Indian J. Pure Appl. Math., **46**(5): 613-631, October 2015

DOI: 10.1007/s13226-015-0141-8

THE SPECTRAL CHARACTERIZATION OF WIND-WHEEL GRAPHS[1]

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(*Received* 12 *June* 2014; *accepted* 29 *October* 2014)

Let $G_{s,t}$ denote the wind-wheel graph on n vertices obtained by appending s triangle(s) to a pendant vertex of a path P_{t+1} with just a vertex in common. In this paper, we prove that all wind-wheel graphs are determined by their Laplacian spectra as well as signless Laplacian spectra. As $G_{s,t}$ is the well-known friendship graph if $t = 0$, our results include that the friendship graph is determined by its Laplacian spectrum as well as signless Laplacian spectrum, which provides of a new proofs of the results in [15].

Key words : Graph; adjacency spectrum; Laplacian spectrum; signless Laplacian spectrum; cospectral graphs.

1. Introduction

Throughout this paper, we concern only with simple undirected graphs (loops and multiple edges are not allowed). Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$ where $|V(G)| = n$, $|E(G)| = m$. Let $M = M(G)$ be a corresponding *graph matrix* defined in a prescribed way. The M-polynomial of G is defined as $\det(xI - M)$, where I is the identity matrix. The M-eigenvalues of G are the eigenvalues of its M-polynomial. The M-spectrum, denoted by $Spec_M(G)$, of G is a multiset consisting of the M-eigenvalues. The M-spectral radius (or M-index) of G is the largest M-eigenvalue of G. Graphs with the same spectrum with respect to a graph matrix M are called M-cospectral graphs. A graph G is said to be determined by its M-spectrum (or we say that G is a DMS-graph for short) if there

[1]This work is supported by NSFC Grant No. 11261059.

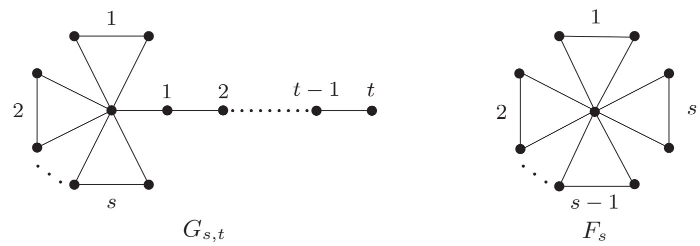

Figure 1: Wind-wheel graph $G_{s,t}$ and friendship graph F_s

is no other non-isomorphic graph with the same spectrum, that is, $Spec_M(H) = Spec_M(G)$ implies $H \cong G$ for any graph H. An M-cospectral mate of G is a graph cospectral with but not isomorphic to G.

Let D be the degree diagonal matrix of G. The graph matrix $M = M(G)$ is respectively called the *adjacency matrix*, *Laplacian matrix* and *signless Laplacian matrix* of G if M equals $A(G)$, $L(G) = D - A(G)$ and $Q(G) = D + A(G)$. The polynomials of G with respect to $A(G)$, $L(G)$ and $Q(G)$ are, respectively, presented by $\phi(G;\lambda) = \det(\lambda I - A(G)) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, $\varphi(G;\mu) = \det(\mu I - L(G)) = l_0\mu^n + l_1\mu^{n-1} + \cdots + l_{n-1}\mu$ and $\psi(G;\nu) = \det(\nu I - Q(G)) = q_0\nu^n + q_1\nu^{n-1} + \cdots + q_{n-1}\nu + q_n$. Conventionally, the *adjacency eigenvalues*, *Laplacian eigenvalues* and *signless Laplacian eigenvalues* of graph G are ordered respectively in non-increased sequence as follows: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$ and $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_n \geq 0$. The DAS-graph, DLS-graph and DQS-graph are DMS-graph for M equals $A(G)$, $L(G)$ and $Q(G)$, respectively.

A *coalescence* $G \cdot H$ is the graph obtained from graphs G and H by identifying a vertex u of G with a vertex v of H. Let F_s denote the *friendship graph* consisting of s triangles intersecting in a single vertex (see Fig. 1). A *wind-wheel graph* $G_{s,t}$ on $n = 2s + t + 1$ vertices is the graph obtained by appending s triangle(s) to a pendent vertex of *path* P_{t+1} (see Fig. 1). By $L(p, n - p)$ we denote the *lollipop graph* obtained by appending C_p to an end vertex of P_{n-p}. Clearly, the *wind-wheel graph* and *lollipop graph* can be obtained respectively by the *coalescence* operation.

Which graphs are determined by their spectra? This question originates from Chemistry and raise by Günthadr and primas (see [11] in the reference [15]), and it was proposed again by Dam and Haemers in [1]. This research has drawn much attention recently, and more and more results on this item have been reported. For a survey of the subject, one can consult [1, 3]. From our knowledge, most known DMS-graphs are characterized separately. In general, the DMS-property is no long preserved under the graph operation. However the coalescence operation of some DMS-graphs will produce new DMS-graphs. There are some related results such as: *starlike tree* [6-13], *friendship graph* F_s and *butterfly graph* $B_{r,s}$ [14-16], W_n and $S(n,c,k)$ [14, 31], K_n^m and $U_{n,p}$ [32], *firefly graph* $F_{s,t,n-2s-2t-1}$ [33], etc. Thus, to consider *whether the coalescence of some DMS-graphs is also DMS* seems an interesting problem.

As our knowledge, the DMS-property of path is preserved under the *coalescence* operation except for A-spectrum (see [6-13]). Also the DMS-property for triangles remains unchanged by the *coalescence* operation (see [14-16]). In this paper, we focus on considering if the DMS-property is preserved for the *coalescence operation* of triangles and paths? We prove that all wind-wheel graphs, the coalescence of s triangles and one path, are determined by their Laplacian spectra as well as signless Laplacian spectra, but not determined by their adjacency spectra from a known example. Moreover, our results include that the friendship graph is determined by its Laplacian spectrum as well as signless Laplacian spectrum, which provides of a new proofs of the results [15].

