{n,\overline{K}}$.

We also have a morphism $\theta_{\overline{K}} : \mathbb{A}^+_{\mathrm{inf},\overline{K}} \to \hat{\bar{\mathcal{O}}}_{\mathfrak{X}_{\overline{K}}}$ of objects of $\mathrm{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$; we construct it at the beginning of section 3.

$\mathbb{A}^+_{\mathrm{inf},\overline{K}}$ and $\mathbb{A}^+_{\mathrm{inf},s,\overline{K}}$ are endowed with an operator, φ, which is the canonical Frobenius associated to the Witt vector construction and are sheaves of $\mathcal{O}_K$-algebras.

We are able now to construct the sheaves $\mathbb{A}^\nabla_{\max,\overline{K}}$ and $\mathbb{A}'^\nabla_{\max,\overline{K}}$.

Firstly, let $\mathbb{A}^\nabla_{\max,n,\overline{K}} := A_{\max,n} \otimes_{W_n} \mathbb{W}_{n,\overline{K}} = W_n[\delta]/(p\delta - \xi_n) \otimes_{W_n} \mathbb{W}_{n,\overline{K}}$ i.e. $\mathbb{A}^\nabla_{\max,n,\overline{K}}$ is the sheaf on $\mathfrak{X}_{\overline{K}}$ associated to the pre-sheaf given by

$$(\mathcal{U},\mathcal{W}) \mapsto A_{\max,n} \otimes_{W_n} \mathbb{W}_{n,\overline{K}}(\mathcal{U},\mathcal{W}) \text{ for } (\mathcal{U},\mathcal{W}) \in \mathfrak{X}_{\overline{K}}.$$

Consider the map $r_{n+1} : \mathbb{W}_{n+1,\overline{K}} \to \mathbb{W}_{n,\overline{K}}$ given by the natural projection composed with Frobenius. This induces a natural map $r_{n+1,\overline{K}} : \mathbb{A}^\nabla_{\max,n+1,\overline{K}} \to \mathbb{A}^\nabla_{\max,n,\overline{K}}$.

Let $\mathbb{A}^\nabla_{\max,\overline{K}}$ be the sheaf in $\mathrm{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$ defined by the family $\{\mathbb{A}^\nabla_{\max,n,\overline{K}}\}_n$ with transition maps $\{r_{n+1,\overline{K}}\}_n$. Secondly, let $\mathbb{A}'^\nabla_{\max,n,\overline{K}}$ be the sheaf on $\mathfrak{X}_{\overline{K}}$ associated to the pre-sheaf given by $(\mathcal{U},\mathcal{W}) \mapsto A_{\max}/p^n A_{\max} \otimes_{W_n} \mathbb{W}_{n,\overline{K}}(\mathcal{U},\mathcal{W})$ for $(\mathcal{U},\mathcal{W}) \in \mathfrak{X}_{\overline{K}}$.

As for $\mathbb{A}^\nabla_{\max,n,\overline{K}}$, r_{n+1} induces a natural map $r'_{n+1,\overline{K}} : \mathbb{A}'^\nabla_{\max,n+1,\overline{K}} \mapsto \mathbb{A}'^\nabla_{\max,n,\overline{K}}$.

Similarly, let $\mathbb{A}'^\nabla_{\max,\overline{K}}$ be the sheaf in $\mathrm{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$ defined by the family $\{\mathbb{A}'^\nabla_{\max,n,\overline{K}}\}_n$ with transition maps $\{r'_{n+1,\overline{K}}\}_n$. Also, note that $\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$ is the sheaf associated to the pre-sheaf $(\mathcal{U},\mathcal{W}) \mapsto \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{U},\mathcal{W})/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{U},\mathcal{W})$.

In order to simplify the notations denote by $\mathbb{A}^\nabla_{\max} := \mathbb{A}^\nabla_{\max,\overline{K}}$, $\mathbb{A}^\nabla_{\max,n} := \mathbb{A}^\nabla_{\max,n,\overline{K}}$, $\mathbb{A}'^\nabla_{\max} := \mathbb{A}'^\nabla_{\max,\overline{K}}$, $\mathbb{A}'^\nabla_{\max,n} := \mathbb{A}'^\nabla_{\max,n,\overline{K}}$, $\bar{\mathcal{O}}_{\mathfrak{X}} := \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$, $\mathbb{W}_n := \mathbb{W}_{n,\overline{K}}$ and $\mathbb{A}^+_{\mathrm{inf}} := \mathbb{A}^+_{\mathrm{inf},\overline{K}}$.

Let $r''_{n+1} : A_{\max,n+1} \to A_{\max,n}$ be the map of rings defined by the natural projection composed with Frobenius, where $A_{\max,n}$ is defined in (2.1). We denote by δ the variable in the definition of $A_{\max,n+1}$ and by α the one in $A_{\max,n}$, and let $r''_{n+1}(\alpha) = \delta$. Let also

$$\widetilde{p}_n := [\widetilde{p}^{1/p^{n-1}}] \in W_n.$$

To check that the map is well defined, we have

$$r''_{n+1}\left(\frac{\xi_{n+1}}{p}\right) = \frac{r''_{n+1}(\xi_{n+1})}{p} = \frac{r''_{n+1}(\widetilde{p}_{n+1}-p)}{p} = \frac{\widetilde{p}_n - p}{p} = \frac{\xi_n}{p}.$$

Let us remark now that, since $A'_{\max,1} = (\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})[\delta]/(\mathrm{pr}_1(\xi_2))$, we have a nice description of $\mathbb{A}'^{\nabla}_{\max,1}$, namely

$$\begin{aligned}\mathbb{A}'^{\nabla}_{\max,1} = A'_{\max,1} \otimes_{W_1} \mathbb{W}_1 &= (\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})[\delta]/(\mathrm{pr}_1(\xi_2)) \otimes_{\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}} (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})\\ &= (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]/(\mathrm{pr}_1(\xi_2)) = (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]/(p^{1/p}).\end{aligned}$$

We use this fact in the proof of the following lemma:

2.4 *A short exact sequence*

Lemma 2.8 — For every n we have an exact sequence of sheaves:

$$0 \longrightarrow \mathbb{A}'^{\nabla}_{\max,n} \xrightarrow{f} \mathbb{A}'^{\nabla}_{\max,n+1} \xrightarrow{g} \mathbb{A}'^{\nabla}_{\max,1} \longrightarrow 0$$

where f is the map of sheaves associated to the Verschiebung $\mathbb{V} : \mathbb{W}_n \mapsto \mathbb{W}_{n+1}$ and $g = r'_{2,\overline{K}} \circ r'_{3,\overline{K}} \circ ... \circ r'_{n+1,\overline{K}}$.

Proof : Firstly, let us fix an object $(\mathcal{U}, \mathcal{W})$ of $\mathfrak{X}$ and denote by $S = \bar{\mathcal{O}}_{\mathfrak{X}}(\mathcal{U}, \mathcal{W})$.

For $(s_0, s_1, ..., s_{n-1}) \in \mathbb{W}_n(S/pS)$, since $(r_2 \circ r_3 \circ ...r_{n+1})(0, s_0, ..., s_{n-1}) = (r_2 \circ r_3 \circ ...r_n)(0, s_0^p, ..., s_{n-2}^p) = ... = (r_2 \circ r_3)(0, s_0^{p^{n-2}}, s_1^{p^{n-2}}) = r_2(0, s_0^{p^{n-1}}) = 0$, one obtains that $g \circ f = 0$.

In order to check the exactness in the middle it remains to show that $\ker(g) \subseteq \mathrm{Im}(f)$:

For this we consider the exact sequence of sheaves:

$0 \to \mathbb{W}_n \to \mathbb{W}_{n+1} \to \mathbb{W}_1 \to 0$ where the first map is the map of sheaves associated to the Verschiebung and the second one is the natural projection. By tensoring it with $A_{\max}/p^{n+1}A_{\max}$ over W_{n+1}, we obtain the exact sequence:

$$A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_n \to A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_{n+1} \to A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_1 \to 0$$

One further identifies $A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_1$ with the appropriate cokernel and also proves the exactness on the left. Since tensoring is right exact, the exactness in the middle for the sequence displayed in the statement of the lemma follows.

Let us prove the surjectivity of g. Denote by $h : \mathbb{W}_{n+1} \mapsto \mathbb{W}_1 = \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ the natural projection and by h' the induced map of sets $\mathbb{W}_{n+1}(S/pS) \xrightarrow{h'} \mathbb{W}_1(S/pS)$ sending $(s_0, s_1, ..., s_n)$ to s_0. Since $\ker(h') = \{(s_0, s_1, ..., s_n) \in (S/pS)^{n+1}/s_0 = 0\} \cong \mathbb{W}_n(S/pS) = (S/pS)^n$, it is clear that $\ker(h)$ is identified with $\mathbb{W}_n$ via Verschiebung. Note that $\ker(h)$ is a W_{n+1}-module via the projection map $W_{n+1} \mapsto W_n$ composed with Frobenius on W_n and since $\mathbb{W}_n$ is a W_n-module. We obtain that

$$A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \ker(h) \cong A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_n.$$

Since $h'(\xi_{n+2}) = h'(\tilde{p}_{n+2} - p) \equiv p^{1/p^{n+1}} \pmod{p}$, it follows that

$$A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_1 \cong \bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p^{n+1}}\bar{\mathcal{O}}_{\mathfrak{X}}[\delta] \tag{5}$$

Now, since $S = \bar{\mathcal{O}}_{\mathfrak{X}}(\mathcal{U}, \mathcal{W})$ is a normal ring, Frobenius to the n-th power $\varphi^n : S/p^{1/p^n}S \to S/pS$ is injective. On the other hand, by [2, Lemma 4.4.1 (v)], Frobenius on $\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ is surjective with kernel $p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ hence we have an isomorphism $\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}} \cong \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$. Consequently, Frobenius to the nth power on $\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ is surjective with kernel $p^{1/p^n}\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ hence we have an isomorphism

$$\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p^n}\bar{\mathcal{O}}_{\mathfrak{X}} \cong \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}. \tag{6}$$

From (5) and (6), one obtains that

$$A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_1 \cong \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}[\delta].$$

Since $\varphi^n \circ h = r_2 \circ r_3 \circ ... \circ r_{n+1} : \mathbb{W}_{n+1} \mapsto \mathbb{W}_1 = \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ is surjective, after tensoring with $A_{\max}/p^{n+1}A_{\max}$ over W_{n+1}, and since tensoring is right exact, we obtain a surjective map $\mathbb{A}'^{\nabla}_{\max,n+1} \xrightarrow{g} (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})[\delta] \cong (\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]$ where the last isomorphism follows from (6).