2. Preliminaries

We summarize some results of [1] and [4] in the following lemma.

Lemma 2.1 — Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be obtained from the spectrum.

(i) The number of vertices.

(ii) The number of edges.

(iii) Whether G is regular.

(iv) Whether G is regular with any fixed girth.

For the adjacency matrix the following follows from the spectrum.

(v) The number of closed walks of any length.

(vi) Whether G is bipartite.

For the Laplacian matrix the following follows from the spectrum.

(vii) The number of spanning trees.

(viii) The number of components.

(ix) The sum of the square of degrees of vertices.

Lemma 2.2 [2] — Suppose that N is a symmetric $n \times n$ matrix with eigenvalues $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n$. Then the eigenvalues $\kappa'_1 \geq \kappa'_2 \geq \ldots \geq \kappa'_m$ of a principal submatrix of N of size m satisfy $\kappa_i \geq \kappa'_i \geq \kappa_{n-m+i}$ for $i = 1, 2, \ldots, m$.

Let $N_G(H)$ be the number of subgraphs of graph G which are isomorphic to H and let $N_G(i)$ be the number of closed walks of length i in G. Let $N'_H(i)$ be the number of closed walks of H of length i which contain all the edges of H and $S_i(G)$ be the set of all the connected subgraphs H of G such that $N'_H(i) \neq 0$. Then

$$N_G(i) = \sum_{H \in S_i(G)} N_G(H) N'_H(i). \tag{1}$$

Eq. (1) provides some formulae for calculating the number of closed walks of length $2, 3, 4$ for any graph.

Lemma 2.3 [11] — The number of closed walks of length k $(k = 2, 3, 4)$ of a graph G are giving in the following, where m is the number of edges of G.

(i) $N_G(2) = 2m, \quad N_G(3) = 6N_G(C_3)$.

(ii) $N_G(4) = 2m + 4N_G(P_3) + 8N_G(C_4)$.

For a graph G, we denote $\bar{G}$ by the complement of G.

Lemma 2.4 [5] — Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$ and $\bar{\mu}_1 \geq \bar{\mu}_2 \geq \ldots \geq \bar{\mu}_n = 0$ be the Laplacian spectra of G and $\bar{G}$, respectively. Then $\mu_i + \bar{\mu}_{n-i} = n$ for any $i \in \{1, 2, \ldots, n-1\}$.

Let $d_i = d_i(G)$ be the degree of vertex v_i of graph G for $i = 1, \ldots, n$. In the sequel, we enumerate the degrees in non-increasing order, i.e., $d_1 \geq d_2 \geq \ldots \geq d_n$.

Lemma 2.5 [17] — Let G be a graph with at least one edge. Then $\mu_1(G) \geq d_1(G) + 1$. Moreover, if G is connected, then the equality holds if and only if $d_1(G) = n - 1$.

Lemma 2.6 [18, 19] — Let G be a connected graph. Then $\mu_1(G) \leq \max\{d(v) + m(v) : v \in V\}$ where $m(v) = \sum d(u)/d(v)$, and the sum is extended to the neighborhood of v. Moreover, the equality holds if and only if G is a regular bipartite graph or bipartite semi-regular graph.

Lemma 2.7 [20] — Let G be a graph with $n \geq 2$ vertices. Then $\mu_1(G) \leq d_1(G) + d_2(G)$.

Lemma 2.8 [21] — Let G be a connected graph with $n \geq 3$ vertices. Then $d_2(G) \leq \mu_2(G)$.

Lemma 2.9 [4] — Let G be a graph with n vertices and m edges. Then the first four coefficients in $\varphi(G;\mu)$ are

$$l_0 = 1, \qquad l_1 = -2m, \qquad l_2 = 2m^2 - m - \tfrac{1}{2}\sum_{i=1}^{n} d_i^2,$$
$$l_3 = \tfrac{1}{3}\big(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + 6N_G(C_3)\big).$$

From Lemma 2.9 we easily obtain the following.

Lemma 2.10 — If G and H are L-cospectral and have the same degree sequences, then $N_G(C_3) = N_H(C_3)$.

Let G and H be the two graphs with $deg(G) = (d_1, d_2, \ldots, d_n)$, $deg(H) = (d_1', d_2', \ldots, d_n')$, and adjacency matrices $A(G)$ and $A(H)$, respectively. Note that $tr(A(G)^3) = 6N_G(C_3)$. If G and H are L-cospectral, then by Lemma 2.1 and Lemma 2.9, we have

$$tr(A(G)^3) - \sum_{i=1}^{n} d_i^3 = tr(A(H)^3) - \sum_{i=1}^{n} d_i'^3. \tag{2}$$

From Eq. (2) the authors in [28] give a L-cospectral invariant

$$\tilde{\varepsilon}_3(G) = tr(A(G)^3) - \sum_{i=1}^{n}(d_i - 2)^3. \tag{3}$$

Lemma 2.11 [28] — Let G and H be the two graphs with $deg(G) = (d_1, d_2, \ldots, d_n)$, $deg(H) = (d_1', d_2', \ldots, d_n')$, and adjacency matrixes $A(G)$ and $A(H)$, respectively. If G and H have the same Laplacian spectrum, then $\tilde{\varepsilon}_3(G) = \tilde{\varepsilon}_3(H)$.