Also by (6) it follows that $(\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]/(p^{1/p}) \cong (\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]$, in other words $\mathbb{A}'^{\nabla}_{\max,1} \cong (\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]$ and so the right exactness of the displayed sequence is proved.

Now we need to prove the left exactness of our sequence. We will show that it is left exact on stalks. For this, let x be a point of X. Recall that $A'_{\max,n} = W_n[\delta]/(p\delta - \mathrm{pr}_n(\xi_{n+1}))$. Since $\frac{\xi_{n+1}}{p} = \frac{\tilde{p}_{n+1}}{p} - 1$, we have that $A'_{\max,n} \cong W_n[\delta]/(p\delta - \mathrm{pr}_n(\tilde{p}_{n+1}))$ where for the latest isomorphism we use the ring isomorphism: $W_n[\delta] \cong W_n[\delta+1]$ induced by $\delta \mapsto \delta + 1$.

Define $B := W_n(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})[\delta]$, and similarly, denote by $C := W_{n+1}(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})[\delta]$ and by $D := (\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})[\delta]$.

Let us remark that $B/(p\delta - \tilde{p}_{n+1})B$ is the stalk $\mathbb{A}'^{\nabla}_{\max,n,x}$ of $\mathbb{A}'^{\nabla}_{\max,n}$ at x, that $C/(p\delta - \tilde{p}_{n+2})C$ is the stalk $\mathbb{A}'^{\nabla}_{\max,n+1,x}$ of $\mathbb{A}'^{\nabla}_{\max,n+1}$ at x and that $D/\tilde{p}_{n+2}D$ is the stalk $\mathbb{A}'^{\nabla}_{\max,1,x}$ of $\mathbb{A}'^{\nabla}_{\max,1}$ at x ($\mathbb{A}'^{\nabla}_{\max,1,x} = D/p^{1/p}D \cong D/\tilde{p}_{n+2}D$ by using the isomorphism from (6)) and note that for the easiness of reading by the end of the section we don't carry further the projection maps $\mathrm{pr}_n(\tilde{p}_{n+1})$, $\mathrm{pr}_{n+1}(\tilde{p}_{n+2})$ and $\mathrm{pr}_1(\tilde{p}_{n+2})$ respectively.

The following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{f_x} & C & \xrightarrow{s_x} & D & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p\delta - \tilde{p}_{n+1}} & & \downarrow{\scriptstyle p\delta - \tilde{p}_{n+2}} & & \downarrow{\scriptstyle -\tilde{p}_{n+2}} & & \\
0 & \longrightarrow & B & \xrightarrow{f_x} & C & \xrightarrow{s_x} & D & \longrightarrow & 0
\end{array}$$

where f_x is the map sending $\delta \mapsto \delta$ and inducing the Verschiebung $W_n(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x}) \mapsto W_{n+1}(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$ and s_x is the natural projection.

Since the Verschiebung is injective and since B (respectively C) is a free $W_n(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$-module (respectively $W_{n+1}(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$-module), one obtains that the map f_x is injective. Also D is a free $\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x}$-module and the rows in the above diagram are exact.

Let us check now the commutativity of the two squares.

For the first square diagram, since $\delta \mapsto \delta$ it is enough to verify the commutativity on coefficients. Let $s \in W_n(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$, $s = (s_0, s_1, ..., s_{n-1})$. We have that $\tilde{p}_{n+1} \cdot s = (p^{1/p^n}s_0, p^{1/p^{n-1}}s_1, ..., p^{1/p}s_{n-1})$ and since $\tilde{p}_{n+2} \cdot V(s) = (0, p^{1/p^n}s_0, ..., p^{1/p}s_{n-1})$, one obtains that $V(\tilde{p}_{n+1} \cdot s) = \tilde{p}_{n+2} \cdot V(s)$. The composition of the maps on the left lower side of the first square diagram will then be $V(p\delta s - \tilde{p}_{n+1}s) = p\delta V(s) - \tilde{p}_{n+2} \cdot V(s) = (p\delta - \tilde{p}_{n+2})V(s)$, which is exactly what the composition of the maps on the right upper side gives us. We obtain that the first square diagram is commutative. Similarly, for the second one, if $t \in W_{n+1}(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$, $t = (t_0, t_1, ..., t_n)$, then:

$$\begin{array}{ccc}
(t_0, t_1, ..., t_n) & \xmapsto{\quad s_x \quad} & t_0 \\
{\scriptstyle p\delta - \tilde{p}_{n+2}}\downarrow & \equiv & \downarrow{\scriptstyle -\tilde{p}_{n+2}} \\
(p\delta - \tilde{p}_{n+2}) \cdot t & \xmapsto{s_x} & -\tilde{p}_{n+2}t_0 = -p^{1/p^{n+1}}t_0
\end{array}$$

With the same type of argument as for the first square diagram we conclude that the second one is commutative.

Note that the sequence of cokernels $B/(p\delta-\tilde{p}_{n+1})B \mapsto C/(p\delta-\tilde{p}_{n+2})C$ is the map on stalks associated to f. We want to prove its injectivity. By the Snake Lemma in the main diagram this is equivalent to showing that the kernel of the multiplication by $p\delta-\tilde{p}_{n+2}$ on C surjects onto the kernel of the multiplication by $-\tilde{p}_{n+2}$ on D. Let's remark that $\tilde{p}_{n+2}=p^{1/p^{n+1}}$ in $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ and that, since $\cdot p$ itself kills D, the kernel of the multiplication by $p^{1/p^{n+1}}$ on D is $p\cdot p^{-1/p^{n+1}}D=p^{\frac{p^{n+1}-1}{p^{n+1}}}D=\tilde{p}_{n+2}^{p^{n+1}-1}D$. Take now $v\in D$ (so in particular $p^{\frac{p^{n+1}-1}{p^{n+1}}}\cdot v\in\ker(\cdot p^{1/p^{n+1}})$) and let $x\in C$ be the lift of v under s_x defined by taking the Teichmueller lifts of the coefficients of x with respect to a $\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x}$-basis of D. Define $u:=\sum_{i=0}^{p^{n+1}-1}p^i\delta^i\tilde{p}_{n+2}^{p^{n+1}-i-1}v$. We have that:

$$\begin{aligned}(p\delta-\tilde{p}_{n+2})u &= \textstyle\sum_{i=0}^{p^{n+1}-1}p^{i+1}\delta^{i+1}\tilde{p}_{n+2}^{p^{n+1}-i-1}v-\sum_{i=0}^{p^{n+1}-1}p^i\delta^i\tilde{p}_{n+2}^{p^{n+1}-i}v\\ &=\delta^{p^{n+1}}p^{p^{n+1}}v-\tilde{p}_{n+2}^{p^{n+1}}v=0\end{aligned}$$

since $\delta^{p^{n+1}}p^{p^{n+1}}v\equiv 0 \pmod p$ and $\tilde{p}_{n+2}^{p^{n+1}}v=p\cdot v=0$ on D.

On the other hand, $s_x(u)=p^0\delta^0\tilde{p}_{n+2}^{p^{n+1}-1}\cdot v=\tilde{p}_{n+2}^{p^{n+1}-1}\cdot v=p^{\frac{p^{n+1}-1}{p^{n+1}}}\cdot v$ hence the kernel of the multiplication by $p\delta-\tilde{p}_{n+2}$ on C surjects onto the kernel of the multiplication by $-\tilde{p}_{n+2}$ on D which is what we wanted. One uses further Snake Lemma in the main diagram. □

Consider now the map of sheaves $u_{n,\overline{K}}:\mathbb{A}^{\nabla}_{\max,n+1}\to\mathbb{A}'^{\nabla}_{\max,n}$ associated to the map of pre-sheaves induced by $u_n:A_{\max,n+1}\to A_{\max}/p^nA_{\max}$ (defined in §2.1) and by the natural projection $\mathbb{W}_{n+1}(\mathcal{U},\mathcal{W})\to\mathbb{W}_n(\mathcal{U},\mathcal{W})$.

Also consider the map of sheaves

$$q'_{n,\overline{K}}:\mathbb{A}'^{\nabla}_{\max,n}\to\mathbb{A}^{\nabla}_{\max,n}$$

associated to the map of pre-sheaves induced by $q'_n:A_{\max}/p^nA_{\max}\to A_{\max,n}$ (defined as well in §2.1) and by Frobenius $\mathbb{W}_n(\mathcal{U},\mathcal{W})\to\mathbb{W}_n(\mathcal{U},\mathcal{W})$.

Write $q'_{\overline{K}}:=\{q'_{n,\overline{K}}\}_n:\mathbb{A}'^{\nabla}_{\max}\to\mathbb{A}^{\nabla}_{\max}$ and $u_{\overline{K}}:=\{u_{n,\overline{K}}\}_n:\mathbb{A}^{\nabla}_{\max}\to\mathbb{A}'^{\nabla}_{\max}$.

In order to conclude the comparison between $\mathbb{A}'^{\nabla}_{\max,n}$ and $\mathbb{A}^{\nabla}_{\max,n}$ let us prove the following:

Lemma 2.9 — For any positive integers $m\geq n+2$ we have an isomorphism of rings $A_{\max}/p^nA_{\max}\cong A_{\max,m}/p^nA_{\max,m}$ and the map $u_{n,\overline{K}}\circ r_{n+2,\overline{K}}\circ\ldots\circ r_{m,\overline{K}}:\mathbb{A}^{\nabla}_{\max,m}\to\mathbb{A}'^{\nabla}_{\max,n}$ induces an isomorphism $\mathbb{A}^{\nabla}_{\max,m}/p^n\mathbb{A}^{\nabla}_{\max,m}\cong\mathbb{A}'^{\nabla}_{\max,n}$.