The connected graphs with A-index less than 2 are proper subgraphs of the Smith graphs (namely, those graphs whose A-index equal 2; see [26, 29])

Lemma 2.12 — Let $\mathcal{G}_A^2$ denote the set of connected graphs whose A-index is strictly less than 2. Then

$$\mathcal{G}_A^2 = \{P_n(n \geq 1), T_{1,1,n-3}(n \geq 4)\} \cup \{T_{1,2,k} | k = 2, 3, 4\}.$$

3. The L-Spectral Characterization of $G_{s,t}$

In this section, we will show that all wind-wheel graphs $G_{s,t}$ are determined by their Laplacian spectra. First, we recall some notions that can be found in [35] and [36]. A matrix M is said to be nonnegative if $m_{ij} \geq 0$ for all i and j. A matrix is reducible if and only if it can be placed into block upper-triangular form by simultaneous row/column permutations. A square matrix that is not reducible is said to be irreducible. If M is a matrix, denote by $|M|$ the matrix obtained by replacing each entry of M by its absolute value. Denote by $\rho(|M|)$ the spectral radius of $|M|$.

***Theorem* 3.1** [35] — *Let M be irreducible and let η be an eigenvalue of M. Then $|\eta| \leq \rho(|M|)$, with equality if and only if $M = e^{i\phi}N|M|N^{-1}$, where $|N| = I$.*

***Theorem* 3.2** [35, 36] — *Let M be an irreducible nonnegative matrix. Then $\rho(M) \leq \max\{\sum_{j=1}^{n} m_{ij} | i = 1, \ldots, n\}$, with equality iff all row sums are equal.*

Lemma 3.1 — For the graph $G_{s,t}$ we get $\mu_2(G_{s,t}) < 4$.

PROOF : Let v be the vertex with maximum degree of $G_{s,t}$, and M_v be the $(n-1)\times(n-1)$ principal submatrix of $L(G_{s,t})$ formed by deleting the rows and columns corresponding to v. Here, M_v is reducible and contains negative entries. So, we consider $|M_v|$, which is nonnegative. Although, $|M_v|$ is reducible, and it contains $s+1$ irreducible principal submatrices that correspond the components of $G_{s,t} - v$, and each of them has spectral radius strictly less than 4. Thus, the largest eigenvalue of $|M_v|$ is less than 4, and so is M_v. By Lemma 2.2, $\mu_2(G_{s,t}) < 4$. □

Lemma 3.2 — For a graph $G_{s,t}$, we have

(i) if $t = 0$, then $\mu_1(G_{s,t}) = 2s + 1$;

(ii) if $t \geq 1$, then $2s + 2 \leq \mu_1(G_{s,t}) < 2s + 3$.

PROOF : Let v be the vertex with maximum degree of $G_{s,t}$. If $t = 0$, then $d(v) = 2s$, and so $\mu_1(G_{s,t}) = 2s + 1$ by Lemma 2.5; similarly, we have $\mu_1(G_{s,t}) \geq 2s + 2$ if $t \geq 1$. On the other hand, by Lemma 2.6, $\mu_1(G_{s,t}) < 2s + 3$. Thus, $2s + 2 \leq \mu_1(G_{s,t}) < 2s + 3$.

Lemma 3.3 — Suppose that H and $G_{s,t}$ are L-cospectral where $s \geq 2$. Then the degree sequence of H is determined by its L-spectrum.

PROOF : By Lemma 3.1, $\mu_2(H) < 4$, and thus $d_2(H) \leq 3$ by Lemma 2.8. Since H and $G_{s,t}$ are L-cospectral, by Lemma 2.1 we know that H is connected, and they have the same order, size and the sum of the squares of vertex degrees. Assume that H has n_i vertices of degree i for $i = 1, 2, \ldots, d_1(H)$. Then

$$\sum_{i=1}^{d_1(H)} n_i = n(G_{s,t}), \tag{4}$$

$$\sum_{i=1}^{d_1(H)} in_i = 2m(G_{s,t}), \tag{5}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = n'_1 + 4n'_2 + d_1^2(G_{s,t}) \tag{6}$$

where n'_i is the number of vertices of degree i $(i = 1, 2)$ in $G_{s,t}$.

Case $1 : t \geq 1$.

Clearly, $n(G_{s,t}) = n$, $m(G_{s,t}) = n + s - 1$, $n'_1 = 1$, $n'_2 = n - 2$ and $d_1(G) = 2s + 1$. By adding up Eqs. (4), (5) and (6) with coefficients 2, -3 and 1, respectively, we get

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2)n_i = 4s^2 - 2s. \tag{7}$$

By Lemma 3.2 (ii), $2s + 2 \leq \mu_1(G_{s,t}) < 2s + 3$. By Lemma 2.5, $d_1(H) + 1 \leq \mu_1(H) = \mu_1(G_{s,t}) < 2s + 3$, which leads to $d_1(H) \leq 2s + 1$. On the other hand, by Lemmas 3.2 (ii) and 2.7 one gets $2s + 2 \leq \mu_1(G_{s,t}) = \mu_1(H) \leq d_1(H) + d_2(H) \leq d_1(H) + 3$, which leads to $d_1(H) \geq 2s - 1$. So, we have $2s - 1 \leq d_1(H) \leq 2s + 1$. By Lemma 2.11

$$6N_H(C_3) - \sum_{i=1}^{n} (d_i(H) - 2)^3 = \tilde{\varepsilon}_3(H) = \tilde{\varepsilon}_3(G_{s,t}) = 4s^2(3 - 2s) + 2. \tag{8}$$

First we suppose that $d_1(H) = 2s - 1$. If $n_{2s-1} \geq 2$ then $2s - 1 = d_1(H) = d_2(H) \leq 3$, which implies that $s = 2$, and it results $n_3 = 6$ from Eq. (7). Furthermore, by Eqs. (4) and (5) one can obtain $n_1 = 4$ and $n_2 = n - 10$. Consequently we have $N_H(C_3) = -2$ from Eq. (8), a contradiction. If $n_{2s-1} = 1$ then $2s - 1 = d_1(H) \geq 3 \geq d_2(H)$. From Eq. (7) we have

$$((2s - 1)^2 - 3(2s - 1) + 2) + 2n_3 = 4s^2 - 2s \tag{9}$$

Eq. (9) gives $n_3 = 4s - 3$. Then by Eqs. (4) and (5) one can obtain $n_1 = 4s - 4$ and $n_2 = n - 8s + 6$. From Eq. (8) we have $N_H(C_3) = -4s^2 + 9s - 4$, which is less than zero due to $s \geq 2$, a contradiction.