Proof : We defined at the beginning of the chapter the surjective maps q_m and the reduction π_m. Their composition is the surjective map

$$q_m \circ \pi_m : \mathbb{W}(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \twoheadrightarrow \mathbb{W}_m(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \twoheadrightarrow \mathbb{W}_m(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$$

sending $(s_0, s_1, ...)$ to $(s_0^{(m-1)} \pmod p, ..., s_{m-1}^{(m-1)} \pmod p)$, which induces the surjection:

$$\mathbb{W}(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))\{\delta\} \twoheadrightarrow \mathbb{W}_m(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})[\delta] = W_m[\delta]$$

defined by $\sum_{i\geq 0} a_i\delta^i \to \sum_{i\geq 0} \bar{a}_i\delta^i$, where $\bar{a}_i = (q_m \circ \pi_m)(a_i) = q_m(a_i \bmod p^m)$. Further we get a surjective map $\psi_m : A_{\max} \to A_{\max,m}$ and for any integers $m \geq n+2$, $\psi_m(p^n A_{\max}) = p^n A_{\max,m}$ since $\psi_m(p^n \sum_{i\geq 0} a_i\delta^i) = \bar{p^n} \sum'_{i\geq 0} \bar{a}_i\delta^i = p^n \sum'_{i\geq 0} \bar{a}_i\delta^i$ the last sum being finite since the sequence $(a_i)_i$ converges to 0 for the p-adic topology (for the last equality remark that $q_m(p^n \bmod p^m) = (0, ..., 0, 1, 0, ..., 0) \in W_m$ for $m \geq n+2$). The second isomorphism theorem for rings gives us now:

$$A_{\max}/p^n A_{\max} \cong A_{\max,m}/p^n A_{\max,m}.$$

Remark that the finiteness of the sum appears since $a_i \to 0$ in the strong topology of $\mathbb{W}(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$ (the p-adic topology).

One can write $\cdot p$ on $\mathbb{W}_m$ as $\mathbb{V} \circ \varphi$ where $\mathbb{V}$ is the Verschiebung and φ Frobenius. Recall that φ is surjective on $\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ by [2, Lemma 4.4.1(v)]. As in Lemma 2.8 we get an isomorphism $\mathbb{W}_m/p^n\mathbb{W}_m \cong \mathbb{W}_n$ induced by the natural projection on the first n components. One obtains that, via this identification, the map $u_n \circ r_{n+2} \circ ... \circ r_m : \mathbb{W}_m \to \mathbb{W}_n$ is φ^{m-n-1} and that at the level of rings sends $\xi_m \in W_m$ to $\mathrm{pr_n}(\xi_{n+1}) \in W_n$.

We have that $(V^s(\tilde{p}_m))^{p^n} = p^{sp^n} \cdot \tilde{p}_m^{p^{n-s}} = p^{sp^n + p^{n-s}} \frac{\tilde{p}_m^{p^{n-s}}}{p^{p^{n-s}}} = 0$ in $A_{\max,m}/p^n A_{\max,m}$ since $sp^n + p^{n-s} \geq n$.

Now, $\tilde{p}_m^{p^n} \pmod p$ generates the kernel of φ^{m-n-1} on $\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$. On one hand, $\varphi^{m-n-1}(\tilde{p}_m^{p^n} \pmod p) = (p) = 0$ on S/pS (recall that $S = \bar{\mathcal{O}}_{\mathfrak{X}}(\mathcal{U}, \mathcal{W})$). For the other inclusion let $x \in \ker(\varphi^{m-n-1})$ so $x^{p^{m-n-1}} = p \cdot y$ for some $y \in S$. Since S is normal it follows that $x = p^{1/p^{m-n-1}} \cdot y'$, $y' \in S$, hence $x \in (\tilde{p}_m^{p^n})$. We obtain that $\{\mathbb{V}^s(\tilde{p}_m^{p^n})\}_{0\leq s\leq n}$ generates the kernel of φ^{m-n-1} on $\mathbb{W}_n$.

Similarly it follows that $W_m/p^n W_m \cong W_n$ and that $\{V^s(\tilde{p}_m^{p^n})\}_{0\leq s\leq n}$ generates the kernel of φ^{m-n-1} on W_n.

Let us prove now that $p^n\mathbb{A}^{\nabla}_{\max,m} = \ker(u_{n,\overline{K}} \circ r_{n+2,\overline{K}} \circ ... \circ r_{m,\overline{K}})$.

Firstly, let $x \otimes_{W_m} y \in A_{\max,m} \otimes_{W_m} \mathbb{W}_m(\mathcal{U}, \mathcal{W})$. Since $p^n \in W_m$ we have $p^n(x \otimes_{W_m} y) = p^n x \otimes_{W_m} y = x \otimes_{W_m} p^n y \in \ker(u_{n,\overline{K}} \circ r_{n+2,\overline{K}} \circ ... \circ r_{m,\overline{K}})$ clearly.

Secondly, let $\sum_i x_i \otimes_{W_m} y_i \in \ker(u_{n,\overline{K}} \circ r_{n+2,\overline{K}} \circ ... \circ r_{m,\overline{K}})$. The element $\sum_i x_i \otimes_{W_m} y_i$ is mapped to $\sum_i \bar{x}_i \otimes_{W_n} \mathrm{pr_n}(y_i) = 0 \in A_{\max,m}/p^n A_{\max,m} \otimes_{W_n} \mathbb{W}_n(\mathcal{U}, \mathcal{W})$ (here we use the isomorphism $A_{\max}/p^n A_{\max} \cong A_{\max,m}/p^n A_{\max,m}$). We conclude that $\sum_i x_i \otimes_{W_m} y_i \in p^n(A_{\max,m} \otimes_{W_m} \mathbb{W}_m(\mathcal{U}, \mathcal{W}))$ and so the second inclusion also holds. The second claim of the Lemma follows.

2.5 *Localization over small affines*

We study now the localization of $\mathbb{A}^{\nabla}_{\max}$ over small affines.

Let $\mathcal{U}$=Spf$(R_{\mathcal{U}})$ be a small affine open of the étale site X^{et} on X. This is an object such that $R_{\mathcal{U}} \otimes_{\mathcal{O}_K} k$ is geometrically irreducible over k and there are parameters $T_1, T_2, ..., T_d \in R_{\mathcal{U}}^{\times}$ such that the map $R_0 := \mathcal{O}_K\{T_1^{\pm 1}, T_2^{\pm 1}, ..., T_d^{\pm 1}\} \subset R_{\mathcal{U}}$ is formally étale. Fix now an algebraic closure Ω of the fraction field of $X_{\overline{K}}$ and denote by $\overline{R}_{\mathcal{U}}$ the union of all $R_{\mathcal{U}} \otimes_{\mathcal{O}_K} \overline{K}$-subalgebras S of Ω, such that S is normal and $R_{\mathcal{U}} \otimes_{\mathcal{O}_K} \overline{K} \subset S[1/p]$ is finite and étale.

Let $\mathcal{R}(\overline{R}_{\mathcal{U}}) := \varprojlim \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ where the transition maps are given by Frobenius.

We define $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ to be the p-adic completion of the sub-$\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))$-algebra of $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))[\frac{1}{p}]$ generated by $p^{-1}\ker(\vartheta)$ where the map ϑ is defined as follows:

For every n, let ϑ_n be the composition of the projection (reduction modulo p^n map): $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))$, of the map $\mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})$ induced by the projection $\mathcal{R}(\overline{R}_{\mathcal{U}}) = \varprojlim \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} \to \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ on the nth component (see Proposition 2.1) and of $\theta_n : \mathbb{W}_n(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}) \to \overline{R}_{\mathcal{U}}/p^n\overline{R}_{\mathcal{U}}$ (defined at the beginning of §2.1).

Then define $\vartheta : \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \widehat{\overline{R}}_{\mathcal{U}} = \varprojlim \overline{R}_{\mathcal{U}}/p^n\overline{R}_{\mathcal{U}}$ to be the map $x \to \varprojlim \vartheta_n(x)$.

In [4, §6] it is proved that $\ker(\vartheta)$ is the principal ideal generated by ξ. We also have a Frobenius φ on $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ induced by the Frobenius on $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))$. Remark that if $x \in \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))$ belongs to $\ker(\vartheta)$ and if $n \in \mathbb{N}_{>0}$, one can write $x^{[n]} = p^{[n]}(x/p)^n \in A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ (where $x^{[n]}$ is the n-th divided power of x i.e. $x^n/n!$) and hence there exists a natural homomorphism $A^{\nabla}_{\mathrm{cris}}(\overline{R}_{\mathcal{U}}) \to A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ (which is injective according to [5, Proposition 2.3.2]). $A^{\nabla}_{\mathrm{cris}}(\overline{R}_{\mathcal{U}})$ is the p-adic completion of the $\mathbb{W}(k)$-DP envelope of $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))$ with respect to the kernel of the map ϑ defined above (see [1, §2.3] or [4, §6] for details).

Note that ϑ makes sense since the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) & \xrightarrow{\vartheta_{n+1}} & \overline{R}_{\mathcal{U}}/p^{n+1}\overline{R}_{\mathcal{U}} \\ & \searrow^{\vartheta_n} & \downarrow (\bmod p^n) \\ & & \overline{R}_{\mathcal{U}}/p^n\overline{R}_{\mathcal{U}}. \end{array}$$

Let g_n be the composite of the projection (reduction modulo p^n map) $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))$ and of the map $v_n : \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})$ induced by the projection $\mathcal{R}(\overline{R}_{\mathcal{U}}) = \varprojlim \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} \to \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ on the $n+1$th component (similar to q_n). Denote by $R := \mathcal{R}(\overline{R}_{\mathcal{U}})$. Since $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) = \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))[\delta]/(p\delta - \xi)$ we have that:

$$\begin{aligned} A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) &\cong \frac{\mathbb{W}(R)[\delta]/(p\delta-\xi)}{(p^n, p\delta-\xi)\mathbb{W}(R)[\delta]/(p\delta-\xi)} \\ &\cong \frac{\mathbb{W}(R)[\delta]/p^n\mathbb{W}(R)[\delta]}{(p^n, p\delta-\xi)\mathbb{W}(R)[\delta]/p^n\mathbb{W}(R)[\delta]} \\ &\cong \mathbb{W}_n(R)[\delta]/(p\delta - \xi \pmod{p^n}) \end{aligned} \tag{7}$$

so $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))[\delta]/(p\delta - \xi \pmod{p^n}))$ and since $g_n(\xi) = \xi_{n+1}$, we get a map $g'_n : A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}'^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}}) = A_{\max}/p^n A_{\max} \otimes_{W_n} (\mathbb{W}_n(\overline{R}_{\mathcal{U}}))$ and recall that $\mathbb{W}_n := W_n(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})$ is the sheaf $(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})^n$.