Next we assume that $d_1(H) = 2s$. If $n_{2s} \geq 2$, then $s \leq \frac{3}{2}$ since $2s = d_1(H) = d_2(H) \leq 3$, it contradicts $s \geq 2$. Thus, $n_{2s} = 1$. By simple calculation as in Eq. (9), we get $n_3 = 2s - 1$. Combining Eqs. (4) and (5) one can obtain $n_1 = 2s - 1$, $n_2 = n - 4s + 1$. Thus from Eq. (8) we have $N_H(C_3) = -2s^2 + 4s - 1$, which is less than zero due to $s \geq 2$, a contradiction.

At last we suppose that $d_1(H) = 2s+1$. Clearly, $n_{2s+1} = 1$ since $d_1(H) = 2s+1 > 3 \geq d_2(H)$. By Eq. (7) we get $n_3 = 0$. Combining Eqs. (4) and (5) one can obtain $n_1 = 1$, $n_2 = n-2$. Thus $deg(H) = deg(G_{s,t})$.

Case 2 : $t = 0$.

Obviously, $n = 2s+1$, $m = 3s$, $n'_1 = 0$, $n'_2 = n-1$ and $d_1(G_{s,t}) = 2s$. Similarly as Eq. (7) in Case 1, we get

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2)n_i = 4s^2 - 6s + 2. \tag{10}$$

From Lemma 3.2 (i) we get $\mu_1(H) = 2s+1$, thus $d_1(H) \leq 2s$ by Lemma 2.5. On the other hand, Lemma 3.2 (ii) and Lemma 2.7 give that $2s+1 = \mu_1(H) \leq d_1(H)+d_2(H) \leq d_1(H)+3$, that is, $d_1(H) \geq 2s-2$. Consequently we have $2s-2 \leq d_1(H) \leq 2s$.

First suppose that $d_1(H) = 2s-2$. If $n_{2s-2} \geq 2$, then $s = 2$ since $2s-2 = d_1(H) = d_2(H) \leq 3$. By Eqs. (4) and (5) we get $n_1 = 2-2s < 0$, a contradiction. Thus, $n_{2s-2} = 1$. Here, $s \geq 3$ since $d_1(H) = 2s-2 \geq 3 \geq d_2(H)$. By Eq. (10) we have

$$((2s-2)^2 - 3(2s-2) + 2) + 2n_3 = 4s^2 - 6s + 2. \tag{11}$$

Eq. (11) gets $n_3 = 4s-5$. Substituting it into Eq. (4) one can get $n_1 + n_2 = 5-2s < 0$, a contradiction.

Next assume that $d_1(H) = 2s-1$. If $n_{2s-1} \geq 2$, then $s = 2$ since $2s-1 = d_1(H) = d_2(H) \leq 3$. Similar to Eq. (11) we have $2n_3 = 6$, that is, $n_3 = 3$. Putting it into Eqs. (4) and (5) we get $n_1 = n_2 = 1$. By Lemma 2.11 we have $N_H(C_3) = \frac{2}{3}$, a contradiction. Thus $n_{2s-1} = 1$. By simple calculation as Eq. (11) we get $n_3 = 2s-2$. In addition, $s \geq 3$ since otherwise $s = 2$, then $n_{2s-1} = 2$, it contradicts our assumption. Furthermore, by Eqs. (4) and (5) we have $n_1 = 2s-3$, and $n_2 = n-4s+4$. Then from Lemma 2.11 we get

$$6N_H(C_3) - (8s^3 - 36s^2 + 54s - 26) = \tilde{\varepsilon}_3(H) = \tilde{\varepsilon}_3(G_{s,t}) = 6s - (8s^3 - 24s^2 + 24s - 8),$$

that is, $N_H(C_3) = -2s^2 + 6s - 3$, which is less than zero due to $s \geq 3$, a contradiction.

A last suppose that $d_1(H) = 2s$, then $n_{2s} = 1$ since $d_1(H) = 2s > 3 \geq d_2(H)$. By Eq. (10) we get $n_3 = 0$. Combining Eqs. (4) and (5), one can obtain that $n_1 + n_2 = n-1$, and $n_1 + 2n_2 = n+s$, and so, $n_1 = 0$, and $n_2 = 2s$. Hence $deg(H) = deg(G_{s,t})$.

The proof is completed. □

***Theorem* 3.3** — *All wind-wheel graphs are determined by their Laplacian spectra.*

PROOF : $G_{s,t}$ will be a path if $s = 0$. Clearly, $G_{0,t} = P_t$ is determined by its L-spectrum.

If $s = 1$, then $G_{1,t}$ is a lollipop graph with cycle C_3, which is also determined by its L-spectrum (see literature [30] Theorem 6.1).

For $s \geq 2$, let H be L-cospectral with $G_{s,t}$. Then from Lemma 2.1 H is connected graph. By Lemma 3.3 $deg(H) = deg(G_{s,t})$. Thus, H has the form of $\mathbb{G}_{s,k}$ (see Fig. 2), say $H = \mathbb{G}_{s,k}$. By Lemma 2.10 H also has s triangles. Thus, H is isomorphic to $G_{s,t}$. We complete the proof. □

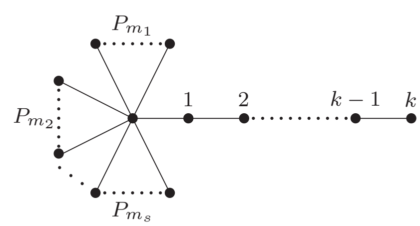

Figure 2: Graph $\mathbb{G}_{s,k}$

Interesting, $G_{s,0}$ is the well-known friendship graph F_s (see Fig. 1). It immediately follows the result.