We have the following important result:

Proposition 2.10 — The ring $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ is p-torsion free.

PROOF : We observe that $\mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))[\xi/p]$ has no p-torsion being a subring of $\mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))[p^{-1}]$. Consequently, its p-adic completion namely $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ is p-torsion free. □

We will use this result in the proof of Theorem 1.2:

PROOF : a) We have that $\overline{R}_{\mathcal{U}}$ is a normal ring and that Frobenius is surjective on $\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ by [4, Proposition. 2.0.1] and as in the proof of Proposition 2.1 we get that the kernel of the projection $\mathcal{R}(\overline{R}_{\mathcal{U}}) = \varprojlim \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} \to \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ on the $n+1$-th component is generated by $\tilde{p}^{p^n}$.

As in the proof of Lemma 2.9 we have that $(V^s([\tilde{p}]))^{p^n} = (p^{sp^n}[\tilde{p}])^{p^{n-s}} = p^{(1+sp^n)p^{n-s}} \frac{[\tilde{p}]^{p^{n-s}}}{p^{p^{n-s}}}$ $= 0$ in $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$, $0 \leq s \leq n$. Now, via Proposition 2.2, we obtain that $\{\mathbb{V}^s([\tilde{p}])\}^{p^n}_{0\leq s\leq n}$ generate the kernel of v_n. Via Proposition 2.4, it follows that:

$$A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{W}_n(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})[\delta]/(p\delta - \mathrm{pr}_n(\xi_{n+1})) \tag{8}$$

where the isomorphism is induced by the map $g_n : \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))$.

We prove a) by induction on n. For $n=1$ the map

$$A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})/pA_{\max}^{\nabla}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}'^{\nabla}_{\max,1}(\overline{R}_{\mathcal{U}}) \text{ becomes}$$
$$(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})[\delta]/(p\delta - \mathrm{pr}_1(\xi_2)) \to ((\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})(\overline{R}_{\mathcal{U}}))[\delta]/(\mathrm{pr}_1(\xi_2))$$

via the above isomorphism and the remark before Lemma 2.8. By using now [1, Proposition 2.13] and [1, Proposition 2.14] we have an injective map

$$\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} = \bar{\mathcal{O}}_{\mathfrak{X}}(\overline{R}_{\mathcal{U}})/p\bar{\mathcal{O}}_{\mathfrak{X}}(\overline{R}_{\mathcal{U}}) \to (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})(\overline{R}_{\mathcal{U}}) \text{ hence}$$
$$(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})[\delta]/(p^{1/p}) \to ((\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})(\overline{R}_{\mathcal{U}}))[\delta]/(p^{1/p})$$ is injective and so the case $n=1$ is proved.

By Proposition 2.10, $A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})$ has no p-torsion hence we have the exact sequence:

$$0 \longrightarrow \frac{A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})}{p^n A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})} \xrightarrow{\cdot p} \frac{A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})}{p^{n+1} A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})} \longrightarrow \frac{A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})}{p A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})} \longrightarrow 0$$

This is compatible with the exact sequence obtained by taking the localizations in the exact sequence of Lemma 2.8 i.e. we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})}{p^n A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})} & \xrightarrow{\cdot p} & \frac{A_{max}^{\nabla}(\overline{R}_{\mathcal{U}})}{p^{n+1} A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})} & \longrightarrow & \frac{A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})}{p A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})} & \longrightarrow & 0 \\ & & \downarrow g'_n & & \downarrow g'_{n+1} & & \downarrow g'_1 & & \\ 0 & \longrightarrow & \mathbb{A}'^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}}) & \xrightarrow{f'} & \mathbb{A}'^{\nabla}_{\max,n+1}(\overline{R}_{\mathcal{U}}) & \xrightarrow{g'} & \mathbb{A}'^{\nabla}_{\max,1}(\overline{R}_{\mathcal{U}}) & \longrightarrow & 0 \end{array} \tag{9}$$

where the maps $f' = f_{\overline{R}_{\mathcal{U}}}$ and $g' = g_{\overline{R}_{\mathcal{U}}}$ are induced by f and g respectively (see Lemma 2.8).

The second square diagram of the main one is commutative since:

$$\begin{array}{ccc} \sum b_i(\frac{\xi}{p})^i \pmod{p^{n+1}} & \longmapsto & \sum b_i(\frac{\xi}{p})^i \pmod{p} \\ \downarrow g'_{n+1} & \equiv & \downarrow g'_1 \\ \sum b_i(\frac{\mathrm{pr}_{n+1}(\xi_{n+2})}{p})^i \pmod{p^{n+1}} \otimes 1 & \longmapsto & \sum b_i(\frac{\mathrm{pr}_1(\xi_2)}{p})^i \pmod{p} \otimes 1 \end{array}$$

where the bottom map is induced by Frobenius to the n-th power φ^n composed with the projection and we have that $(proj \circ \varphi^n)(\mathrm{pr}_{n+1}(\xi_{n+2})) = \mathrm{pr}_1(\xi_2)$ and for the vertical maps we use the fact that $g'_n(\xi \pmod{p^n}) = \mathrm{pr}_n(\xi_{n+1})$.

The first square diagram of the main one is also commutative since:

$$\begin{array}{ccc}
\sum b_i(\frac{\xi}{p})^i \pmod{p^n} & \xrightarrow{\cdot p} & \sum p\cdot b_i(\frac{\xi}{p})^i \pmod{p^{n+1}} \\
\downarrow g'_n & \equiv & \downarrow g'_{n+1} \\
\sum b_i(\frac{\mathrm{pr}_n(\xi_{n+1})}{p})^i \pmod{p^n} & \xrightarrow{f'} & \sum p\cdot b_i(\frac{\mathrm{pr}_{n+1}(\xi_{n+2})}{p})^i \pmod{p^{n+1}}.
\end{array}$$

For the commutativity of the above diagram one uses the fact that f' induces the Verschiebung at the level of the Witt vectors.

Now we apply the inductive hypothesis (g'_n injective) and use the Snake Lemma in the main diagram, (9), so at the level of kernels we get:

$0 \longrightarrow \ker(g'_{n+1}) \longrightarrow 0$ hence g'_{n+1} is injective (one can also see this directly by diagram chase). Claim a) follows.

b) We prove that for every $n\in\mathbb{N}^*$ we have $q'_{n,\overline{K}}\circ u_{n,\overline{K}} = r_{n+1,\overline{K}}$ and $u_{n,\overline{K}}\circ q'_{n+1,\overline{K}} = r'_{n+1,\overline{K}}$.

For the first relation, let's remark that the following diagram is commutative:

$$\begin{array}{ccc}
A_{\max,n+1}\otimes_{W_{n+1}}\mathbb{W}_{n+1}(\overline{R}_{\mathcal{U}}) & \xrightarrow{u_{n,\overline{K}}} & A_{\max,n}\otimes_{W_n}\mathbb{W}_n(\overline{R}_{\mathcal{U}}) \\
 & \searrow^{r_{n+1,\overline{K}}} & \downarrow q'_{n,\overline{K}} \\
 & & A_{\max}/p^nA_{\max}\otimes_{W_n}\mathbb{W}_n(\overline{R}_{\mathcal{U}})
\end{array}$$

since
$$\begin{array}{ccc}
\xi_{n+1}\otimes_{W_{n+1}}1 & \xrightarrow{u_{n,\overline{K}}} & pr_n(\xi_{n+1})\otimes_{W_n}1 \\
 & \searrow^{r_{n+1,\overline{K}}} & \downarrow q'_{n,\overline{K}} \\
 & & \xi_n\otimes_{W_n}1
\end{array}
\quad \text{and also} \quad
\begin{array}{ccc}
(s_0,s_1,...,s_n) & \xrightarrow{u_n} & (s_0,s_1,...,s_{n-1}) \\
 & \searrow^{r_{n+1}} & \downarrow q'_n \\
 & & (s_0^p,s_1^p,...,s_{n-1}^p)
\end{array}$$

For the second relation, we obtain similarly that the following diagram is commutative:

$$\begin{array}{ccc}
A_{\max}/p^{n+1}A_{\max}\otimes_{W_{n+1}}\mathbb{W}_{n+1}(\overline{R}_{\mathcal{U}}) & \xrightarrow{q'_{n+1,\overline{K}}} & A_{\max,n+1}\otimes_{W_{n+1}}\mathbb{W}_{n+1}(\overline{R}_{\mathcal{U}}) \\
 & \searrow^{r'_{n+1,\overline{K}}} & \downarrow u_{n,\overline{K}} \\
 & & A_{\max}/p^nA_{\max}\otimes_{W_n}\mathbb{W}_n(\overline{R}_{\mathcal{U}}).
\end{array}$$

By taking now $\varprojlim$, the two above mentioned relations give us: $q'_{\overline{K}}\circ u_{\overline{K}} = id$ and $u_{\overline{K}}\circ q'_{\overline{K}} = id$ respectively. Claim b) follows; $u_{\overline{K}}$ defines the inverse of $q'_{\overline{K}}$. □

Corollary 2.11 — The induced map $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \longrightarrow \mathbb{A}^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) = \varprojlim \mathbb{A}^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}})$ is an isomorphism.

PROOF : From [1, Lemma 2.17] the image of the map $(\bar{\mathcal{O}}_{\mathfrak{X}}/p^{n+1}\bar{\mathcal{O}}_{\mathfrak{X}})(\overline{R}_{\mathcal{U}}) \to (\bar{\mathcal{O}}_{\mathfrak{X}}/p^{n}\bar{\mathcal{O}}_{\mathfrak{X}})(\overline{R}_{\mathcal{U}})$ factors via $\overline{R}_{\mathcal{U}}/p^n\overline{R}_{\mathcal{U}} \subset (\bar{\mathcal{O}}_{\mathfrak{X}}/p^{n}\bar{\mathcal{O}}_{\mathfrak{X}})(\overline{R}_{\mathcal{U}})$. By using now the description we provided in (8) and since the transition maps $\mathbb{A}'^{\nabla}_{\max,n+1}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}'^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}})$ are induced by the map $\mathbb{W}_{n+1}(\overline{R}_{\mathcal{U}}) \to \mathbb{W}_{n}(\overline{R}_{\mathcal{U}})$ given by the the natural projection composed with Frobenius, we obtain that the maps $\mathbb{A}'^{\nabla}_{\max,n+1}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}'^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}})$ factor via $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^nA^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ for all $n \geq 1$. By taking projective limit and further using the fact that $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ is complete, one obtains that $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{A}'^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$. By Theorem 1.2b) we have the isomorphism $\mathbb{A}'^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{A}^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ and consequently we obtain that $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{A}^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$. □

3. THE SHEAF $\mathbb{A}_{\max}$

Let $p > 0$ be a prime integer, K a finite, unramified extension of $\mathbb{Q}_p$ with residue field k, $\mathcal{O}_K$ the ring of integers of K and denote by K^{unr} the maximal unramified subfield of $\overline{K}$ and by $\mathcal{O}_{K^{\mathrm{unr}}}$ its ring of integers.

We have a morphism $\theta_{\overline{K}} : \mathbb{A}^{+}_{\inf,\overline{K}} \to \hat{\bar{\mathcal{O}}}_{\mathfrak{X}_{\overline{K}}}$ of objects of $\mathrm{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$ constructed as follows: let $(\mathcal{U}, \mathcal{W})$ be an object of $\mathfrak{X}_{\overline{K}}$. Denote by $S = \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{U}, \mathcal{W})$ and for fixed $n \in \mathbb{N}$, consider the diagram of sets:

$$\begin{array}{ccc} (S/p^nS)^n & \xrightarrow{a_n} & S/p^nS \\ & \searrow^{b_n} & \uparrow{\scriptstyle \exists! c_n} \\ & & (S/pS)^n \end{array}$$

where b_n is the natural projection and $a_n(s_0, s_1, ..., s_{n-1}) := \sum_{i=0}^{n-1} p^i s_i^{p^{n-1-i}}$.

There exists a unique map of sets, call it $c_n : (S/pS)^n \to S/p^nS$ making the diagram commutative i.e. $c_n \circ b_n = a_n$.

We have that $c_n(s_0, s_1, ..., s_{n-1}) := \sum_{i=0}^{n-1} p^i \tilde{s_i}^{p^{n-1-i}}$, where $\tilde{s_i} \in S/p^nS$ is a lift of $s_i \in S/pS$ for all $0 \leq i \leq n-1$ and let us remark that c_n is well defined:

For this, let $(c_0, c_1, ..., c_n) \in (S/pS)^n$ such that $c_i \equiv s_i \pmod{p}$ for all $0 \leq i \leq n-1$. Then $c_i^{p^{n-1-i}} \equiv s_i^{p^{n-1-i}} \pmod{p^{n-i}}$ and by multiplying the latest relation by p^i we obtain that $p^i c_i^{p^{n-1-i}} \equiv p^i s_i^{p^{n-1-i}} \pmod{p^n}$ for all $0 \leq i \leq n-1$. It follows that $\sum_{i=0}^{n-1} p^i \tilde{c_i}^{p^{n-1-i}} \equiv \sum_{i=0}^{n-1} p^i \tilde{s_i}^{p^{n-1-i}} \pmod{p^n}$, in other words c_n is well defined.

The map c_n induces a ring homomorphism $c_{n,(\mathcal{U},\mathcal{W})} : \mathbb{W}_n(S/pS) \to S/p^nS$, which is functorial in $(\mathcal{U}, \mathcal{W})$, in other words a morphism of presheaves $\mathbb{W}_{n,\overline{K}} \xrightarrow{c_n} \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p^n\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$. One denotes by $\theta_{n,\overline{K}}$ the induced morphism on the associated sheaves and let:

$$\theta_{\overline{K}} := \{\theta_{n,\overline{K}}\} : \mathbb{A}^+_{\mathrm{inf},\overline{K}} = \varprojlim \mathbb{W}_{n,\overline{K}} \to \hat{\bar{\mathcal{O}}}_{\mathfrak{X}_{\overline{K}}} = \varprojlim (\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p^n\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})$$

Assume that X is a smooth scheme over $\mathcal{O}_K$.

Let $\mathcal{O}_X$ be the sheaf on the site $\mathfrak{X}_{\overline{K}}$ defined by $\mathcal{O}_X(\mathcal{U},\mathcal{W}) := \mathcal{O}_X(\mathcal{U})$.

For every $n \geq 1$ one defines the sheaf $\mathbb{W}_{X,n,\overline{K}} := \mathbb{W}_n(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}) \otimes_{\mathcal{O}_K} \mathcal{O}_X$ of $\mathcal{O}_{K^{\mathrm{unr}}}$-algebras and also the morphism of sheaves of $\mathcal{O}_{K^{\mathrm{unr}}} \otimes_{\mathcal{O}_K} \mathcal{O}_X$-algebras $\theta_{X,n,\overline{K}} : \mathbb{W}_{X,n,\overline{K}} \to \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p^n\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$ associated to the following map of presheaves: firstly take an object $(\mathcal{U},\mathcal{W})$ of $\mathfrak{X}_{\overline{K}}$ such that $\mathcal{U}$=Spf($R_{\mathcal{U}}$) is affine (i.e. $R_{\mathcal{U}} = \mathcal{O}_X(\mathcal{U},\mathcal{W})$). Clearly $S = \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{U},\mathcal{W})$ has a natural $R_{\mathcal{U}}$-algebra structure. Define now:

$$\theta_{n,(\mathcal{U},\mathcal{W})} : \mathbb{W}_n(S/pS) \otimes_{\mathcal{O}_K} R_{\mathcal{U}} \to S/p^nS \text{ by } (x \otimes r) \to c_n(x)r.$$

Let now $\mathcal{U}$=Spf($R_{\mathcal{U}}$) be a small affine open of the étale site X^{et} of X, with parameters $T_1, T_2, ..., T_d \in R_{\mathcal{U}}^{\times}$ (recall the definition of small affines from the previous chapter). Further, for $n \geq 0$, let $R_{\mathcal{U},n} := R_{\mathcal{U}}[\zeta_n, T_1^{1/p^n}, ..., T_d^{1/p^n}]$, where $R_{\mathcal{U},0} = R_{\mathcal{U}}$, ζ_n is a primitive p^nth root of unity with $\zeta_{n+1}^p = \zeta_n$ and such that T_i^{1/p^n} is a fixed p^nth root of T_i in $\overline{R}_{\mathcal{U}}$ with $(T_i^{1/p^{n+1}})^p = T_i^{1/p^n}$ for any $1 \leq i \leq d$. Moreover, consider the category $\mathfrak{U}_{n,\overline{K}}$ consisting of morphisms $(\mathcal{V},\mathcal{W}) \to (\mathcal{U}, \mathrm{Spf}(R_{\mathcal{U},n}) \otimes_{\mathcal{O}_K[\zeta_n]} \overline{K})$ in $\mathfrak{X}_{\overline{K}}$. The morphisms of this category are the morphisms as objects over $(\mathcal{U}, \mathrm{Spf}(R_{\mathcal{U},n}) \otimes_{\mathcal{O}_K[\zeta_n]} \overline{K})$ and the covering families of an object $(\mathcal{V},\mathcal{W})$ are the covering families of $(\mathcal{V},\mathcal{W})$ regarded as object of $\mathfrak{X}_{\overline{K}}$. Given a sheaf $\mathcal{F}$ on $\mathfrak{X}_{\overline{K}}$, one writes $\mathcal{F}|_{\mathfrak{U}_{n,\overline{K}}}$ for $u_*(\mathcal{F})$ where $u : \mathfrak{U}_{n,\overline{K}} \to \mathfrak{X}_{\overline{K}}$ is the forgetful functor.

Let now $(\mathcal{V},\mathcal{W}) \in \mathfrak{U}_{n,\overline{K}}$ with $\mathcal{V} = \mathrm{Spf}(R_{\mathcal{V}})$ affine and let $S := \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{V},\mathcal{W})$. Remark that $T_i^{1/p^n} \in R_{\mathcal{U},n} \subset S$ for all $1 \leq i \leq d$ since S is the normalization of $\Gamma(\mathcal{V},\mathcal{O}_{\mathcal{V}}) = R_{\mathcal{V}}$ in $\Gamma(\mathcal{W},\mathcal{O}_{\mathcal{W}})$. Also denote by:

$$\widetilde{T}_i := ([T_i], [T_i^{1/p}], ..., [T_i^{1/p^n}], ...) \in \varprojlim \mathbb{W}_n(R_{\mathcal{U},n}/pR_{\mathcal{U},n})$$

the inverse limit being taken with respect to the map $\mathbb{W}_{n+1}(R_{\mathcal{U},n+1}/pR_{\mathcal{U},n+1}) \to \mathbb{W}_n(R_{\mathcal{U},n}/pR_{\mathcal{U},n})$ defined as the composition between the natural projection

$\mathbb{W}_{n+1}(R_{\mathcal{U},n+1}/pR_{\mathcal{U},n+1}) \to \mathbb{W}_n(R_{\mathcal{U},n+1}/pR_{\mathcal{U},n+1})$ and the map induced by the Frobenius: $R_{\mathcal{U},n+1}/pR_{\mathcal{U},n+1} \to R_{\mathcal{U},n}/pR_{\mathcal{U},n}$. Note that the image of $\tilde{T}_i$ in $\mathbb{W}_n(R_{\mathcal{U},n}/pR_{\mathcal{U},n})$ is $(T_i^{1/p^n}, 0, ..., 0)$ i.e. the Teichmueller lift of T_i^{1/p^n}. For all $1 \leq i \leq d$, define now:

$$X_i := 1 \otimes T_i - \widetilde{T}_i \otimes 1 \in \mathbb{W}_n(R_{\mathcal{U},n}/pR_{\mathcal{U},n}) \otimes_{\mathcal{O}_K} R_{\mathcal{U}}$$

and remark that these elements also live in $\mathbb{W}_n(S/pS)\otimes_{\mathcal{O}_K} R_{\mathcal{V}}$.