Corollary 3.1 — The friendship graph is determined by its Laplacian spectrum.

Remark 1 : Liu *et al.*, in [14] (see Theorem 3.1) prove that all multifan graphs, namely the join (or complete) product of a vertex with a disjoint union of paths, are DLS-graphs. So they can also obtain that the friendship graph is DLS. However, our method is different.

For a graph, by Lemma 2.4 we know that its Laplacian eigenvalues determine the eigenvalues of its complement. Thus, we have the following Corollary.

Corollary 3.2 — The complements of wind-wheel graphs, and friendship graph are determined by their Laplacian spectra.

4. Some Lemmas on Q-Spectrum

In this section we give some useful lemmas that are needed in the next section.

Lemma 4.1 [22] — Let G be a graph with second maximum degree $d_2(G)$. Then $\nu_2(G) \geq d_2(G) - 1$. If the equality holds, then the maximum and the second maximum degree vertices are adjacent and $d_1(G) = d_2(G)$. Moreover, if G is connected, then $\nu_n(G) < d_n(G)$.

Lemma 4.2 [23, 24] — Let G be a connected graph of order $n \geq 2$. Then
(i) $\nu_1(G) \leq \max\{d(v) + m(v) : v \in (G)\}$, where $m(v) = \sum_{u \in N(v)} d(u)/d(v)$;
(ii) $\nu_1(G) \geq d_1(G) + 1$, with equality if and only if G is the star $K_{1,n-1}$.

Lemma 4.3 [22] — Let G be a connected graph on n vertices. Then $\nu_1(G) \leq d_1(G)+d_2(G)$ with equality holding if and only if G is a regular graph or $G \cong K_{1,n-1}$.

Lemma 4.4 [24, 25] — Let G be graph with n vertices, m edges, $N_G(C_3)$ triangles and $deg(G) = (d_1, d_2, \ldots, d_n)$. Let $T_k = \sum\limits_{i=1}^{n} \nu_i^k$, $(k = 0, 1, 2, \ldots,)$ be the kth spectral moment for the Q-spectrum. Then

$$T_0 = n, \quad T_1 = \sum_{i=1}^{n} d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6N_G(C_3) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

Based on the above Lemma 4.4 we have the following Lemma.

Lemma 4.5 — Let H be a graph Q-cospectral to G. Then

(i) G and H have the same number of vertices.

(ii) G and H have the same number of edges.

(iii) $\sum\limits_{i=1}^{n} d_i(G)^2 = \sum\limits_{i=1}^{n} d_i(H)^2$.

(iv) $6N_G(C_3) + \sum\limits_{i=1}^{n} d_i(G)^3 = 6N_H(C_3) + \sum\limits_{i=1}^{n} d_i(H)^3$.

From Lemma 4.5 we easily obtain the following Corollary.

Corollary 4.1 — If G and H are Q-cospectral and have the same degree sequences, then $N_G(C_3) = N_H(C_3)$.

Similar to Eq. (3), we also give a Q-cospectral invariant

$$\tilde{q}_3(G) = tr(A(G)^3) + \sum_{i=1}^{n} (d_i - 2)^3.$$

Thus, we have the Lemma 4.6 as follows.

Lemma 4.6 — Let G and H be the two graphs with $deg(G) = (d_1, d_2, \ldots, d_n)$, $deg(H) = (d_1{}', d_2{}', \ldots, d_n{}')$, and adjacency matrices $A(G)$ and $A(H)$, respectively. If G and H have the same signless Laplacian spectrum, then $\tilde{q}_3(G) = \tilde{q}_3(H)$.

PROOF : Since G and H are Q-cospecral, Lemma 4.5 (ii), (iii) and (iv) imply that

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d_i{}', \quad \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} d_i'^2 \tag{12}$$

and

$$tr(A(G)^3) + \sum_{i=1}^{n} d_i^3 = tr(A(H)^3) + \sum_{i=1}^{n} {d_i'}^3. \tag{13}$$

From the two equations of (12) we have

$$-\sum_{i=1}^{n} 6d_i^2 + \sum_{i=1}^{n} 12d_i - 8n = -\sum_{i=1}^{n} 6{d_i'}^2 + \sum_{i=1}^{n} 12d_i' - 8n. \tag{14}$$

Note that

$$\sum_{i=1}^{n}(d_i - 2)^3 = \sum_{i=1}^{n} d_i^3 - \sum_{i=1}^{n} 6d_i^2 + \sum_{i=1}^{n} 12d_i - 8n.$$

Thus by adding the both sides of Eqs. (13) and (14) we get

$$tr(A(G)^3) + \sum_{i=1}^{n}(d_i - 2)^3 = tr(A(H)^3) + \sum_{i=1}^{n}(d_i' - 2)^3,$$

and so the result follows. □

Lemma 4.7 [24] — Let $\nu_1(G)$ be the Q-index of graph G, then

(i) $\nu_1(G) = 0$ if and only if G has no edges;

(ii) $0 < \nu_1(G) < 4$ if and only if all components of G are paths;

(iii) For a connected graph G, we have $\nu_1(G) = 4$ if and only if G is a cycle C_n or star $K_{1,3}$.

Recall that for any graph G, $\nu_n(G) \geq 0$ holds. The following lemma is taken from [24].

Lemma 4.8 — In any graph the multiplicity of the eigenvalue 0 in the Q-spectrum is equal to the number of bipartite components.

Let $S(G)$ be the *subdivision graph* of G obtained by replacing each edge of G by a path of length two. The lemma below gives the relation between the Q-polynomial of a graph and the A-polynomials of its subdivision graph $S(G)$.

Lemma 4.9 [27] — Let G be a graph of order n and size m. Then

$$\phi(S(G); \lambda) = \lambda^{m-n} \psi(G; \lambda^2).$$

5. The Q-Spectral Characterization of $G_{s,t}$

In this section, we will give a Q-characterization of wind-wheel graphs. Firstly, we prove some lemmas which will be used in the proof of our main results.

Lemma 5.1 — For a graph $G_{s,t}$, we have $\nu_2(G_{s,t}) < 4$, and

(i) if $t \geq 1$, then $2s + 2 < \nu_1(G_{s,t}) < 2s + 3$;

(ii) if $t = 0$, then $\nu_n(G_{s,0}) = 1$, and $\nu_1(G_{s,0}) = \frac{2s+3+\sqrt{(2s+3)^2-16s}}{2}$.

Proof : Let v be the vertex with maximum degree of $G_{s,t}$. Then $G_{s,t} = sK_2 \cup P_t$. By Lemma 4.8, $\lambda_1(S(G_{s,t}) - v) < 2$. The well-know interlacing implies that $\lambda_2(S(G_{s,t})) < 2$. Furthermore, by Lemma 4.9 we have $\nu_2(G_{s,t}) < 4$.

For $t \geq 1$, since $d_1(G_{s,t}) = d(v) = 2s + 1$, $d_2(G_{s,t}) = 2$, we have $2s + 2 < \mu_1(G_{s,t}) < d_1(G_{s,t}) + d_2(G_{s,t}) = 2s + 3$ from Lemmas 4.2 and 4.3. It follows (i).

For $t = 0$, by direct calculation we get

$$\psi(G_{s,0}, \nu) = (\nu - 1)^s(\nu - 3)^{s-1}(\nu^2 - (2s + 3)\nu + 4s), \tag{15}$$

which gives $\nu_n(G_{s,0}) = 1$ and $\nu_1(G_{s,0}) = \frac{2s+3+\sqrt{(2s+3)^2-16s}}{2}$. It follows (ii). □

Lemma 5.2 — Let H be Q-cospectral with $G_{s,t}$. Then H is a connected.

Proof : Assume that $H = \bigcup_{i=1}^{k} H_i$, where H_i $(1 \leq i \leq k)$ is a connected component. Since $G_{s,t}$ is non-bipartite graph, by Lemma 4.8 each H_i $(1 \leq i \leq k)$ is non-bipartite. In view of $\psi(H) = \prod_{i=1}^{k} \psi(H_i) = \psi(G_{s,t})$, we obtain from Lemma 4.9 that

$$\phi(S(H)) = \prod_{i=1}^{k} \phi(S(H_i)) = \phi(S(G_{s,t})),$$

which implies that there exists some component, say H_1 such that $\lambda_1(S(H)) = \lambda_1(S(H_1)) = \lambda_1(S(G_{s,t}))$. Furthermore, by the proof of Lemma 5.1(i) we have

$$\lambda_2(S(H)) = \max\{\lambda_2(S(H_1)), \lambda_1(S(H_i))|2 \leq i \leq k\} = \lambda_2(S(G_{s,t})) < 2.$$

Therefore, by Lemma 2.12 we get $S(H_i) \in \mathcal{G}_A^2$ for $2 \leq i \leq k$. Then $S(H_i)$ is a tree and thus H_i is also a tree for $2 \leq i \leq k$, a contradiction. □

Let G be a connected graph with n vertices and m edges. If $m = n + k - 1$, then G is a *k-cyclic graph*. Clearly, $G_{s,t}$ is a s-cyclic graph.

Corollary 5.1 — Let H be Q-cospectral with $G_{s,t}$. Then H at most has s triangles.

PROOF : From Lemma 5.2, H is a connected graph. Because H and $G_{s,t}$ are Q-cospectral, they have the same number of vertices and the same number of edges. Since $G_{s,t}$ is a s-cyclic graph and $\nu_2(G_{s,t}) < 4$, so the conclusion is obvious. □

Lemma 5.3 — Graphs H and $G_{s,0}$ ($s \geq 2$) are not Q-cospectral if $d_1(H) \leq 2s - 1$.

PROOF : Assume that H and $G_{s,0}$ are Q-cospectral. Then by Lemmas 5.2, H is a connected. From Lemma 5.1 we known that $\nu_n(H) = 1$, $\nu_2(H) \leq 4$ and $2s + 1 < \nu_1(H) = \frac{2s+3+\sqrt{(2s+3)^2-16s}}{2} < 2s + 2$. Furthermore, $d_2(H) - 1 \leq \nu_2(H) < 4$ and $1 = \nu_n(H) < d_n(H)$ by Lemma 4.1, that is, $2 \leq d_n(H) \leq d_2(H) \leq 4$. Suppose that $\max\{d(v) + m(v) : v \in V(H)\}$ occurs at the vertex v_0 of H.

If $d(v_0) = 2$ then $\nu_1(H) \leq d(v_0) + m(v_0) \leq d(v_0) + d_1(H) \leq d(v_0) + 2s - 1 \leq 2s + 1$, a contradiction.

If $d(v_0) = 3$, we distinct two cases as follows:

(a) when $s = 2$, we have $n = 5$, and $m = 6$. By Lemma 4.5(ii), H has 6 edges. Note that $\nu_1(H) = \nu_1(G_{s,0}) \approx 5.56$, and $d_n(H) \geq 2$. Because $d(v_0) = 3$, there is a non-adjacent vertex of v_0 in H. Then $\nu_1(H) \leq d(v_0) + m(v_0) \leq d(v_0) + \frac{2m-d(v_0)-d_n}{d(v_0)} = 3 + \frac{12-3-2}{3} \approx 5.33 < 5.56$, it is impossible.

(b) when $s \geq 3$, since $d_1(H) = 2s - 1 > 4 \geq d_2(H)$, we get $\nu_1(H) \leq d(v_0) + m(v_0) \leq 3 + \frac{2s-1+4\times 2}{3} = \frac{2s+16}{3} < \frac{2s+3+\sqrt{(2s+3)^2-16s}}{2} = \nu_1(H)$, a contradiction.