Recall now that $\xi_n = \tilde{p}_n - p = [p^{1/p^{n-1}}] - p$. We will need the following:

Lemma 3.1 ([1, Lemma 2.28]) — The kernel of the map $\theta_{n,(\mathcal{V},\mathcal{W})} : \mathbb{W}_n(S/pS)\otimes_{\mathcal{O}_K} R_{\mathcal{V}} \to S/p^nS$ is the ideal generated by $(\xi_n, X_1, ..., X_d)$.

3.1 *The existence of* $\mathbb{A}_{\max}$ *and its localization over small affines*

We are ready now to prove the main theorem of this section, Theorem 1.4, restated below:

***Theorem* 1.4** *There exists a unique continuous sheaf* $\mathbb{A}_{\max}$ *on* $\mathfrak{X}_{\overline{K}}$ *of* $\mathbb{A}^{\nabla}_{\max}$*-algebras such that for every small affine* $\mathcal{U} = Spec(R_{\mathcal{U}})$ *of* $X^{\rm et}$ *we have a canonical isomorphism as* $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$*-algebras:* $\mathbb{A}_{\max}(\overline{R}_{\mathcal{U}}) \cong A_{\max}(\overline{R}_{\mathcal{U}})$*. Here the algebra* $A_{\max}(\overline{R}_{\mathcal{U}})$ *is the one defined in [4, Remark 8.3.5]: the separated completion for the p-adic topology of* $\mathbb{W}(\underline{\mathcal{R}}(\overline{R}_{\mathcal{U}}))\otimes_{W(R)}\overline{R}_{\mathcal{U}}$*-subalgebra of* $\mathbb{W}(\underline{\mathcal{R}}(\overline{R}_{\mathcal{U}}))\otimes_{W(R)}\overline{R}_{\mathcal{U}}[p^{-1}]$ *generated by* $p^{-1}\cdot\ker(\theta_{\overline{R}_{\mathcal{U}}})$.

Proof Let us fix a small affine $\mathcal{U} = \mathrm{Spec}(R_{\mathcal{U}})$ and a choice of $\overline{R}_{\mathcal{U}}$. Let us now fix $n \geq 0$ and let us recall that we defined at the beginning of this section a certain category $\mathfrak{U}_{\overline{K},n}$. Fix $T_1, T_2, ..., T_d$ parameters of $R_{\mathcal{U}}$ let us recall that we have chosen for every $1 \leq i \leq d$ a compatible family of p-power roots $(T_i^{1/p^n})_{n=0}^{\infty}$ and also a compatible family of p-power roots on 1, $\varepsilon := (\zeta_n)_{n=0}^{\infty}$. With these choices let us recall that we have defined the elements $X_i := 1\otimes T_i - \widetilde{T}_i \otimes 1 \in \mathbb{W}_{X,n,\overline{K}}(\mathcal{V},\mathcal{W})$ for any $(\mathcal{V},\mathcal{W})$ in $\mathfrak{U}_{\overline{K},n}$. We define the presheaf $\mathcal{A}_{\mathcal{U},n}$ on $\mathfrak{U}_{\overline{K},n}$ by

$$(\mathcal{V},\mathcal{W}) \longrightarrow \mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W}) := \mathbb{W}_{X,n,\overline{K}}(\mathcal{V},\mathcal{W})[Y_0, Y_1, Y_2, ..., Y_d]/(pY_0 - \xi_n, pY_i - X_i)_{1\leq i\leq d},$$

for $(\mathcal{V},\mathcal{W})$ in $\mathfrak{U}_{\overline{K},n}$. If we denote by $y_1^{(n)}, y_2^{(n)}, ..., y_d^{(n)}$ the images of $Y_1, Y_2, ..., Y_d$ in $\mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W})$, let us remark that $\mathbb{A}^{\nabla}_{\max,n}(\mathcal{V},\mathcal{W}) \subset \mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W})$ and moreover we have that $\mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W}) = \mathbb{A}^{\nabla}_{\max,n}(\mathcal{V},\mathcal{W})[y_1^{(n)}, ..., y_d^{(n)}]$. In fact $\mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W})$ is a free $\mathbb{A}^{\nabla}_{\max,n}(\mathcal{V},\mathcal{W})$-module with basis the monomials in $y_1^{(n)}, y_2^{(n)}, ..., y_d^{(n)}$, therefore the presheaf $\mathcal{A}_{\mathcal{U},n}$ is in fact a sheaf on $\mathfrak{U}_{\overline{K},n}$.

Let us first remark that we have a natural morphism of $\mathcal{O}_K$-algebras:

$R_0 := \mathcal{O}_K[T_1^{\pm1}, T_2^{\pm1}, ..., T_d^{\pm1}] \longrightarrow \mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W})$ given by $T_i \mapsto \widetilde{T}_i\otimes 1 + X_i$, for $1 \leq i \leq d$. We remark that as $\widetilde{T}_i$ is a unit in $\mathbb{W}_n(\mathcal{O}_{\mathfrak{X}}/p\mathcal{O}_{\mathfrak{X}})(\mathcal{V},\mathcal{W})$ and as $X_i = py_i$ in $\mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W})$ and therefore nilpotent in that ring, it follows that $\widetilde{T}_i\otimes 1 + X_i \in \mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W})^{\times}$ and so the definition makes sense.

We extend the morphism $\theta_n : \mathbb{A}^{\nabla}_{\max,n}|_{\mathfrak{U}_{\overline{K},n}} \longrightarrow (\mathcal{O}_{\mathfrak{X}}/p^n\mathcal{O}_{\mathfrak{X}})|_{\mathfrak{U}_{\overline{K},n}}$ to a morphism

$\theta_{\mathcal{U},n}:\mathcal{A}_{\mathcal{U},n}\longrightarrow(\mathcal{O}_{\mathfrak{X}}/p^n\mathcal{O}_{\mathfrak{X}})|_{\mathfrak{U}_{\overline{K},n}}$ by sending $y_i^{(n)}$ to 0, for all $1\le i\le d$.

For each $(\mathcal{V},\mathcal{W})$ in $\mathfrak{U}_{\overline{K},n}$ we have a diagram of rings and ring homomorphisms

$$\begin{array}{ccc}\mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W}) & \stackrel{f_{n,1}}{\longrightarrow} & \mathcal{A}_{\mathcal{U},1}(\mathcal{V},\mathcal{W})\\ \uparrow & & \uparrow\\ R_0 & \longrightarrow & R_{\mathcal{V}}\end{array}$$

Let us recall that $\mathcal{A}_{\mathcal{U},1}(\mathcal{V},\mathcal{W})=\mathbb{A}^{\nabla}_{\max,1}(\mathcal{V},\mathcal{W})[y_1^{(1)},...,y_d^{(1)}]=(\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p\mathcal{O}_{\mathfrak{X}_{\overline{K}}})(\mathcal{V},\mathcal{W})[y_1^{(1)},...,y_d^{(1)}]$ and so the morphism $R_{\mathcal{V}}\longrightarrow\mathcal{A}_{\mathcal{U},1}$ in the diagram is the natural one. With this definition the diagram is commutative and moreover $\mathrm{Ker}(f_{n,1})$ is a nilpotent ideal of $\mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W})$. As $R_{\mathcal{V}}$ is étale over R_0, there is a unique R_0-morphism

$$R_{\mathcal{V}}\longrightarrow\mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W}),$$

making the two triangles commute and so we obtain a morphism of sheaves on $\mathfrak{U}_{\overline{K},n}$, $h_{\mathcal{U},n}:\mathbb{W}_{X,n,\overline{K}}|_{\mathfrak{U}_{\overline{K},n}}\longrightarrow\mathcal{A}_{\mathcal{U},n}$.

Now let us denote by $\mathfrak{U}_{\overline{K}}$ the full subcategory of $\mathfrak{X}_{\overline{K}}$ consisting of pairs $(\mathcal{V},\mathcal{W})$ such that the map $\mathcal{V}\longrightarrow X$ factors through $\mathcal{U}$. We endow $\mathfrak{U}_{\overline{K}}$ with the topology induced from $\mathfrak{X}$ and consider $\mathfrak{U}_{\overline{K},n}$ as a sub-topology of it. Our construction proceeds in several steps, as follows:

Step 1 : *The sheaf* $\mathcal{A}_{\mathcal{U},n}$ *on* $\mathfrak{U}_{\overline{K},n}$ *extends uniquely to a sheaf which we denote* $\mathbb{A}_{\max,\mathfrak{U},n}$ *on the whole of* $\mathfrak{U}_{\overline{K}}$.

For this let us fix an étale open $\mathcal{V}$ of X^{et} such that the structure map $\mathcal{V}\longrightarrow X$ factors through $\mathcal{U}$ and let $\mathcal{V}^{\mathrm{fet}}$ (respectively $\mathcal{V}_n^{\mathrm{fet}}$) denote the sub-site of $\mathfrak{U}_{\overline{K}}$ consisting of pairs $(\mathcal{V},\mathcal{W})$ (respectively consisting of pairs $(\mathcal{V},\mathcal{W})$ such that the structure map $\mathcal{W}\longrightarrow\mathcal{V}$ factors through $\mathrm{Spf}(R_{\mathcal{V},n})\otimes_{\mathcal{O}_K[\zeta_n]}K$. We recall that $R_{\mathcal{V},n}=R_{\mathcal{V}}[\zeta_n,T_1^{1/p^n},...,T_d^{1/p^n}])$.

To prove the claim it is enough to prove that the restriction of $\mathcal{A}_{\mathcal{U},n}$ to $\mathcal{V}_n^{\mathrm{fet}}$ extends uniquely to $\mathcal{V}^{\mathrm{fet}}$, for all $\mathcal{V}$ as above. Let $\Delta_{\mathcal{V}}:=\pi_1^{\mathrm{alg}}(\mathcal{V}_{\overline{K}},\overline{\eta})$, and by Δ_n its open subgroup of elements which fix $R_{\mathcal{V},n}$.