If $4 \leq d(v_0) \leq 2s - 1$, here H has $m = 3s$ edges by Lemmas 4.5 (ii), and $s \geq 3$. Then

$$\nu_1(H) \leq d(v_0) + m(v_0) \leq d(v_0) + \frac{2m - d(v_0) - 2}{d(v_0)} = d(v_0) - 1 + \frac{6s - 2}{d(v_0)}.$$

Let $f(x) = x - 1 + \frac{6s-2}{x}$, where $4 \leq x \leq 2s - 1$. It is easy to show that $f(x)$ is a monotone increasing function, so we get

$$f(x) \leq f(2s - 1) = 2s + 1 + \frac{1}{2s - 1} < \frac{2s + 3 + \sqrt{(2s + 3)^2 - 16s}}{2} = \nu_1(H),$$

it is a contradiction.

The proof is completed. □

Lemma 5.4 — Suppose that H and $G_{s,t}$ are Q-cospectral where $s \geq 2$. Then the degree sequence of H is determined by the shared Q-spectrum.

Proof : By Lemma 5.2 H is a connected graph. Furthermore, we have $\nu_2(H) < 4$ in light of Lemma 5.1, it follows from Lemma 4.1 that $d_2(H) \leq 4$. Also since H and $G_{s,t}$ are Q-cospectral, by Lemma 4.5 they have the same order, size and the sum of the squares of degrees of vertices. Assume that H has n_i vertices of degree i for $i = 1, 2, \ldots, d_1(H)$. Then

$$\sum_{i=1}^{d_1(H)} n_i = n(G_{s,t}), \tag{16}$$

$$\sum_{i=1}^{d_1(H)} in_i = m(G_{s,t}), \tag{17}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = n'_1 + 4n'_2 + d_1^2(G_{s,t}) \tag{18}$$

where n'_i is the number of vertices of degree i $(i = 1, 2)$ in $G_{s,t}$.

Case 1 : $t \geq 1$.

Obviously, $n(G_{s,t}) = n$, $m(G_{s,t}) = n + s - 1$, $n'_1 = 1$, $n'_2 = n - 2$ and $d_1(G) = 2s + 1$. By adding up Eqs. (16), (17) and (18) with coefficients 2, -3 and 1, respectively, we have

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2)n_i = 4s^2 - 2s. \tag{19}$$

Since $\nu_1(H) = \nu_1(G_{s,t})$, by Lemma 5.1 (i) $2s+2 < \nu_1(H) < 2s+3$, it follows from Lemma 4.2 that $d_1(H) \leq 2s + 1$. On the other hand, by Lemma 4.3, one can get $2s + 2 < \nu_1(H) \leq d_1(H) + d_2(H) \leq d_1(H) + 4$, that is, $2s - 1 \leq d_1(H)$. Consequently, $2s - 1 \leq d_1(H) \leq 2s + 1$. By Lemma 4.6 we have

$$6N_H(C_3) + \sum_{i=1}^{n} (d_i - 2)^3 = \tilde{q}_3(H) = \tilde{q}_3(G) = 6s + (2s - 1)^3 - 1. \tag{20}$$

First suppose that $d_1(H) = 2s - 1$. If $n_{2s-1} \geq 2$, then $s = 2$ since $2s - 1 = d_1(H) = d_2(H) \leq 4$. Hence, $d_1(H) = 3$. From Eq. (19), $n_3 = 6$. Putting it into Eqs. (16) and (17) we have $n_1 = 4$, $n_2 = n - 10$. By Eq. (20)

$$6N_H(C_3) + 2 = 2 + (5 - 2)^3 - 1,$$

that is, $N_H(C_3) = \frac{13}{3}$, it is impossible. Thus, $n_{2s-1} = 1$, and $s \geq 3$ since $d_1(H) \geq 4 \geq d_2(H)$. Combining Eqs. (16), (17) and (19) we have

$$\begin{cases} n_1 = 4s - 4 - n_4 \\ n_2 = n + 3n_4 - 8s + 6 \\ n_3 = 4s - 3n_4 - 3 \end{cases}$$

So we have $\frac{8s-6-n}{3} \leq n_4 \leq \frac{4s-3}{3}$.

Furthermore, by Eq. (20) we get

$$6N_H(C_3) + (4s - 3n_4 - 3) - (4s - 4 - n_4) + 8n_4 + (2s - 3)^3 = 6s + (2s - 1)^3 - 1.$$

By direct computation, $N_H(C_3) = 4s^2 - 7s + 4 - n_4$, which leads to $4s^2 - 7s + 4 - n_4 \leq s$ by Corollary 5.1, i.e., $n_4 \geq 4s^2 - 8s + 4$. However, $n_4 \geq 4s^2 - 8s + 4 > \frac{4s-3}{3} \geq n_4$ for $s \geq 2$, a contradiction.

Next assume that $d_1(H) = 2s$. If $n_{2s} \geq 2$, then $s = 2$ since $2s = d_1(H) = d_2(H) \leq 4$. Hence, $d_1(H) = 4$ and $n_4 \geq 2$. From Eq. (19) $n_3 + 3n_4 = 6$. Thus $n_4 = 2$ and $n_3 = 0$. Putting it into Eqs. (16) and (17) we have $n_1 = 2$, $n_2 = n - 4$. By Eq. (20) $N_H(C_3) = \frac{11}{3}$, it is impossible. Hence, $n_{2s} = 1$. Combining Eqs. (16), (17) and (19) we have

$$\begin{cases} n_1 = 2s - 1 - n_4 \\ n_2 = n - 4s + 1 + 3n_4 \\ n_3 = 2s - 1 - 3n_4 \end{cases} \tag{21}$$

So we have $\frac{4s-n-1}{3} \leq n_4 \leq \frac{2s-1}{3}$.

From Eq. (20) we get $N_H(C_3) = 2s^2 - 2s + 1 - n_4$. Since H is a connected, by Corollary 5.1 we get $2s^2 - 2s + 1 - n_4 \leq s$, i.e., $n_4 \geq 2s^2 - 3s + 1$. However, $n_4 \geq 2s^2 - 3s + 1 > \frac{2s-1}{3} \geq n_4$ for $s \geq 2$, a contradiction.

At last suppose that $d_1(H) = 2s + 1$. Then $n_{2s+1} = 1$ since $d_1(H) = 2s + 1 > 4 \geq d_2(H)$. Putting it into Eq. (19) we get $n_3 = n_4 = 0$. Combining Eqs. (16) and (17), one can obtain $n_1 = 1$, and $n_2 = n - 2$. Consequently, $deg(H) = deg(G_{s,t})$.

Case $2:\ t = 0$.

Clearly, $n = 2s + 1$, $m = 3s$, $n'_1 = 0$, $n'_2 = n - 1$ and $d_1(G_{s,t}) = 2s$ of $G_{s,0}$. Similarly as Eq. (19) in Case 1. we get

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2)n_i = 4s^2 - 6s + 2. \tag{22}$$

Since H and $G_{s,t}$ are Q-cospectral, by Lemma 4.2(ii) we have $2s + 1 < \nu_1(H) < 2s + 2$. From Lemma 4.2 $\nu_1(H) \geq d_1(H) + 1$, thus we have $d_1(H) \leq 2s$. On the other hand, $d_1(H) \geq 2s$ by Lemma 5.3. So we have $d_1(H) = 2s$. By Eq. (22) we get $n_{2s} = 1$. Furthermore, combining Eqs. (16) and (17) we get $n_1 = 0$, and $n_2 = n - 1$. Thus, $deg(H) = deg(G_{s,t})$.

The proof is completed. □

***Theorem* 5.1** — *All wind-wheel graphs are determined by their signless Laplacian spectra.*

PROOF : $G_{s,t}$ will be a path if $s = 0$. Then $G_{0,t} = P_t$ is determined by its Q-spectrum by Lemma 4.7.

If $s = 1$, then $G_{1,t}$ is a lollipop graph with cycle C_3, which is also determined by its Q-spectrum (see [34] Theorem 3.4).

For $s \geq 2$, let H be Q-cospectral with $G_{s,t}$. Then H is a connected graph from Lemma 5.2. By Lemma 5.4 $deg(H) = deg(G_{s,t})$. Thus, H has the form of $\mathbb{G}_{s,k}$ (see Fig. 2), say $H = \mathbb{G}_{s,k}$. By Corollary 4.1 H also has s triangles. Thus, H is isomorphic to $G_{s,t}$. The proof completes. □

For the graph $G_{s,t}$, if $t = 0$, then we immediately obtain a result due to Wang *et al.* (see [15] Lemma 4.7).

Corollary 5.2 [15] — The friendship graph is determined by its Q-spectrum.

Remark 2 : For the graph $G_{s,t}$, if $t = 1$, then $G_{s,1} \cong S(2s+2, s, 0)$; if $t = 2$, then $G_{s,2} \cong S(2s+3, s, 1)$; by Theorem 5.1 we also get partly Liu's results [31].

Remark 3 : For the adjacency characterization of wind-wheel graphs, we easily prove that no two non-isomorphic wind-wheel graphs are A-cospectral (in fact, suppose that $G_{s',t'}$ and $G_{s,t}$ are A-cospectral. Then they have the same number of triangles by Lemma 2.3, and so $s' = N_{G_{s',t'}}(3) = N_{G_{s,t}}(3) = s$. Furthermore, by Lemma 2.1 they have the same number of vertices. So we have $2s' + t' + 1 = 2s + t + 1$, that is, $t' = t$. Thus, $G_{s',t'}$ is isomorphic to $G_{s,t}$). However there exist A-cospectral wind-wheel graphs that are non-isomorphic. In fact, a pair of G' and $G_{2,1}$ shown in Fig. 3 is an example given in literature [2]. In additon, for each fixed s ($s \geq 3$), to determine the degree sequence of a graph which is cospectral to some $G_{s,t}$ w.r.t. adjacency spectrum is more complicated, and the method used here is invalid. Thus, some new techniques are needed to characterize whether the graphs $G_{s,t}$ are determined by their A-spectrum.

As a further consideration of our results, studying the DMS-property of $G_{s,t_1,\ldots,t_k}$, the coalescence of s triangles and k ($k \geq 2$) paths P_{t_i} (see Fig. 3), is an interesting problem. Applying our methods, one can also prove that its degree sequence is determined by L-spectrum as well as Q-spectrum, but we encounter trouble to show that no two non-isomorphic $G_{s,t_1,\ldots,t_k}$ are cospectral. It is worth to mention that we don't find any cospectral mates in

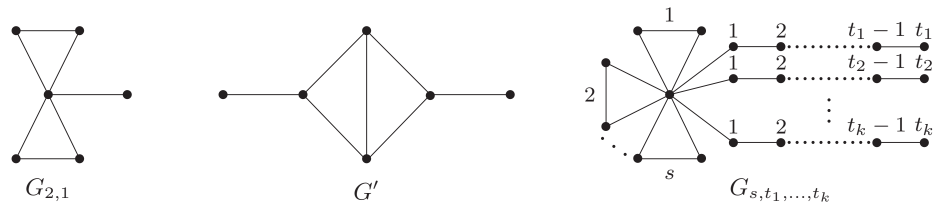

Figure 3: Graphs $G_{2,1}$, G' and $G_{s,t_1,\dots,t_k}$

$G_{s,t_1,\dots,t_k}$ for L-spectrum as well as Q-spectrum, and so we present an open problem below.

Open Problem 1 : Graphs $G_{s,t_1,\dots,t_k}$ are determined by their Laplacian spectra as well as signless Laplacian spectra.

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