We have the following natural diagram of categories and functors:

$$\begin{array}{ccc}\mathrm{Sh}(\mathcal{V}^{\mathrm{fet}}) & \stackrel{\mathrm{Res}}{\longrightarrow} & \mathrm{Sh}(\mathcal{V}_n^{\mathrm{fet}})\\ \downarrow\mathcal{L} & & \downarrow\mathcal{L}_n\\ \mathrm{Rep}(\Delta_{\mathcal{V}}) & \stackrel{\mathrm{Res}}{\longrightarrow} & \mathrm{Rep}(\Delta_n)\end{array}$$

where $\mathcal{L}$ and $\mathcal{L}_n$ are the localization functors: if $\mathcal{F}$ is a sheaf on $\mathcal{V}^{\mathrm{fet}}$, respectively on $\mathcal{V}_n^{\mathrm{fet}}$, then $\mathcal{L}(\mathcal{F}) := \mathcal{F}(\overline{R}_{\mathcal{V}})$, respectively $\mathcal{L}_n(\mathcal{F}) := \mathcal{F}(\overline{R}_{\mathcal{V}})$. Therefore we have $\mathcal{L}_n(\mathrm{Res}(\mathcal{F})) \cong \mathrm{Res}(\mathcal{L}(\mathcal{F}))$ and so the diagram is commutative. Both $\mathcal{L}$ and $\mathcal{L}_n$ are equivalences of categories, therefore in order to prove that $\mathcal{A}_{\mathcal{U},n}$ (seen as sheaf on $\mathcal{V}_n^{\mathrm{fet}}$) extends uniquely to a sheaf on $\mathcal{V}^{\mathrm{fet}}$ it is enough to show that the Δ_n-action on $A_{\mathcal{V},n} := \mathcal{L}_n(\mathcal{A}_{\mathcal{U},n})$ extends uniquely to a $\Delta_{\mathcal{V}}$-action.

Let us remark that $\mathbb{A}^{\nabla}_{\max,n}(\overline{R}_{\mathcal{V}})[y_1, ..., y_d] = A^{\nabla}_{\max,n}(\overline{R}_{\mathcal{V}})[y_1, ..., y_d]$, where until the end of this section we put $y_i := y_i^{(n)}, 1 \leq i \leq d$. As $A^{\nabla}_{\max,n}(\overline{R}_{\mathcal{V}})$ has a canonical action of $\Delta_{\mathcal{V}}$, we only need to define the action on y_i, $1 \leq i \leq d$. For this let us denote by $c_i : \Delta_{\mathcal{V}} \longrightarrow \mathbb{Z}_p$ the cocycle defined by: if $\sigma \in \Delta_{\mathcal{V}}$

$$\sigma\big((T_i^{1/p^m})_{m=0}^{\infty}\big) = (T_i^{1/p^m})_{m=0}^{\infty}\varepsilon^{c_i(\sigma)}.$$

Let us remark that after we fixed the choices of p-power roots of T_i and of 1, the cocycles c_i are uniquely determined for every $1 \leq i \leq d$. Let us denote for every such i and every $\sigma \in \Delta_{\mathcal{V}}$ by $e_i(\sigma) \in A_{\max,n}$ the image under the natural map $A_{\max} \longrightarrow A_{\max,n}$ of the element

$$(1 - [\varepsilon]^{c_i(\sigma)})/p \in A_{\max}.$$

Then, for every $\sigma \in \Delta_{\mathcal{V}}$, we define

$$\sigma(y_i) := y_i + e_i(\sigma)\widetilde{T}_i \otimes 1 \in A_{\mathcal{V},n}.$$

By the definition above, $A_{\mathcal{V},n}$ is now a representation of $\Delta_{\mathcal{V}}$ and so let us denote by $\mathbb{A}_{\max,\mathfrak{U},n}$ the unique sheaf on $\mathfrak{U}_{\overline{K}}$ such that for every $\mathcal{V}$ as above we have natural isomorphisms as $\Delta_{\mathcal{V}}$-representations $\mathbb{A}_{\max,\mathfrak{U},n}(\overline{R}_{\mathcal{V}}) \cong A_{\mathcal{V},n}$. It follows that $\mathbb{A}_{\max,\mathfrak{U},n}|_{\mathfrak{U}_{\overline{K},n}} = \mathcal{A}_{\mathcal{U},n}$.

Step 2: extension of the morphisms $h_{\mathcal{U},n}$ and $\theta_{\mathcal{U},n}$

We show that $h_{\mathcal{U},n} : \mathbb{W}_n(\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K},n}} \longrightarrow \mathcal{A}_{\mathcal{U},n}$ and $\theta_{\mathcal{U},n} : \mathcal{A}_{\mathcal{U},n} \longrightarrow (\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p^n\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K},n}}$ extend uniquely to morphisms of sheaves $h_{\mathcal{U}} : \mathbb{W}_n(\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K}}} \longrightarrow \mathbb{A}_{\max,\mathfrak{U},n}$ and $\theta_{\mathcal{U},n} : \mathbb{A}_{\max,\mathfrak{U},n} \longrightarrow (\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p^n\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K}}}$ respectively.

a) *The extension of $h_{\mathcal{U},n}$.* As the natural inclusion $\mathbb{W}_n(\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p\mathcal{O}_{\mathfrak{X}_{\overline{K}}}) \longrightarrow \mathbb{A}^{\nabla}_{\max,n}$ is in fact defined over all $\mathfrak{X}_{\overline{K}}$, it is enough to show that the natural morphism induced by $h_{\mathcal{U},n}$, $\mathcal{O}_X|_{\mathfrak{U}_{\overline{K},n}} \longrightarrow \mathcal{A}_{\mathcal{U},n}$ extends to the whole of $\mathfrak{U}_{\overline{K}}$. Let us fix $\mathcal{V}$ as above, then it is enough to show that the map induced by $h_{\mathcal{U},n}$, $R_{\mathcal{V}} \longrightarrow A_{\mathcal{V},n}$ is $\Delta_{\mathcal{V}}$-invariant. But this map is completely determined by the map $R_0 \longrightarrow A_{\mathcal{V},n}$. In the end we have to prove that the images of T_i,

$1 \le i \le d$, are $\Delta_{\mathcal{V}}$-invariant. Let us recall, $h_{\mathcal{U},n}(T_i) = \widetilde{T}_i \otimes 1 + X_i = \widetilde{T}_i \otimes 1 + py_i$. Therefore,

$$\begin{aligned} \sigma(h_{\mathcal{U},n}(T_i)) &= \sigma(\widetilde{T}_i) \otimes 1 + p\sigma(y_i) = [\varepsilon]^{c_i(\sigma)}\widetilde{T}_i \otimes 1 + p(e_i(\sigma)\widetilde{T}_i \otimes 1 + y_i) \\ &= [\varepsilon]^{c_i(\sigma)}\widetilde{T}_i \otimes 1 + (1 - [\varepsilon]^{c_i(\sigma)})\widetilde{T}_i \otimes 1 + X_i = h_{\mathcal{U},n}(T_i). \end{aligned}$$

b) *The extension of* $\theta_{\mathcal{U},n}$.

Following the same line of arguments as above, after fixing a small affine $\mathcal{V}$, we need to prove that the map induced by $\theta_{\mathcal{U},n}$, $A_{\mathcal{V},n} \longrightarrow (\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p^n\mathcal{O}_{\mathfrak{X}_{\overline{K}}})(\overline{R}_{\mathcal{V}})$ is $\Delta_{\mathcal{V}}$-equivariant. It is then enough to look at the images of $y_i, 1 \le i \le d$. Let us choose such an i and let $\sigma \in \Delta_{\mathcal{V}}$. We have

$$\theta_{\mathcal{U},n}(\sigma(y_i)) = \theta_{\mathcal{U},n}(y_i + e_i(\sigma)\widetilde{T}_i \otimes 1) = \theta_{\mathcal{U},n}(y_i) + \theta_{\mathcal{U},n}(e_i(\sigma))\theta_{\mathcal{U},n}(\widetilde{T}_i \otimes 1) = T_i\theta_n(e_i(\sigma)).$$

Now $e_i(\sigma) \in A_{\max,n}$ and we have $(1 - [\varepsilon]^{c_i(\sigma)})/p = a_i(\sigma)(\xi/p)$ in $A_{\max}$, with $a_i(\sigma) \in A^+_{\text{inf}}$, we have that $e_i(\sigma) = b_i(\sigma)\delta_n$, where $b_i(\sigma) \in W_n$ is the image of $a_i(\sigma)$ and $\delta_n \in A_{\max,n}$ is the image of Y_0. Therefore $\theta_n(e_i(\sigma)) = \theta_n(b_i(\sigma))\theta_n(\delta_n) = 0$ and so $\theta_{\mathcal{U},n}(\sigma(y_i)) = 0 = \sigma(\theta_{\mathcal{U},n}(y_i))$.

Now let us remark that for every $n \ge 0$, we have natural morphisms of sheaves $\mathbb{A}_{\max,\mathfrak{U},n+1} \longrightarrow \mathbb{A}_{\max,\mathfrak{U},n}$ induced by the natural morphism $\mathbb{A}^{\nabla}_{\max,n+1}|_{\mathfrak{U}} \longrightarrow \mathbb{A}^{\nabla}_{\max,n}|_{\mathfrak{U}}$, which make the family $\mathbb{A}_{\max,\mathfrak{U}} := \{\mathbb{A}_{\max,\mathfrak{U},n}\}_{n\ge 0}$ into a projective system of torsion sheaves, i.e. a continuous sheaf. Moreover, the family of maps $\{h_{\mathcal{U},n}\}_{n\ge 0}$ induces a morphism of continuous sheaves $h_{\mathcal{U}} : \mathcal{O}_{\hat{\mathcal{U}}} \longrightarrow \mathbb{A}_{\max,\mathfrak{U}}$ and the family $\{\theta_{\mathcal{U},n}\}_{n\ge 0}$ induces a morphism of continuous sheaves $\theta_{\mathcal{U}} : \mathbb{A}_{\max,\mathfrak{U}} \longrightarrow \hat{\mathcal{O}}_{\mathfrak{U}_{\overline{K}}}$. Here we have denoted by $\mathcal{O}_{\hat{\mathcal{U}}}$ the continuous sheaf $\{\mathcal{O}_{\mathcal{U}}/p^n\mathcal{O}_{\mathcal{U}}\}_{n\ge 0}$ and $\hat{\mathcal{O}}_{\mathfrak{U}_{\overline{K}}}$ is the continuous sheaf $\{(\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p^n\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K}}}\}_{n\ge 0}$.

Step 3: Gluing of $\mathbb{A}_{\max,\mathfrak{U}_{\overline{K}},n}$.

We choose a covering $\{\mathcal{U}_j\}_j$ of X by small affines. For each j, we have defined unique continuous sheaves $\mathbb{A}_{\max,\mathfrak{U}_j}$ on $\mathfrak{U}_{j,\overline{K}}$. By uniqueness, these sheaves glue to give a unique continuous sheaf $\mathbb{A}_{\max}$ on $\mathfrak{X}_{\overline{K}}$, together with morphisms of sheaves $h : \mathbb{A}^+_{\text{inf}} \longrightarrow \mathbb{A}_{\max}$, $\mathbb{A}^{\nabla}_{\max} \longrightarrow \mathbb{A}_{\max}$ and $\theta : \mathbb{A}_{\max} \longrightarrow \hat{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$, such that for every j, their restrictions to $\mathfrak{U}_{i,\overline{K}}$ are the ones defined above. □

3.2 *Further properties*

The continuous sheaf $\mathbb{A}_{\max}$ constructed above has nice properties summarized in Theorem 1.5, which we restate and prove below:

***Theorem* 1.5** — *Let us fix $n \geq 1$.*

1) *The sheaf $\mathbb{A}_{\max}$ has a decreasing filtration by sheaves of ideals $\mathrm{Fil}^r\mathbb{A}_{\max} := (Ker(\theta))^r$, for all $r \geq 0$.*

2) *There is a unique connection $\nabla := \{\nabla_n\}_{n\geq 1} : \mathbb{A}_{\max} \longrightarrow \mathbb{A}_{\max} \otimes_{\mathcal{O}_{\hat{X}}} \Omega^1_{\hat{X}/\mathcal{O}_K}$ such that*

(a) $\nabla|_{\mathbb{A}^{\nabla}_{\max}} = 0$

(b) for every $n \geq 0$ and every small affine $\mathcal{U}$ of X with parameters $T_1, T_2, ..., T_d$ and for every pair $(\mathcal{V}, \mathcal{W})$ in $\mathfrak{U}_{\overline{K},n}$, if we denote as before the elements $y_1, y_2, ..., y_d \in \mathbb{A}_{\max,n}(\mathcal{V}, \mathcal{W})$, then $\nabla_n(y_i) = 1 \otimes dT_i \in \mathbb{A}_{\max,n}(\mathcal{V}, \mathcal{W}) \otimes_{R_{\mathcal{V}}} \widehat{\Omega}^1_{R_{\mathcal{V}}/\mathcal{O}_K}$.

3) *The connection described at 2) has the property that it is integrable and $\mathbb{A}^{\nabla}_{\max} = \big(\mathbb{A}_{\max}\big)^{\nabla=0}$.*

4) *We have $\nabla(\mathrm{Fil}^r\mathbb{A}_{\max}) \subset \mathrm{Fil}^{r-1}\mathbb{A}_{\max} \otimes_{\mathcal{O}_{\hat{X}}} \Omega^1_{\hat{X}/\mathcal{O}_K}$ for every $r \geq 1$, i.e. ∇ satisfies the Griffith transversality property.*

Proof : Let us first remark that the properties 2) a) and b) define a unique connection on the restrictions of the sheaf $\mathbb{A}_{\max,n}$ to $\mathfrak{U}_{\overline{K},n}$. We show that it extends uniquely to a connection on the whole of $\mathfrak{U}_{\overline{K}}$. For this it is enough to show that if we fix an affine open $\mathcal{V}$ of X^{et} such that the structure map $\mathcal{V} \longrightarrow X$ factors through $\mathcal{U}$, the connection $\nabla_n : A_{\mathcal{U},n} \longrightarrow A_{\mathcal{U},n} \otimes_{R_{\mathcal{V}}} \widehat{\Omega}^1_{R_{\mathcal{V}}/\mathcal{O}_K}$ induced by ∇_n is $\Delta_{\mathcal{V}}$-equivariant. It is enough to check on the elements y_i, $1 \leq i \leq d$. Let $\sigma \in \Delta_{\mathcal{V}}$. Then on one hand we have $\sigma(\nabla_n(y_i)) = \sigma(1 \otimes dT_i) = 1 \otimes dT_i$. On the other hand $\nabla_n(\sigma(y_i)) = \nabla\big(y_i + e_i(\sigma)\widetilde{T}_i \otimes 1\big) = \nabla(y_i) = 1 \otimes dT_i$, which shows that indeed ∇_n is $\Delta_{\mathcal{V}}$-equivariant.

Properties 3), 4) are local therefore it is enough to verify them on the restriction $\mathcal{A}_{\mathcal{U},n}$ of $\mathbb{A}_{\max,n}$ to $\mathfrak{U}_{\overline{K},n}$, and in that case $\mathcal{A}_{\mathcal{U},n}$ is a free $\mathbb{A}^{\nabla}_{\max}|_{\mathfrak{U}_{\overline{K},n}}$-module with basis the monomials in $y_1, y_2, ..., y_d$. Therefore everything follows from the local definition of ∇_n.

4. Open Questions

Let X be a smooth proper scheme over $\mathcal{O}_K$ with geometrically connected fibers. We would like to construct a functor which makes a (Riemann-Hilbert) correspondence between the category of locally constant sheaves on X^{et}_K and the category of sheaves of $\mathcal{O}_{X_K}$-modules endowed with an integrable connection, a filtration and a Frobenius endomorphism on $\hat{X}$, where by $\hat{X}$ we mean the completion of X along the special fiber X_k.

Let $u : \mathfrak{X} \to X^{\mathrm{et}}_{\overline{K}}$ and $v : X^{\mathrm{et}} \to \mathfrak{X}$ be the functors defined by: $u(\mathcal{U}, \mathcal{W}) = \mathcal{W}$ and $v(\mathcal{U}) = (\mathcal{U}, \mathcal{U}_{\overline{K}})$ respectively.

One further defines the morphisms $u_* : \mathrm{Sh}(X^{\mathrm{et}}_{\overline{K}}) \to \mathrm{Sh}(\mathfrak{X})$ and $v_* : \mathrm{Sh}(\mathfrak{X}) \to \mathrm{Sh}(X^{\mathrm{et}})$ analogous to the push-forward in the following way: $u_*(\mathbb{L})(\mathcal{U}, \mathcal{W}) = \mathbb{L}(\mathcal{W})$ and $v_*(\mathcal{F})(\mathcal{U}) = \mathcal{F}(\mathcal{U}, \mathcal{U}_{\overline{K}})$ respectively, where $\mathbb{L}$ is a sheaf on $X^{\mathrm{et}}_{\overline{K}}$ and $\mathcal{F}$ a sheaf on $\mathfrak{X}$.

Denote now by $\mathbb{L}$ a locally constant $\mathbb{Q}_p$-sheaf on X^{et}_K which we view via base change as a sheaf on $X^{\mathrm{et}}_{\overline{K}}$ and then applying u_* as a sheaf on $\mathfrak{X}$. Put:

$$\mathbb{D}^{\mathrm{ar}}_{\mathrm{max}}(\mathbb{L}) = v_*(\mathbb{L} \otimes \mathbb{A}_{\mathrm{max}})^{G_K}.$$

We then make the following:

Conjecture 4.1 — We have an isomorphism $\mathbb{D}^{\mathrm{ar}}_{\mathrm{max}}(\mathbb{L}) \cong \mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})$ as sheaves of $\mathcal{O}_{X_K}$-modules on X^{et}_K.

The sheaf $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})$ was defined by Andreatta and Iovita in [1] by setting $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L}) = v_*(\mathbb{L} \otimes \mathbb{A}_{\mathrm{cris}})^{G_K}$ and $\mathbb{A}_{\mathrm{cris}}$ is a sheaf on $\mathfrak{X}$ also constructed in [1].

This conjecture is supported by the fact that if V is a p-adic representation of G_K then

$$(V \otimes_{\mathbb{Q}_p} B_{\mathrm{max}})^{G_K} \cong (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K} = D_{\mathrm{cris}}(V)$$

(*cf* [6, Theorem 2.3.13]).

Moreover, we believe that the sheaves $\mathbb{A}_{\mathrm{max}}$ and $\mathbb{A}^{\nabla}_{\mathrm{max}}$ can be defined even when K is ramified over $\mathbb{Q}_p$, our theory from the previous sections can be extended and one can prove "localization over small affines"-equivalent theorems for this general case. Concretely, we expect that the localizations $\mathbb{A}^{\nabla}_{\mathrm{max}}(\overline{R}_{\mathcal{U}})$ and $\mathbb{A}_{\mathrm{max}}(\overline{R}_{\mathcal{U}})$ are respectively isomorphic to the rings $A^{\nabla}_{\mathrm{max}}(\overline{R}_{\mathcal{U}})$ and $A_{\mathrm{max}}(\overline{R}_{\mathcal{U}})$ for a "small" affine $\mathcal{U} = \mathrm{Spec}(R_{\mathcal{U}})$. If X over $\mathcal{O}_K$ is a smooth, proper and connected scheme, such that there exists a scheme X_0 defined over $\mathcal{O}_{K_0}$ (K_0 being the maximal absolutely unramified subfield of K and $\mathcal{O}_{K_0}$ its ring of integers), such that $X \cong X_0 \times_{\mathcal{O}_{K_0}} \mathcal{O}_K$ then one can define $\mathbb{A}^{\nabla}_{\mathrm{max}}$ and $\mathbb{A}_{\mathrm{max}}$ by extending scalars to K. We leave open the problem of constructing $\mathbb{A}^{\nabla}_{\mathrm{max}}$ and $\mathbb{A}_{\mathrm{max}}$ for the case when X is not obtained by base change from a scheme defined over $\mathcal{O}_{K_0}$.

Finally, we make the following:

Conjecture 4.2 — There are isomorphisms (compatible with filtrations, G_K-actions and

Frobenii):

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\max} \cong \mathrm{H}^n(\mathfrak{X}, \mathbb{A}^{\nabla}_{\max}) \otimes_{A_{\max}} B_{\max} \cong \mathrm{H}^n_{\text{cris}}(\overline{X}, K_0) \otimes_{K_0} B_{\max}.$$

One obtains in this way a new proof of Faltings' theorem (see [7]). By taking G_K-invariants, one has:

Corollary 4.3 — $D_{\text{cris}}(\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong \mathrm{H}^n_{\text{cris}}(\overline{X}, K_0)$.

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